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CHARACTERIZATION OF SUBDIAGONAL ALGEBRAS ON NONCOMMUTATIVE LORENTZ SPACES

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Abstract. Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra, $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$. We prove that $\mathcal{A}$ has $L^{p,q}$-factorization if and only if $\mathcal{A}$ is a subdiagonal algebra. We also obtain some characterizations of subdiagonal algebras.

1 Introduction

First, we recall the definition of the classical Lorentz spaces. Given a measure space (X, Σ, ν), $0 < p, q \leq \infty$ and a measurable function f on (X, Σ, ν), define

$$\|f\|_{L^{p,q}(X)} = \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}, & q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), & q = \infty, \end{cases}$$

where $f^*(t)$ is the non-increasing rearrangement of f. The classical Lorentz space $L^{p,q}(X)$ is the set all measurable functions f on (X, Σ, ν) with $\|f\|_{L^{p,q}(X)} < \infty$. We refer to [5, 8, 9, 10] for more information about $L^{p,q}(X)$.

In [3, 4], among other things, Blecher and Labuschagne proved that a tracial subalgebra $\mathcal{A}$ has L^∞-factorization if and only if $\mathcal{A}$ is a subdiagonal algebra and Bekjan [2] obtained that if a tracial subalgebra has L^p-factorization ($0 < p < \infty$), then it is a subdiagonal algebra.

In this paper we will consider the $L^{p,q}$-factorization property of a tracial subalgebra. The organization of this paper is as follows. Section 2 contains some preliminaries and notation on tracial subalgebra and noncommutative Lorentz spaces. In Section 3, we prove that a tracial subalgebra $\mathcal{A}$ has $L^{p,q}$-factorization if and only if $\mathcal{A}$ is a subdiagonal algebra.

2 Preliminaries

We use standard notation and notions from theory of noncommutative L_p spaces. Our main references are [7, 12, 13] (see [13] for more historical references). Let $\mathcal{M}$ be a finite von Neumann algebra on a Hilbert space $\mathcal{H}$ with a normal finite faithful trace τ

and denote the lattice of (orthogonal) projections in $\mathcal{M}$ by P($\mathcal{M}$). A closed densely defined linear operator x in $\mathcal{H}$ with domain $D(x)$ is said to be affiliated with $\mathcal{M}$ if and only if $u^*xu = x$ for all unitary operators u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$, then x is said to be τ-measurable if for every $\varepsilon > 0$ there exists a projection $e \in P(\mathcal{M})$ such that $e(H) \subseteq D(x)$ and $\tau(e^\perp) < \varepsilon$ (where for any projection e we let $e^\perp = 1 - e$). The set of all τ-measurable operators will be denoted by $L_0(\mathcal{M};\tau)$ or just by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with the sum and product to be the respective closure of the algebraic sum and product.

The measure topology t_τ in $L_0(\mathcal{M})$ is given by the system

$$V(\varepsilon,\delta) = \{x \in L_0(\mathcal{M}) : \|xe\|_\infty \le \delta \ \text{ for some } e \in P(\mathcal{M}) \text{ with } \tau(e^\perp) \le \varepsilon\},$$

$\varepsilon > 0, \delta > 0$, of neighborhoods of zero. Note that if one replaces the condition $\|xe\|_\infty \le \delta$ above by the generally weaker condition $\|exe\|_\infty \le \delta$, then the corresponding family of neighborhoods of zero generates the same topology t_τ (see [6]).

The trace τ can be extended to the positive cone $L_0^+(\mathcal{M})$ of $L_0(\mathcal{M})$:

$$\tau(x) = \int_0^\infty \lambda d\tau(e_\lambda(x)),$$

where $x = \int_0^\infty \lambda de_\lambda(x)$ is the spectral decomposition of x.

Given $0 < p < \infty$, we define

$$\|x\|_p = \tau(|x|^p)^{1/p}, \quad x \in \mathcal{M},$$

where $|x| = (x^*x)^{\frac{1}{2}}$. Then $(\mathcal{M}, \|\cdot\|_p)$ is a normed (or quasi-normed for $p < 1$) space, whose completion is the noncommutative L^p-space associated with $(\mathcal{M},\tau)$, satisfying all the expected properties such as duality (see [13]), denoted by $L^p(\mathcal{M},\tau)$ or just by $L^p(\mathcal{M})$. As usual, we set $L^\infty(\mathcal{M},\tau) = \mathcal{M}$ and denote by $\|\cdot\|_\infty (= \|\cdot\|)$ the usual operator norm.

