ISSN 2077–9879

Eurasian Mathematical Journal

2015, Volume 6, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http : //publicationethics.org/files/u2/New$_C$ode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana
Kazakhstan

KORDAN NAURYZKHANOVICH OSPANOV

(to the 60th birthday)

On 25 September 2015 Kordan Nauryzhanovich Ospanov, professor of the Department "Fundamental Mathematics" of the L.N. Gumilyov Eurasian National University, Doctor of Physical and Mathematical Sciences (2000), a member of the Editorial Board of our journal, celebrated his 60th birthday.

He was born on September 25, 1955, in the village Zhanatalap of the Zhanaarka district of the Karaganda region. In 1976 he graduated from the Kazakh State University, and in 1981 he completed his postgraduate studies at the Abay Kazakh Pedagogical Institute.

Scientific works of K.N. Ospanov are devoted to application of methods of functional analysis to the theory of differential equations. On the basis of a local approach to the resolvent representation he has found weak conditions for the solvability of the singular generalized Cauchy-Riemann system and established coercive estimates for its solution. He has obtained a criterion of the spectrum discreteness for the resolvent of the system and the exact in order estimates of singular values and Kolmogorov widths. He has original research results on the coercive solvability of the quasilinear singular generalized Cauchy-Riemann system and degenerate Beltrami-type system. He has established important smoothness and approximation properties of non strongly elliptic systems. K.N. Ospanov has found separability conditions in Banach spaces for singular linear and quasi-linear second-order differential operators with growing intermediate coefficients and established a criterion for the compactness of its resolvent and finiteness of the resolvent type.

His results have contributed to a significant development of the theory of two-dimensional singular elliptic systems, degenerate differential equations and non strongly elliptic boundary value problems.

K.N. Ospanov has published more than 140 scientific papers. The list of his most important publications one may see on the

http://mmf.enu.kz/images/stories/photo/pasport/fm/ospanov

K.N. Ospanov is an Honoured Worker of Education of the Republic of Kazakhstan, and he was awarded the state grant "The best university teacher".

The Editorial Board of the Eurasian Mathematical Journal is happy to congratulate Kordan Nauryzkhanovich Ospanov on occasion of his 60th birthday, wishes him good health and further productive work in mathematics and mathematical education.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 6, Number 4 (2015), 19 – 28

ON CONDITIONS OF THE SOLVABILITY OF NONLOCAL MULTI-POINT BOUNDARY VALUE PROBLEMS FOR QUASI-LINEAR SYSTEMS OF HYPERBOLIC EQUATIONS

A.T. Assanova, A.E. Imanchiev

Communicated by K.N. Ospanov

Key words: nonlocal problem, multi-point condition, quasi-linear hyperbolic equation, solvability.

AMS Mathematics Subject Classification: 35L20, 35L55, 34B08, 35L72.

Abstract. A nonlocal multi-point boundary value problem for a system of quasi-linear hyperbolic equations is investigated. Based on the results for linear problems coefficient conditions are established ensuring the existence of classical solutions to nonlocal multi-point boundary value problem for a system of quasi-linear hyperbolic equations, and algorithms of finding these solutions are suggested.

1 Introduction

We consider the following nonlocal multi-point boundary value problem on $\bar{\Omega} = [0,T] \times [0,\omega]$ for a second-order system of quasilinear hyperbolic equations

$$\frac{\partial^2 u}{\partial t \partial x} = A(t,x)\frac{\partial u}{\partial x} + f\left(t, x, u, \frac{\partial u}{\partial t}\right), \qquad u \in R^n, \tag{1.1}$$

$$\sum_{i=0}^{m} \left\{ P_i(x)\frac{\partial u(t_i,x)}{\partial x} + S_i(x)\frac{\partial u(t_i,x)}{\partial t} + U_i(x)u(t_i,x) \right\} = \varphi(x), \quad x \in [0,\omega], \tag{1.2}$$

$$u(t,0) = \psi(t), \qquad t \in [0,T], \tag{1.3}$$

where $u(t,x) = col(u_1(t,x), u_2(t,x), ..., u_n(t,x))$ is the unknown function, the $n \times n$ matrices $A(t,x)$, $P_i(x)$, $S_i(x)$, $U_i(x)$, $i = \overline{0,m}$, and the n-vector function $f(t,x,u,\frac{\partial u}{\partial t})$ are continuous on $\bar{\Omega} \times R^n \times R^n$, the n-vector function φ is continuous on $[0,\omega]$, and the n-vector function ψ is continuously differentiable on $[0,T]$, $0 = t_0 < t_1 < ... < t_{m-1} < t_m = T$.

