EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 1, Number 4 (2010), 95 – 115

INVERSE EXTREMAL PROBLEM FOR VARIATIONAL FUNCTIONALS

I.V. Orlov

Communicated by A.A. Shkalikov

Key words: variational functional, integrand, local extremum, non-local extremum, compact extremum, Sobolev space, Legendre-Jacobi condition, compact derivative, dominating mixed smoothness.

AMS Mathematics Subject Classification: 49J05, 49L99.

Abstract. We investigate an inverse extremal problem for the variational functionals: to describe, under certain conditions, all types of variational functionals having a local extremum (in case of the space $C^1[a;b]$) or a compact extremum (in case of the Sobolev space $W^{1,2}[a;b] = H^1[a;b]$) at a given point of the corresponding function space. The non-locality conditions for a compact extrema of variational functionals are described as well.

1 Introduction. Preliminaries

The classical scheme of proving the existence of a local extremum for the one-dimensional Euler-Lagrange variational functional

$$\Phi(y) = \int_a^b f(x, y, y')dx \mapsto \text{extr} \quad (y \in C^1[a;b])$$

at an extremal point y assumes [5], [4] checking the strengthened Legendre condition $f_{y'y'}(x, y, y') \neq 0$ and the Jacobi condition $U(x) \neq 0$ $(a < x \leq b)$ for the Hamilton–Jacobi equation:

$$-\frac{d}{dx}\Big[f_{y'y'}(x, y, y')U'\Big] + \Big[-\frac{d}{dx}\big(f_{yy'}(x, y, y')\big) + f_{y^2}(x, y, y')\Big]U = 0$$
$$(U(a) = 0,\ U'(a) = 1).$$

The second step is the most laborious, it requires to solve a rather complicated equation in order to get, in fact, a very small information about the behaviour of the solution U.

Moreover, the initial conditions $U(a) = 0$, $U'(a) = 1$, have as a consequence automatical fulfilment of the Jacobi condition near a. The only question is – how long is the appropriate interval?

In the recent author's paper [8], it was shown that the interval satisfying the Jacobi condition can be chosen depending only on the form of the integrand f and not depending on a concrete extremal. More precisely, the main result distinguishes two cases depending on the range of the coefficients in the Hamilton-Jacobi equation. In the first case, an extremum is guaranteed without any restriction on the length of $[a;b]$, in the second one, such a restriction is present. This result remains valid also for the case of a compact extremum in the Sobolev space $H^1[a;b]$.

Let us formulate these results considering first the classical C^1–case.

Theorem 1. *Let the variational functional*

$$\Phi(y) = \int_a^b f(x,y,y')dx \quad (y \in C^1[a;b],\ y(a) = y(b) = 0,\ f \in C^2,\ f_{yz} \in C^1) \tag{1.1}$$

satisfy at a point $y_0(\cdot) \in C^2[a;b]$ the Euler-Lagrange equation

$$f_y(x, y_0, y_0') - \frac{d}{dx}\big[f_z(x, y_0, y_0')\big] = 0\,. \tag{1.2}$$

Denote

$$p := \min_{a \le x \le b} f_{z^2}(x, y_0(x), y_0'(x));$$

$$q := \min_{a \le x \le b}\left[f_{y^2}(x, y_0(x), y_0'(x)) - \frac{d}{dx}\big(f_{yz}(x, y_0(x), y_0'(x))\big)\right].$$

Then, under the boundary conditions $y(a) = y_0(a)$, $y(b) = y_0(b)$,

1) *for $p > 0$, $q \ge 0$, $\Phi(y)$ attains a strong local minimum at $y_0(\cdot)$ (without any restriction on the length of $[a;b]$);*

2) *for $p > 0$, $q < 0$, and under the restriction*

$$b - a < \frac{\pi}{4}\sqrt{\frac{p}{|q|}}, \tag{1.3}$$

on the length of $[a;b]$, $\Phi(y)$ attains a strong local minimum at $y_0(\cdot)$ as well.

An analogous result holds for a compact extrema of a variational functional in the Sobolev H^1-case.