Let x be a τ-measurable operator and $t > 0$. The " t-th singular number (or generalized s-number)" of x $\mu_t(x)$ is defined by

$$\mu_t(x) = \inf\{\|xe\| : e \in P(\mathcal{M}), \tau(e^\perp) \le t\}.$$

It is clear that, if x is a τ-measurable operator, then $\mu_t(x) < \infty$ for every $t > 0$. See [7] for more information about generalized s-numbers.

Definition 1. Let x be a τ-measurable operator affiliated with a finite von Neumann algebra $\mathcal{M}$, and $0 < p, q \le \infty$. Define

$$\|x\|_{L^{p,q}(\mathcal{M})} = \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}}\mu_t(x))^q \frac{dt}{t}\right)^{\frac{1}{q}}, & q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}}\mu_t(x), & q = \infty. \end{cases}$$

The set of all $x \in L_0(\mathcal{M})$ with $\|x\|_{L^{p,q}(\mathcal{M})} < \infty$ is denoted by $L^{p,q}(\mathcal{M})$ and is called the noncommutative Lorentz space with indices p and q.

It is easy to check that $(L^{p,q}(\mathcal{M}), \|\cdot\|_{L^{p,q}(\mathcal{M})})$ is a quasi-Banach space. Moreover, if $p>1$, $q\geq 1$, and equipped with the equivalent norm

$$\|x\|^*_{L^{p,q}(\mathcal{M})} = \begin{cases} \left(\int_0^\infty \left[t^{-1+\frac{1}{p}}\int_0^t \mu_s(x)ds\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}, & q<\infty, \\ \sup_{t>0} t^{-1+\frac{1}{p}}\int_0^t \mu_s(x)ds, & q=\infty, \end{cases}$$

then $(L^{p,q}(\mathcal{M}), \|\cdot\|^*_{L^{p,q}(\mathcal{M})})$ is a Banach space (see [14]).

Remark 1. (i) *If $p=q$, then $L^{p,p}(\mathcal{M}) = L^p(\mathcal{M})$.*

(ii) *If $1<p<\infty$, $1\leq q<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$. Then by Lemma 4.1 of [15], we obtain the following result*

$$(L^{p,q}(\mathcal{M}))^* = L^{p',q'}(\mathcal{M}).$$

For a subset K of $L^{p,q}(\mathcal{M})$, put $J(K)=\{x^*:\ x\in K\}$, $K^{-1}=\{x:\ x,x^{-1}\in K\}$, $K^+=\{x:\ x\geq 0, x\in K\}$, and $[K]_{p,q}$ the closed linear span of K in $L^{p,q}(\mathcal{M})$.(Here $[K]_\infty$ is the weak* closure of K).

Given a von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$, an expectation $\mathcal{E}:\mathcal{M}\to\mathcal{N}$ is defined to be a positive linear map which preserves the identity and satisfies $\mathcal{E}(xy)=x\mathcal{E}(y)$ for all $x\in\mathcal{N}$ and $y\in\mathcal{M}$. Since $\mathcal{E}$ is positive it is hermitian, i.e. $\mathcal{E}(x)^*=\mathcal{E}(x^*)$ for all $x\in\mathcal{M}$. Hence $\mathcal{E}(yx)=\mathcal{E}(y)x$ for all $x\in\mathcal{N}$ and $y\in\mathcal{M}$. In order to get a more profound study of $\mathcal{E}$ we reference the readers to [1, 11].

Definition 2. Let $\mathcal{A}$ be a *weak** closed unital subalgebra of $\mathcal{M}$. If there is a linear projection $\mathcal{E}$ from $\mathcal{A}$ onto $\mathcal{D}=\mathcal{A}\cap J(\mathcal{A})$ such that

(i) $\mathcal{E}$ is multiplicative on $\mathcal{A}$, i.e. $\mathcal{E}(ab)=\mathcal{E}(a)\mathcal{E}(b)$ for all $a,b\in\mathcal{A}$;

(ii) $\tau\circ\mathcal{E}=\tau$,

then $\mathcal{A}$ is called a tracial subalgebra of $\mathcal{M}$.