The study of nonlocal boundary value problems for hyperbolic-type equations was initiated in the 1960s (see [12, 17, 19, 20, 22] and references therein). Sufficient conditions for the existence and uniqueness of solutions to such problems have been obtained by various methods. In the last several decades nonlocal multi-point problems for hyperbolic equations appeared to be of great interest to specialists [11, 18, 20, 24]. Motivated by this, in [9, 10] the linear problem corresponding to (1.1)–(1.3) has been inestigated. Necessary and sufficient conditions for the well-posedness of linear problem were found in terms of the initial data. The main results of [9] were based on the equivalence of the well-posedness of nonlocal multi-point boundary value problem for a system of hyperbolic equations and a family of multi-point boundary value problems for the systems of ordinary differential equations. Using the parametrization method [14] necessary and sufficient conditions of the unique solvability of a family of multi-point boundary value problems for a system of ordinary differential equations are established in terms of the initial data. The coefficient criteria of unique solvability of the corresponding linear nonlocal multi-point boundary value problem (1.1)–(1.3) are obtained.

In [2,3] a nonlocal boundary value problem with data on the characteristics of the corresponding system of hyperbolic equations is considered for $t_0 = 0$, $t_1 = T$. Sufficient conditions are given for the existence and uniqueness of a classical solution to problem (1.1)–(1.3), when $m = 1$, in terms of the initial data. In [4-6, 8, 13] this problem, by introducing new functions, was reduced to a family of the two-point boundary value problems for ordinary differential equations and functional relations. Coefficient criteria of well-posedness of the considered problem were obtained. Problem (1.1)–(1.3), when $m = 1$, is investigated by this method in [7].

In this paper, we establish sufficient coefficient conditions of the unique solvability of problem (1.1)–(1.3) by introducing some additional functions and applying related results for families of multi-point boundary value problems for systems of ordinary differential equations; we suggest an algorithm for finding a solution.

Let $C(\bar{\Omega}, R^n)$ be the space of all continuous functions $u : \bar{\Omega} \to R^n$ on $\bar{\Omega}$ with the norm

$$||u||_0 = \max_{(t,x)\in\bar{\Omega}} ||u(t,x)||,$$

and $C^{1,1}(\bar{\Omega}, R^n)$ be the space of all continuous functions $u : \bar{\Omega} \to R^n$ on $\bar{\Omega}$ continuously differentiable with respect to t and x with the norm $||u||_1 = \max\Big(||u||_0, \Big|\Big|\frac{\partial u}{\partial x}\Big|\Big|_0, \Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_0\Big)$.

A function $u(t,x) \in C(\bar{\Omega}, R^n)$, that has partial derivatives $\frac{\partial u(t,x)}{\partial x} \in C(\bar{\Omega}, R^n)$, $\frac{\partial u(t,x)}{\partial t} \in C(\bar{\Omega}, R^n)$, $\frac{\partial^2 u(t,x)}{\partial t \partial x} \in C(\Omega, R^n)$ is called a classical solution to problem (1.1)–(1.3) if it satisfies system (1.1) for all $(t,x) \in \bar{\Omega}$ and meets boundary conditions (1.2) and (1.3).

2 Family of multi-point boundary value problem for systems of ordinary differential equations. Main result

We introduce new unknown functions $v(t,x) = \frac{\partial u(t,x)}{\partial x}$ and $w(t,x) = \frac{\partial u(t,x)}{\partial t}$ and reduce problem (1.1)–(1.3) to the equivalent problem

$$\frac{\partial v}{\partial t} = A(t,x)v + f(t,x,u,w), \qquad (t,x) \in \bar{\Omega}, \tag{2.1}$$

$$\sum_{i=0}^{m} P_i(x)v(t_i,x) = \varphi(x) - \sum_{i=0}^{m}\Big\{S_i(x)w(t_i,x) + U_i(x)u(t_i,x)\Big\}, \quad x \in [0,\omega], \tag{2.2}$$

$$u(t,x) = \psi(t) + \int_0^x v(t,\xi)d\xi, \quad w(t,x) = \dot{\psi}(t) + \int_0^x \frac{\partial v(t,\xi)}{\partial t}d\xi, \tag{2.3}$$

where $(t,x) \in \bar{\Omega}$.