Theorem 2. *Let the variational functional*

$$\Phi(y) = \int_a^b f(x,y,y')dx \quad (y \in H^1[a;b],\ y(a) = y(b) = 0,\ f \in W_K^2(z)) \tag{1.4}$$

satisfy at a $W^{2,2}$-smooth point $y_0(\cdot)$ the Euler-Lagrange equation. Then, under the conditions and notation of Theorem 1, *1) for* $p > 0$, $q \geq 0$, $\Phi(y)$ *attains a strong* K*–minimum at* $y_0(\cdot)$ *(without any restriction on the length of* $[a; b]$*);*

2) for $p > 0$, $q < 0$, *and under the restriction*

$$b - a < \frac{\pi}{4}\sqrt{\frac{p}{|q|}}$$

on the a length of $[a; b]$, $\Phi(y)$ *attains a strong* K*–minimum at* $y_0(\cdot)$ *as well.*

(The Weierstrass class $W^2_K(z)$ will be defined below).

These results simplify essentially finding of both a local extremum in the C^1-case and a K-extremum in the H^1-case for the variational functionals of type (1.1), (1.4). Thisallows setting the inverse problem: *to describe a general form of a variational functional having extremum at a given point (under the Euler-Lagrange and Legendre conditions).*

The paper is devoted to solution of this problem. The first item contains a solution of the inverse problem in $C^1[a; b]$ (Theorems 3, 4 below). The second item contains a solution of the inverse problem in $H^1[a; b]$ (Theorems 7 – 9 below).

The third item is auxiliary for the further investigation of K-extrema. It contains a description of integrands in the Weierstrass class $W^2K_2(z)$, which provides the appropriate analytical properties of variational functionals in H^1 (Theorem 10). The fourth item contains the so-called "stationary form of the Legendre-Jacobi conditions" (SLG) guaranteeing the existence of a K-extremum under a weaker restriction on the length of $[a; b]$. On the basis of that results, the fifth item contains a solution of the inverse problem in $H^1[a; b]$ under the (SLJ)-condition and the exponential (SLJ)-condition.

Next, the sixth item of the work consider the main properties of the mappings from the Weierstrass class $W^2_K(z)$ that allow to construct easily the extensive classes of the integrands from the Weierstrass class $W^2K_2(z)$. The final, seventh item of the work contains a description of an extensive enough class of the variational functionals having non–local compact extrema in $H^1[a, b]$.

2 Inverse extremal problem for variational functionals in $C^1[a; b]$

Let us set up the following problem: to find a general form of the variational functional (1.1) possessing local minimum at zero under the strengthened Legendre condition.

1) We shall write integrands f of functional (1.1) the form

$$f(x, y, z) = P(x, y) + Q(x, y) \cdot z + \frac{1}{2}R(x, y, z) \cdot z^2 \,. \tag{2.1}$$

Then

$$P(x, y) = f(x, y, 0), \quad Q(x, y) = f_z(x, y, 0), \quad R(x, y, 0) = f_{z^2}(x, y, 0).$$

Under this notation, the Euler–Lagrange equation at the zero extremal takes the form

$$(Q_x - P_y)(x, 0) = 0 \quad (a \leq x \leq b); \tag{2.2}$$

and the strengthened Legendre condition at the zero extremal takes the form

$$R(x,0,0) =: p(x) > 0 \quad (a \le x \le b). \tag{2.3}$$

2) Choose an arbitrary function $P \in C^2$. Then a general form of Q follows by (2.2):

$$\Big(Q_x(x,0) = P_y(x,0)\Big) \Rightarrow \Big(Q(x,0) = C + \int\limits_a^x P_y(t,0)dt\Big) \Rightarrow$$

$$\Rightarrow \Big(Q(x,y) = C + \int\limits_a^x P_y(t,0)dt + \widetilde{Q}(x,y), \text{ where } \widetilde{Q}(x,0) = 0\Big) \Rightarrow$$

$$\Rightarrow \Big(Q(x,y) = C + \int\limits_a^x P_y(t,0)dt + [q(x,y) - q(x,0)]\Big). \tag{2.4}$$

Here $C \in \mathbb{R}$ and $q \in C^2$ can be chosen arbitrarily.

3) A general form of R easily follows by condition (2.3):

$$\Big(R(x,0,0) = p(x) > 0\Big) \Rightarrow \Big(R(x,y,z) = p(x) + [\rho(x,y,z) - \rho(x,0,0)]\Big), \tag{2.5}$$

where $p(x) > 0$, $p \in C^2$; $\rho \in C^2$ can be chosen arbitrarily.

4) A general form of the integrand f follows now from (2.1), (2.4) and (2.5):

$$f(x,y,z) = P(x,y) + \Big(C + \int\limits_a^x P_y(t,0)dt + [q(x,y) - q(x,0)]\Big) \cdot z +$$

$$+\frac{1}{2}\Big(p(x) + [\rho(x,y,z) - \rho(x,0,0)]\Big) \cdot z^2 \,, \tag{2.6}$$

where $C \in \mathbb{R}$; q, $p \in C^2$ $(p > 0)$ can be chosen arbitrarily. So, the following statement in proved.