Definition 3. Let $\mathcal{A}$ be a *weak** closed unital subalgebra of $\mathcal{M}$ and let $\mathcal{E}$ be a normal faithful conditional expectation from $\mathcal{M}$ onto a von Neumann subalgebra $\mathcal{D}$ of $\mathcal{M}$. $\mathcal{A}$ is called a subdiagonal algebra of $\mathcal{M}$ with respect to $\mathcal{E}$ if the following conditions are satisfied

(i) $\mathcal{A}+J(\mathcal{A})$ is *weak** dense in $\mathcal{M}$;

(ii) $\mathcal{E}(ab)=\mathcal{E}(a)\mathcal{E}(b)$ for all $a,b\in\mathcal{A}$;

(iii) $\mathcal{D}=\mathcal{A}\cap J(\mathcal{A})$.

Let $\mathcal{A}_0=\mathcal{A}\cap\ker(\mathcal{E})$. We call $\mathcal{A}$ τ-maximal, if

$$\mathcal{A}=\{x\in\mathcal{M}:\tau(xy)=0,\ \forall y\in\mathcal{A}_0\}.$$

3 $L^{p,q}$-factorization property of tracial subalgebra

We know that $\mathcal{E}$ can be extended to a contract projection from $L^p(\mathcal{A})$ onto $L^p(\mathcal{D})$ for every $1 \leq p \leq \infty$ (see [11]). Here we give the general results of contractivity of $\mathcal{E}$ from $L^{p,q}(\mathcal{M})$ onto $L^{p,q}(\mathcal{D})$ for $1 \leq p < \infty$ and $1 \leq q < \infty$.

Lemma 3.1. *Let $1 \leq p < \infty$ and $1 \leq q < \infty$. Then*

$$\|\mathcal{E}(x)\|_{L^{p,q}(\mathcal{M})} \leq \|x\|_{L^{p,q}(\mathcal{M})}\ ,\ \forall x \in L^{p,q}(\mathcal{M}).$$

Proof. From Proposition 3.9 of [11], we know that $\mathcal{E}(x)$ is submajorized by x, for all $x \in L^1(\mathcal{M})$. Then

$$\int_0^s \mu_t(\mathcal{E}(x))dt \leq \int_0^s \mu_t(x)dt \quad \forall s \in (0, \tau(1)).$$

Hence

$$\|\mathcal{E}(x)\|_{L^{p,q}(\mathcal{M})} \leq \|x\|_{L^{p,q}(\mathcal{M})}, \quad \forall x \in L^{p,q}(\mathcal{M}).$$

□

Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$. We write $\mathcal{A}_{p,q}$ for $[\mathcal{A}]_{p,q} \cap \mathcal{M}$.

Lemma 3.2. *Let $1 < p < \infty$ and $1 \leq q < \infty$. If $\mathcal{A}$ is a tracial subalgebra of $\mathcal{M}$, then $\mathcal{A}_{p,q}$ is a tracial subalgebra of $\mathcal{M}$.*

Proof. First, we prove that $\mathcal{A}_{p,q}$ is weak* closed in $\mathcal{M}$. Indeed, suppose that there is $x \in \mathcal{M}$ in the *weak** closure of $[\mathcal{A}]_{p,q}$ but not in $[\mathcal{A}]_{p,q}$. Then by (ii) of Remark 1, we could find $z \in L^{p',q'}(\mathcal{M}) \subset L^1(\mathcal{M})$ such that $\tau(zx) \neq 0$ and $\tau(zy) = 0$ for every $y \in [\mathcal{A}]_{p,q}$, where $\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1$. Since x is in the *weak** closure of $[\mathcal{A}]_{p,q}$, $\tau(zx) = 0$. This contradiction shows that $x \in \mathcal{A}_{p,q}$.