A triple $\{v(t,x), u(t,x), w(t,x)\}$ of continuous on $\bar{\Omega}$ functions is called a solution to problem (2.1)–(2.3) if the function $v(t,x) \in C(\bar{\Omega}, R^n)$ has a continuous derivative with respect to t on $\bar{\Omega}$ and satisfies the one-parametered family of multi-point boundary value problems for the system of ordinary differential equations (2.1), (2.2), where the functions $u(t,x)$ and $w(t,x)$ are connected to $v(t,x)$ and $\dfrac{\partial v(t,x)}{\partial t}$ according to the functional relations (2.3).

Let $u^*(t,x)$ be a classical solution to problem (1.1)–(1.3). Then the triple $\{v^*(t,x), u^*(t,x), w^*(t,x)\}$, where $v^*(t,x) = \dfrac{\partial u^*(t,x)}{\partial x}$, $w^*(t,x) = \dfrac{\partial u^*(t,x)}{\partial t}$, becomes a solution to problem (2.1)–(2.3). Conversely, if a triple $\{\widetilde{v}(t,x), \widetilde{u}(t,x), \widetilde{w}(t,x)\}$ is a solution to problem (2.1)–(2.3), then $\widetilde{u}(t,x)$ becomes a classical solution to problem (1.1)–(1.3).

For fixed $w(t,x)$, $u(t,x)$ in problem (2.1)–(2.3) it is necessary to find a solution to a one-parametered family of multi-point boundary value problems for system of ordinary differential equations.

Consider the family of multi-point boundary value problems for the system of ordinary differential equations

$$\frac{\partial v}{\partial t} = A(t,x)v + F(t,x), \qquad t \in [0,T], \quad x \in [0,\omega], \quad v \in R^n, \tag{2.4}$$

$$\sum_{i=0}^{m} P_i(x)v(t_i,x) = \Phi(x), \quad x \in [0,\omega], \tag{2.5}$$

where $F(t,x) \in C(\bar{\Omega}, R^n)$ and $\Phi(x) \in C([0,\omega], R^n)$.

Continuous function $v : \bar{\Omega} \to R^n$ which has a continuous derivative with respect to t on Ω is called a solution to the family of multi-point boundary value problems (2.4)-(2.5) if it satisfies system (2.4) and condition (2.5) for all $(t,x) \in \bar{\Omega}$ and $x \in [0,\omega]$, respectively.

For fixed $x \in [0,\omega]$ problem (2.4)-(2.5) is a linear multi-point boundary value problem for the system of ordinary differential equations. The different types of multi-point boundary value problems for differential equations have been investigated by various methods [1, 15, 16, 21, 23]. Suppose a variable x is changed on $[0,\omega]$; then we obtain a family of multi-point boundary value problems for ordinary differential equations.

Definition 1. A family of multi-point boundary value problems (2.4)-(2.5) is called well-posed solvable with a constant K if for arbitrary $F(t,x) \in C(\bar{\Omega}, R^n)$ and $\Phi(x) \in C([0,\omega], R^n)$ it has a unique solution $v(t,x) \in C(\bar{\Omega}, R^n)$ and for this solution the following estimate holds

$$\max_{t\in[0,T]} ||v(t,x)|| \leq K \max\Big(\max_{t\in[0,T]} ||F(t,x)||, ||\Phi(x)||\Big),$$

where the constant K is independent of $F(t,x)$, $\Phi(x)$ and $x \in [0,\omega]$.

Consider the sets
$G(\psi,\dot{\psi},\rho) = \{(t,x,u,w) : (t,x) \in \bar{\Omega}, ||u-\psi(t)|| < \rho, ||w-\dot{\psi}(t)|| < \rho\}$,
$S(\psi(t),\rho) = \{u \in C^{1,1}(\bar{\Omega},R^n) : ||u-\psi||_1 < \rho\}$.