Theorem 3. *Let, under the conditions of Theorem 1, functional* (1.1) *attain a local minimum at zero under the strengthened Legendre condition. Then the integrand f takes the form* (2.6).

Remark 1. *As it follows from Theorem 3, a general form of the variational functional* (1.1) *taking a local minimum at zero under the strengthened Legendre condition is*

$$\Phi(y) = \int\limits_a^b \Big(P(x,y) + \Big[C + \int\limits_a^x P_y(t,0)dt + q(x,y) - q(x,0)\Big] \cdot y' +$$

$$+\frac{1}{2}\Big[p(x) + \rho(x,y,y') - \rho(x,0,0)\Big] \cdot y'^2\Big) dx \,, \tag{2.7}$$

where $C \in \mathbb{R}$, P, q, $p > 0$, ρ *are the arbitrary functions in* C^2.

Thus, under the strengthened Legendre condition, the inverse extremal variational problem at zero is solved: all the functionals of type (1.1) *taking a local minimum at zero are described.*

Now, let us pass to the general case of an arbitrary C^2-smooth extremal in $C^1[a;b]$.

Let us fix an arbitrary C^2-smooth function $y_0(x)$, $a \le x \le b$, satisfying the boundary conditions $y(a) = y_0(a)$, $y(b) = y_0(b)$.

In order to reduce the problem to the case of the zero extremal considered above, it suffices to introduce the auxiliary variational functional:

$$\widetilde{\Phi}(y) = \Phi(y + y_0) = \int_a^b f(x, y + y_0(x), y' + y_0'(x))dx =: \int_a^b \widetilde{f}(x, y, y')dx$$

$$(y(a) = y(b) = 0).$$

The condition $y_0(\cdot) \in C^2$ guarantees fulfilment of the condition in (1.1) for the auxiliary integrand $\widetilde{f}$.

Application of Theorem 3 to the auxiliary integrand $\widetilde{f}$ leads to a solution of the inverse extremal problem for Φ at an arbitrary point $y_0(\cdot) \in C^2[a;b]$.

Theorem 4. *Let, under the conditions of Theorem 1, the variational functional* (1.1) *attain a local minimum at a point* $y_0(\cdot) \in C^2[a;b]$ *satisfying the boundary conditions* $y(a) = y_0(a)$, $y(b) = y_0(b)$ *and the strengthened Legendre condition. Then the integrand* f *has form*

$$\begin{aligned} f(x,y,z) = {} & P(x, y - y_0(x)) + \\ & + \left(C + \int_a^x P_y(t, -y_0(t))dt + [q(x, y - y_0(x)) - q(x, -y_0(x))]\right) \cdot (z - y_0'(x)) + \\ & + \frac{1}{2}\left(p(x) + [\rho(x, y - y_0(x), z - y_0'(x)) - \rho(x, -y_0(x), -y_0'(x))]\right) \cdot (z - y_0'(x))^2, \end{aligned} \quad (2.8)$$

where $C \in \mathbb{R}$; P, q, $p > 0$, $\rho \in C^2$ *can be chosen arbitrarily.*

This implies the following formula for the general form of functional (1.1) attaining a local minimum in $C^1[a;b]$ at a point $y_0(\cdot) \in C^2[a;b]$ under the strengthened Legendre condition:

$$\begin{aligned} \Phi(y) = \int_a^b \Bigg(& P(x, y - y_0(x)) + \left[C + \int_a^x P_y(t, -y_0(t))dt + q(x, y - y_0(x)) - q(x, -y_0(x))\right] \cdot \\ & \cdot (y' - y_0'(x)) + \frac{1}{2}\Big[p(x) + \rho(x, y - y_0(x), y' - y_0'(x)) - \\ & - \rho(x, -y_0(x), -y_0'(x))\Big] \cdot (y' - y_0'(x))^2 \Bigg) dx, \end{aligned} \quad (2.9)$$

where $C \in \mathbb{R}$; P, q, $p > 0$, $\rho \in C^2$ can be chosen arbitrarily, arises.

Thus, under the strengthened Legendre condition, the inverse extremal variational problem at an arbitrary point $y_0(\cdot) \in C^2$ is solved: all functionals of type (1.1), attaining a local minimum at a point $y_0(\cdot)$, are described.