It is clear that$\mathcal{A}_{p,q}$ is unital. To see that $\mathcal{A}_{p,q}$ is a subalgebra, we first check that if $x \in \mathcal{A}$, $y \in \mathcal{A}_{p,q}$, then $xy \in \mathcal{A}_{p,q}$. Indeed, if $(y_n) \subset \mathcal{A}$ with $y_n \to y$ in $L^{p,q}(\mathcal{M})$, then $xy_n \in \mathcal{A}$ and $xy_n \to xy$ in $L^{p,q}(\mathcal{M})$. If $x \in \mathcal{A}_{p,q}$, $y \in \mathcal{A}_{p,q}$, then there is $(x_n) \subset \mathcal{A}$ such that $x_n \to x$ in $L^{p,q}(\mathcal{M})$. Hence $x_n y \to xy$ in $L^{p,q}(\mathcal{M})$, so $xy \in \mathcal{A}_{p,q}$, since $x_n y \in \mathcal{A}_{p,q}$.

Next we prove that

$$\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(x), \quad \forall x, y \in \mathcal{A}_{p,q}.$$

Let $y \in \mathcal{A}_{p,q}$, then there is a sequence $(y_n) \subset \mathcal{A}$ such that $y_n \to y$ in $L^{p,q}(\mathcal{M})$. So for all $x \in \mathcal{A}$ we have $xy_n \in \mathcal{A}$, and $xy_n \to xy$ in $L^{p,q}(\mathcal{M})$. Thus, by Lemma 3.1,

$$\mathcal{E}(xy) = \lim_{n\to\infty} \mathcal{E}(xy_n) = \lim_{n\to\infty} \mathcal{E}(x)\mathcal{E}(y_n) = \mathcal{E}(x)\mathcal{E}(y).$$

Hence, by what we just proved,

$$\mathcal{E}(yx) = \lim_{n\to\infty} \mathcal{E}(y_n x) = \lim_{n\to\infty} \mathcal{E}(y_n)\mathcal{E}(x) = \mathcal{E}(y)\mathcal{E}(x),\ \forall x \in \mathcal{A}_{p,q}.$$

□

Definition 4. Let $0 < p < \infty,\ 0 < q < \infty$. Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$. We say that $\mathcal{A}$ has $L^{p,q}$-factorization, if for $x \in L^{p,q}(\mathcal{M})^{-1}$, there is a unitary $u \in \mathcal{M}$ and $a \in [\mathcal{A}]_{p,q}^{-1}$ such that $x = ua$.

Proposition 3.1. *Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$. Let $1 < p_1 \le p < \infty,\ 1 \le q, s < \infty$. If $\mathcal{A}$ has $L^{p,q}$-factorization, then $\mathcal{A}$ has $L^{p_1,s}$-factorization.*

Proof. Let $x \in \mathcal{M}^{-1} \subset (L^{p,q}(\mathcal{M}))^{-1}$. Then there exist a unitary $u \in \mathcal{M}$ and $a \in [\mathcal{A}]_{p,q}^{-1}$ such that $x = ua$, since $\mathcal{A}$ has $L^{p,q}$-factorization. So $a \in \mathcal{A}_{p,q}^{-1}$, and therefore $\mathcal{A}_{p,q}$ has L^{∞}-factorization. By Theorem 1.1 of [4], $\mathcal{A}_{p,q}$ is a subdiagonal algebra of $\mathcal{M}$. Let $x \in L^{p_1,q}(\mathcal{M})^{-1}$. By Theorem 3.3 of [14], there exist a unitary $u \in \mathcal{M}$ and $a \in [\mathcal{A}_{p,q}]_{p_1,q}^{-1}$ such that $x = ua$. On the other hand, by Lemma 2.4 of [14], we have $L^{p,q}(\mathcal{M}) \subset L^{p_1,s}(\mathcal{M})$. Hence

$$[\mathcal{A}]_{p_1,s} \subset [\mathcal{A}_{p,q}]_{p_1,s} \subset [[\mathcal{A}]_{p,q}]_{p_1,s} = [\mathcal{A}]_{p_1,s}.$$

Thus $a \in [\mathcal{A}]_{p_1,s}^{-1}$, and so $\mathcal{A}$ has $L^{p_1,s}$-factorization. □

Theorem 3.1. *Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$. Then the following conditions are equivalent.*

(i) *$\mathcal{A}$ is a subdiagonal subalgebra of $\mathcal{M}$.*

(ii) *For all $0 < p < \infty,\ 0 < q < \infty$, $\mathcal{A}$ has $L^{p,q}$-factorization.*

(iii) *For some $1 < p < \infty,\ 1 \le q < \infty$, $\mathcal{A}$ has $L^{p,q}$-factorization.*

Proof. $(i) \Rightarrow (ii)$ follows by Theorem 3.3 of [14].