Condition Lip. Function $f(t,x,u,w)$ is continuous in $(t,x) \in \bar{\Omega}$ for fixed u and w and satisfies the Lipschitz condition with respect to u and w on the set $G(\psi,\dot{\psi},\rho)$, i.e.

$$||f(t,x,u,w) - f(t,x,\bar{u},\bar{w})|| \leq l_1(t,x)||u-\bar{u}|| + l_2(t,x)||w-\bar{w}||,$$

where $l_i(t,x) \geq 0$ are functions continuous on $\bar{\Omega}$ for $i = 1,2$.
We set $F^0(t,x) = f(t,x,\psi(t),\dot{\psi}(t))$, $\quad \Phi^0(x) = \varphi(x) - \sum\limits_{i=0}^{m}\Big\{S_i(x)\dot{\psi}(t_i) + U_i(x)\psi(t_i)\Big\}$,
$\tilde{L}(x) = \sum\limits_{i=0}^{m}\Big\{||S_i(x)|| + ||U_i(x)||\Big\}$,
$l_0(x) = \max\Big\{\max\limits_{t\in[0,T]} l_1(t,x) + \max\limits_{t\in[0,T]} l_2(t,x), \tilde{L}(x)\Big\}$, $\alpha(x) = \max\limits_{t\in[0,T]} ||A(t,x)||$,
$\rho_1(x) = \max(K, \alpha(x)K+1)l_0(x)$,
$\rho_2(x) = \max(K, \alpha(x)K+1)\max\Big\{\max\limits_{t\in[0,T]} ||F^0(t,x)||, ||\Phi^0(x)||\Big\}$,
and $\rho_3(x) = \rho_2(x)\exp\{x \max\limits_{x\in[0,\omega]} \rho_1(x)\}$.

To find conditions for the existence of a unique classical solution to problem (1.1)–(1.3), we consider the equivalent problem (2.1)–(2.3) and suggest an algorithm for solving it.

Step 0. Solving the family of multi-point boundary value problems (2.1)-(2.2) for $u(t,x) = \psi(t)$ and $w(t,x) = \dot{\psi}(t)$, we obtain $v^{(0)}(t,x)$ for $(t,x) \in \bar{\Omega}$. Functional relations (2.3) with $v(t,x) = v^{(0)}(t,x)$ and $\dfrac{\partial v(t,x)}{\partial x} = \dfrac{\partial v^{(0)}(t,x)}{\partial x}$ determine $u^{(0)}(t,x)$ and $w^{(0)}(t,x)$ for $(t,x) \in \bar{\Omega}$.

Step 1. Solving the family of multi-point boundary value problems (2.1)-(2.2) for $u(t,x) = u^{(0)}(t,x)$ and $w(t,x) = w^{(0)}(t,x)$, we obtain $v^{(1)}(t,x)$ for $(t,x) \in \bar{\Omega}$. Functional relations (2.3) with $v(t,x) = v^{(1)}(t,x)$ and $\dfrac{\partial v(t,x)}{\partial x} = \dfrac{\partial v^{(1}(t,x)}{\partial x}$ determine $u^{(1)}(t,x)$ and $w^{(1)}(t,x)$ for $(t,x) \in \bar{\Omega}$.

Continuing in this way, we obtain $v^{(k)}(t,x)$, $u^{(k)}(t,x)$ and $w^{(k)}(t,x)$ for $(t,x) \in \bar{\Omega}$ at the kth step, where $k = 0,1,2,\ldots$.

Sufficient conditions for the implementation and convergence of the algorithm and the existence of a unique classical solution to problem (1.1)–(1.3) are ensured by the following theorem.

Theorem 2.1. *Suppose that*

(i) $f(t,x,u,w)$ is a function satisfying Condition Lip;

(ii) the family of multi-point boundary value problems (2.4), (2.5) is well-posed solvable with the constant K;

(iii) for some $\rho > 0$ $\displaystyle\int_0^x \rho_3(\xi)d\xi \leq \rho$ for all $x \in [0,\omega]$.

Then, problem (1.1)–(1.3) has a unique classical solution $u^*(t,x)$ belonging to $S(\psi(t),\rho)$.

Proof. Consider problem (2.1)–(2.3) which is equivalent to problem (1.1)–(1.3). Using the method of successive approximations, we find a solution $v(t,x)$. For the initial approximation of $u(t,x)$ and $w(t,x)$ we take $\psi(t)$, $\dot{\psi}(t)$ respectively, and then find $v^{(0)}(t,x)$ from the problem

$$\frac{\partial v}{\partial t} = A(t,x)v + F^0(t,x), \tag{2.6}$$

$$\sum_{i=0}^{m} P_i(x)v(t_i,x) = \Phi^0(x), \qquad x \in [0,\omega]. \tag{2.7}$$

Problem (2.6)–(2.7) is a family of multi-point boundary value problems for a system of ordinary differential equations. This problem is investigated in [9] by the parametrization method [14]. Necessary and sufficient conditions of unique and well-posed solvability of the family of boundary value problems (2.4)–(2.5) were established in terms of initial data. An estimate of the solution to investigated problem in terms of the initial data was also obtained.

By assumption (ii) of the theorem, problem (2.6)–(2.7) is well-posed solvable. Therefore, problem (2.6)–(2.7) has a unique solution $v^{(0)}(t,x)$ and the following estimates hold

$$\max_{t\in[0,T]} ||v^{(0)}(t,x)|| \le K \max(\max_{t\in[0,T]} ||F^0(t,x)||, ||\Phi^0(x)||),$$

$$\max_{t\in[0,T]} \left|\left|\frac{\partial v^{(0)}(t,x)}{\partial t}\right|\right| \le [\alpha(x)K + 1] \max(\max_{t\in[0,T]} ||F^0(t,x)||, |||\Phi^0(x)||).$$

Using relations (2.3) we find $u^{(0)}(t,x)$ and $w^{(0)}(t,x)$:

$$u^{(0)}(t,x) = \psi(t) + \int_0^x v^{(0)}(t,\xi)d\xi, \quad w^{(0)}(t,x) = \dot{\psi}(t) + \int_0^x \frac{\partial v^{(0)}(t,\xi)}{\partial t} d\xi.$$

The following estimates are satisfied:

$$\max_{t\in[0,T]} ||u^{(0)}(t,x) - \psi(t)|| \le \int_0^x \max_{t\in[0,T]} ||v^{(0)}(t,\xi)||d\xi,$$

$$\max_{t\in[0,T]} ||w^{(0)}(t,x) - \dot{\psi}(t)|| \le \int_0^x \max_{t\in[0,T]} \left|\left|\frac{\partial v^{(0)}(t,\xi)}{\partial t}\right|\right| d\xi.$$

Then, we have

$$\max(\max_{t\in[0,T]} ||u^{(0)}(t,x) - \psi(t)||, \max_{t\in[0,T]} ||w^{(0)}(t,x) - \dot{\psi}(t)||)$$

$$\le \int_0^x \max\left(\max_{t\in[0,T]} ||v^{(0)}(t,\xi)||, \max_{t\in[0,T]} \left|\left|\frac{\partial v^{(0)}(t,\xi)}{\partial t}\right|\right|\right) d\xi$$

$$\leq \int_0^x \max(K, \alpha(\xi)K+1)\max(\max_{t\in[0,T]}||F^0(t,\xi)||, ||\Phi^0(\xi)||)d\xi = \int_0^x \rho_2(\xi)d\xi.$$

Suppose $u^{(k-1)}(t,x)$ and $w^{(k-1)}(t,x)$ are known. Then $v^{(k)}(t,x)$ can be found from the problem (2.1)–(2.2), where $w(t,x) = w^{(k-1)}(t,x)$, $u(t,x) = u^{(k-1)}(t,x)$, $m = 1,2,...,$ namely

$$\frac{\partial v^{(k)}}{\partial t} = A(t,x)v^{(k)} + f(t,x,u^{(k-1)}(t,x), w^{(k-1)}(t,x)), \tag{2.8}$$

$$\sum_{i=0}^{m} P_i(x)v^{(k)}(t_i,x) = \varphi(x) - \sum_{i=0}^{m}\Big\{S_i(x)w^{(k-1)}(t_i,x) + U_i(x)u^{(k-1)}(t_i,x)\Big\}, \tag{2.9}$$

where $x \in [0,\omega]$.