$(ii) \Rightarrow (iii)$ is clear.

$(iii) \Rightarrow (i)$. Since $L^p(\mathcal{M}) = L^{p,p}(\mathcal{M})$, by Proposition 3.1, we get that $\mathcal{A}$ has L^p-factorization. Then by Theorem 2.4 of [2] $\mathcal{A}$ is a subdiagonal algebra of $\mathcal{M}$. □

Theorem 3.2. *Let $\mathcal{A}$ be a τ-maximal tracial subalgebra of $\mathcal{M}$. Then the following conditions are equivalent.*

(i) *$\mathcal{A}$ is a subdiagonal algebra of $\mathcal{M}$.*

(ii) *For some $0 < p \le 1,\ 0 < q < \infty$, $\mathcal{A}$ has $L^{p,q}$-factorization and $\mathcal{A}_{p,q}$ is a tracial subalgebra of $\mathcal{M}$.*

Proof. We only need to prove $(ii) \Rightarrow (i)$. By the proof of Proposition 3.1, we know that $\mathcal{A}_{p,q}$ is a subdiagonal algebra of $\mathcal{M}$ and $\mathcal{A} \subset \mathcal{A}_{p,q}$. If $y \in \mathcal{A}_{p,q}$, then $\tau(xy) = \tau(\mathcal{E}(xy)) = \tau(\mathcal{E}(x)\mathcal{E}(y)) = 0$ for each $x \in \mathcal{A}_0$. By the τ-maximality of $\mathcal{A}$ we know that $y \in \mathcal{A}$. Thus $\mathcal{A} = \mathcal{A}_{p,q}$. □

Definition 5. We say a tracial subalgebra $\mathcal{A}$ of $\mathcal{M}$ satisfies L^2-density, if $\mathcal{A} + J(\mathcal{A})$ is dense in $L^2(\mathcal{M})$ in the usual Hilbert space norm on that space.

For more detailed information about L^2-density, see [2, 4].

Theorem 3.3. *Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$ satisfy L^2-density. Then the following conditions are equivalent.*

(i) *$\mathcal{A}$ is a subdiagonal algebra of $\mathcal{M}$.*

(ii) *For some $0 < p \leq 1,\ 0 < q < \infty$, $\mathcal{A}$ has $L^{p,q}$-factorization and $\mathcal{A}_{p,q}$ is a tracial subalgebra of $\mathcal{M}$.*

Proof. *(ii)* $\Rightarrow$ *(i)*. It is clear that $\mathcal{A}_{p,q}$ is a subdiagonal algebra of $\mathcal{M}$. Then

$$L^2(\mathcal{M}) = [\mathcal{A}_{p,q}]_2 \oplus [J((\mathcal{A}_{p,q})_0)]_2.$$

By [13],

$$L^2(\mathcal{M}) = [\mathcal{A}]_2 \oplus [J(\mathcal{A}_0)]_2,\ [\mathcal{A}]_2 \subset [\mathcal{A}_{p,q}]_2,\ [J(\mathcal{A}_0)]_2 \subset [J((\mathcal{A}_{p,q})_0)]_2.$$

So we have $[\mathcal{A}]_2 = [\mathcal{A}_{p,q}]_2$, $[J(\mathcal{A}_0)]_2 = [J((\mathcal{A}_{p,q})_0)]_2$. Since $\mathcal{A}_{p,q}$ is a subdiagonal algebra of $\mathcal{M}$, for $x \in L^2(\mathcal{M})^{-1}$ there exist a unitary $u \in \mathcal{M}$ and $a \in [\mathcal{A}_{p,q}]_2^{-1} = [\mathcal{A}]_2^{-1}$ such that $x = ua$. This implies $\mathcal{A}$ has L^2-factorization. By Theorem 2.4 of [2] we know that $\mathcal{A}$ is a subdiagonal algebra of $\mathcal{M}$. □

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