By assumption (ii) of the theorem, problem (2.8)–(2.9) is well-posed solvable. Therefore, problem (2.8)–(2.9) has a unique solution $v^{(0)}(t,x)$ and there hold the estimates

$$\max_{t\in[0,T]}||v^{(k)}(t,x)|| \leq K\max(\max_{t\in[0,T]}||F^{(k-1)}(t,x)||, ||\Phi^{(k-1)}(x)||),$$

$$\max_{t\in[0,T]}\left|\left|\frac{\partial v^{(k)}(t,x)}{\partial t}\right|\right| \leq [\alpha(x)K+1]\max(\max_{t\in[0,T]}||F^{(k-1)}(t,x)||, |||\Phi^{(k-1)}(x)||),$$

where

$$F^{(k-1)}(t,x) = f(t,x,u^{(k-1)}(t,x), w^{(k-1)}(t,x)),$$

$$\Phi^{(k-1)}(x) = \varphi(x) - \sum_{i=0}^{m}\Big\{S_i(x)w^{(k-1)}(t_i,x) + U_i(x)u^{(k-1)}(t_i,x)\Big\}.$$

Once $v^{(k)}(t,x)$ is found, the successive approximations for $u(t,x)$ and $w(t,x)$ are found from relations (2.3):

$$u^{(k)}(t,x) = \psi(t) + \int_0^x v^{(k)}(t,\xi)d\xi, \quad w^{(k)}(t,x) = \dot\psi(t) + \int_0^x \frac{\partial v^{(k)}(t,\xi)}{\partial t}d\xi. \tag{2.10}$$

The functions $u^{(k)}(t,x)$ and $w^{(k)}(t,x)$ satisfy the following inequalities

$$\max_{t\in[0,T]}||u^{(k)}(t,x) - \psi(t)|| \leq \int_0^x \max_{t\in[0,T]}||v^{(k)}(t,\xi)||d\xi,$$

$$\max_{t\in[0,T]}||w^{(k)}(t,x) - \dot\psi(t)|| \leq \int_0^x \max_{t\in[0,T]}\left|\left|\frac{\partial v^{(k)}(t,\xi)}{\partial t}\right|\right|d\xi.$$

Then, we get

$$\max\Big(\max_{t\in[0,T]}||u^{(k)}(t,x) - \psi(t)||, \max_{t\in[0,T]}||w^{(k)}(t,x) - \dot\psi(t)||\Big)$$

$$\leq \int_0^x \max\left(\max_{t\in[0,T]}||v^{(k)}(t,\xi)||, \max_{t\in[0,T]}\left|\left|\frac{\partial v^{(k)}(t,\xi)}{\partial t}\right|\right|d\xi\right)$$

$$\leq \int_0^x \max\Big[K, \alpha(\xi)K+1\Big]\max\Big(\max_{t\in[0,T]}||F^{(k-1)}(t,\xi)||, ||\Phi^{(k-1)}(\xi)||\Big)d\xi$$

$$\leq \int_0^x \rho_3(\xi)d\xi,$$

i.e. $u^{(k)} \in S(\psi(t), \rho)$.

We consider the differences

$$\Delta v^{(k)}(t,x) = v^{(k+1)}(t,x) - v^{(k)}(t,x),$$

$$\Delta u^{(k)}(t,x) = u^{(k+1)}(t,x) - u^{(k)}(t,x),$$

$$\Delta w^{(k)}(t,x) = w^{(k+1)}(t,x) - w^{(k)}(t,x),$$

and using the well-posedness of problem (2.4)–(2.5) we establish the following estimates

$$\max_{t\in[0,T]} ||\Delta v^{(k)}(t,x)|| \leq Kl_0(x) \max\Big(\max_{t\in[0,T]} ||\Delta u^{(k-1)}(t,x)||, \max_{t\in[0,T]} ||\Delta w^{(k-1)}(t,x)||\Big), \tag{2.11}$$

$$\max_{t\in[0,T]}\Big|\Big|\frac{\partial \Delta v^{(k)}(t,x)}{\partial t}\Big|\Big| \leq$$

$$\leq \Big(\alpha(x)K+1\Big) l_0(x) \max\Big(\max_{t\in[0,T]} ||\Delta u^{(k-1)}(t,x)||, \max_{t\in[0,T]} ||\Delta w^{(k-1)}(t,x)||\Big), \tag{2.12}$$

$$\max_{t\in[0,T]} ||\Delta u^{(k)}(t,x)|| \leq \int_0^x \max_{t\in[0,T]} ||\Delta v^{(k)}(t,\xi)||d\xi,$$

$$\max_{t\in[0,T]} ||\Delta w^{(k)}(t,x)|| \leq \int_0^x \max_{t\in[0,T]}\Big|\Big|\frac{\partial \Delta v^{(k)}(t,\xi)}{\partial t}\Big|\Big|d\xi.$$

This implies the main inequality

$$\max\Big(\max_{t\in[0,T]} ||\Delta u^{(k)}(t,x)||, \max_{t\in[0,T]} ||\Delta w^{(k)}(t,x)||\Big)$$

$$\leq \int_0^x \rho_1(\xi) \max\Big(\max_{t\in[0,T]} ||\Delta u^{(k-1)}(t,\xi)||, \max_{t\in[0,T]} ||\Delta w^{(k-1)}(t,\xi)||\Big)d\xi. \tag{2.13}$$

As inequality (2.13) is fair for $k = 1, 2...$, then consistently substituting the corresponding differences in the right-hand side, we will get

$$\max\Big(\max_{t\in[0,T]} ||\Delta u^{(k)}(t,x)||, \max_{t\in[0,T]} ||\Delta w^{(k)}(t,x)||\Big)$$

$$\leq \frac{1}{(k-1)!}\int_0^x [\xi \max_{\xi_1\in[0,\omega]} \rho_1(\xi_1)]^{k-1}\rho_2(\xi)d\xi.$$

Hereof, we get that at $k \to \infty$ the sequences $u^{(k)}(t,x)$, $w^{(k)}(t,x)$ converge uniformly to $u^*(t,x)$, $w^*(t,x)$ on $\bar{\Omega}$. Then from relations (2.11)–(2.12) it follows that sequences $v^{(k)}(t,x)$, $\dfrac{\partial v^{(k)}(t,x)}{\partial t}$ also converge uniformly on $\bar{\Omega}$ to $v^*(t,x)$, $\dfrac{\partial v^*(t,x)}{\partial t}$, respectively.

This means that the triple of functions $\{v^*(t,x)u^*(t,x)w^*(t,x)\}$ is a solution to problem (2.1)–(2.3) and the following inequalities hold

$$\max\Big(\max_{t\in[0,T]} ||u^*(t,x) - \psi(t)||, \max_{t\in[0,T]} ||w^*(t,x) - \dot{\psi}(t)||\Big) \leq \int_0^\omega \rho_3(\xi)d\xi \leq \rho,$$

$$\max\left(\max_{t\in[0,T]}||v^*(t,x)||, \max_{t\in[0,T]}\left|\left|\frac{\partial v^*(t,x)}{\partial t}\right|\right|\right) \leq \int_0^{\omega} \rho_3(\xi)d\xi \leq \rho,$$

i.e. $u^* \in S(\psi(t),\rho)$.

As problems (2.1)–(2.3) and (1.1)–(1.3) are equivalent, the function $u^*(t,x)$ belonging to $S(\psi(t),\rho)$, will be a classical solution to problem (1.1)–(1.3). The uniqueness of the solution is proved by the contradiction method. □

Acknowledgments

This work supported by Grant No. 0822/ГФ 4 (2015-2017) of the Science Committee of the Ministry of education and Science of the Republic of Kazakhstan.

References

[1] R.P. Agarwal, I. Kiguradze, *On multi-point boundary value problems for linear ordinary differential equations with singularities,* Journal of Mathematical Analysis and Applications, 297 (2004), no. 2, 131–151.

[2] A.T. Asanova, D.S. Dzhumabaev, *Unique Solvability of the boundary value problem for systems of hyperbolic equations with data on the characteristics,* Computational Mathematics and Mathematical Physics, 42 (2002), no. 11, 1609–1621.

[3] A.T. Asanova, D.S. Dzhumabaev, *Unique solvability of nonlocal boundary value problems for systems of hyperbolic equations,* Differential Equations, 39 (2003), no. 10, 1414–1427.

[4] A.T. Asanova, D.S. Dzhumabaev, *Criteria of well-posed solvability of boundary value problem for system of hyperbolic equations,* Izvestia NAN Respubl. Kazakhstan. Ser. phyz.-mathem., (2002), no. 3, 20–26 (in Russian).

[5] A.T. Asanova, D.S. Dzhumabaev, *Correct solvability of a nonlocal boundary value problem for systems of hyperbolic equations,* Doklady Mathematics, 68 (2003), no. 1, 46–49.

[6] A.T. Asanova, D.S. Dzhumabaev, *Well-posed solvability of nonlocal boundary value problems for systems of hyperbolic equations,* Differential Equations, 41 (2005), no. 3, 352–363.

[7] A.T. Asanova, *A nonlocal boundary value problem for systems of quasilinear hyperbolic equations,* Doklady Mathematics, 74 (2006), no. 3, 787–791.

[8] A.T. Asanova, *On the unique solvability of a family of two-point boundary-value problems for systems of ordinary differential equations,* Journal of Mathematical Sciences, 150 (2008), no.-5, 2302–2316.

[9] A.T. Asanova, *On the solvability of a family of multi-point boundary value problems for system of differential equations and their applications to nonlocal boundary value problems,* Mathematical journal, 13 (2013), no. 3, 38–42 (in Russian).

[10] A.T. Asanova, *On multipoint problem for system of hyperbolic equations with mixed derivative,* Nonlinear Oscillations (Neliniini Kolyvannya), 17 (2014), no. 3, 295–313 (in Russian).

[11] L. Byszewski, *Existence and uniqueness of solutions of nonlocal problems for hyperbolic equation* $u_{xt} = F(x, t, u, u_x)$*,* Journal of Applied and Stochastic Analysis, 3 (1990), no. 3, 163–168.

[12] L. Cesari, *Periodic solutions of hyperbolic partial differential equations,* in Proc. Internat. Sympos. Non-linear Vibrations (Kiev 1961), Izd. Akad. Nauk Ukrain. SSR, Kiev, 2 (1963), 440–457.

[13] D.S. Dzhumabaev, A.T. Asanova, *Well-posed solvability of a linear nonlocal boundary value problem for systems of hyperbolic equations,* Dopovidi NAN Ukraine, (2010), no. 4, 7–11 (in Russian).

[14] D.S. Dzhumabaev, *Conditions the unique solvability of a linear boundary value problem for ordinary differential equations,* USSR Computational Mathematics and Mathematical Physics, 29 (1989), no. 1, 34–46.

[15] D.S. Dzhumabaev, A.E. Imanchiev, *Well-posed solvability of a linear multi-point boundary value problem.* Mathematical journal, 5 (2005), no. 1(15), 30–38 (in Russian).

[16] I.T. Kiguradze, *Boundary-value problems for system of ordinary differential equations,* Journal of Soviet Mathematics, 43 (1988), no. 2, 2259–2339.

[17] T. Kiguradze, *Some boundary value problems for systems of linear partial differential equations of hyperbolic type,* Mem. Differential Equations Math. Phys. 1 (1994), 1–144.

[18] Yu.A. Mitropol'skii, L.B. Urmancheva, *About two-point problem for systems of hyperbolic equations,* Ukranian mathematical Journal, 42 (1990), no. 12, 1657–1663 (in Russian).

[19] A.M. Nakhushev, *Problems with replased for partial differential equations,* Nauka, Moscow, 2006 (in Russian).

[20] B.I. Ptashnyck, *Ill-posed boundary value problems for partial differential equations,* Naukova Dumka, Kiev, Ukraine, 1984 (in Russian).

[21] A.M. Samoilenko, V.N. Laptinskii, K.K. Kenzhebayev, *Constructive methods of investigation of periodic and multi-point boundary value problems,* Proceedings of Institute of Mathematics of NAS of Ukraine, Institute of Mathematics NAS Ukraine, Kiev, 29 (1999), 1–220 (in Russian).

[22] A.M. Samoilenko, B.P. Tkach, *Numerical-analytical methods in the theory periodical solutions of equations with partial derivatives,* Naukova Dumka, Kiev, Ukraine, 1992 (in Russian).

[23] A.M. Samoilenko, N.I Ronto, *Numerical-analytical methods for investigation of a solutions of boundary value problems,* Naukova Dumka, Kiev, Ukraine, 1985 (in Russian)

[24] L.B. Urmancheva, *Two-point and multi-point problems for systems of partial differential equations hyperbolic type,* Institute of Mathematics NAS Ukraine, Kiev, 1993. Preprint 93.2, 1–40 (in Russian).

Anar Turmaganbetkyzy Assanova
Department of Differential Equations
Institute of Mathematics and Mathematical Modeling MES RK
125 Pushkin St,
050010 Almaty, Kazakhstan
E-mails: anarasanova@list.ru, anartasan@gmail.com

Askarbek Ermekovich Imanchiev
Faculty of Physics and Mathematics
K. Zhubanov Aktobe Regional State University
34 A. Moldagulova St,
Aktobe 030000, Kazakhstan
E-mail: imanchiev_ae@mail.ru

Received: 18.06.2015