3 Compact extrema of variational functionals in Sobolev space $H^1[a;b]$

To consider the inverse extremal problem in $H^1[a;b]$, let us first introduce the concept of a compact extremum and bring a necessary information on compact extrema of variational functionals in $H^1[a;b]$.

In the Hilbert-Sobolev space $W^{1,2}[a;b] = H^1[a;b]$ equipped with the norm

$$\|y\|^2_{H^1[a;b]} = \int_a^b (y^2 + y'^2)dx \ , \tag{3.1}$$

as is well known, by virtue of I.V. Skrypnik's theorem ([14], Ch.11) variational functionals have practically no non-absolute local extrema. In the our works [11], [9], [10], [6] and in the works by E.V. Bozhonok [1], [2], [3] the general concept of *a compact extremum* (or *K-extremum*) of a functional was studied (see, also, [12]). It has been shown there that the classical, both necessary and sufficient conditions of a local extremum of a variational functional in $C^1[a;b]$ can be extended to the case of a K-extremum in $H^1[a;b]$. In this case, the K-extrema inherit the important properties of the local extrema and can be considered as an analogue of the ones in the case of variational functionals in $H^1[a;b]$.

Definition 5. *Let a real functional $\Phi : H \to \mathbb{R}$ be defined in a Hilbert space H. Say that Φ has a compact minimum (or K-minimum) at a point $y_0 \in H$ if, for each absolutely convex (a.c.) compact set $C \subset H$, the restriction of f to the subspace $(y_0 + \operatorname{span} C)$ has a local minimum at y_0 with respect to the Banach norm $\|\cdot\|_C$ in $\operatorname{span} C$ generated by C. In other words, for each a.c. compactum $C \subset H$ there exists such $\varepsilon = \varepsilon(C) > 0$ that $\varphi(y) \geq \varphi(y_0)$ for all y satisfying $y - y_0 \in \varepsilon \cdot C$.*

Next assume for simplicity $[a;b] = [0;T]$.

Definition 6. *We say that a mapping $\varphi : [0;T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ belongs to the Weierstrass class $W_K(z)$, if φ is uniformly continuous and bounded locally in x and y, and globally in z. We say that φ belongs to the Weierstrass class $W^1_K(z)$ if $\varphi \in W_K(z)$ and gradient $\nabla_{yz}\varphi \in W_K(z)$. We say that φ belongs to the Weierstrass class $W^2_K(z)$ if $\varphi \in W_K(z)$, the gradient $\nabla_{yz}\varphi \in W_K(z)$ and the Hessian $H_{yz}\varphi \in W_K(z)$.*

Next, we say that a function $f : [0;T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ belongs to the class $WK_2(z)$ if a representation pseudoquadratic in z

$$f(x,y,z) = A(x,y,z) + B(x,y,z) \cdot z + C(x,y,z) \cdot z^2, \tag{3.2}$$

can be chosen in such a way that the coefficients A, B, C belong to the class $W_K(z)$. We say that f belongs to the Weierstrass class $W^1K_2(z)$ if the coefficients A, B, C belong to the class $W_K^1(z)$. Finally, we say that f belongs to the Weierstrass class $W^2K_2(z)$ if the coefficients A, B, C belong to the class $W_K^2(z)$.

Note that the classes $W_K(z)$, $W_K^1(z)$, $W_K^2(z)$ can be considered as the appropriate *space with dominating mixed smoothness* (see [13], [15]).

In [7] the following sufficient condition of the twice K-differentiability of a variational functional in Sobolev space $H^1[0;T]$ was obtained.

Theorem 5. *If $f \in W^2K_2(z)$ then the Euler-Lagrange functional*

$$\Phi(y) = \int_0^T f(x, y, y')dx, \quad y(\cdot) \in H^1[0;T] \tag{3.3}$$

is twice K–differentiable everywhere on $H^1[0;T]$. Moreover,

$$\Phi''_K(y)(h,k) = \int_0^T \Big[\frac{\partial^2 f}{\partial y^2} h \cdot k + \frac{\partial^2 f}{\partial y \partial z}(h' \cdot k + h \cdot k') + \frac{\partial^2 f}{\partial z^2} h' \cdot k'\Big] dx. \tag{3.4}$$

The following *generalized Euler–Lagrange equation* [1] serves for finding a K-extremum of functional (3.3) in the space $H^1[0;T]$ as an analogue of the classical necessary condition of a local extremum for a variational functional in $C^1[0;T]$.

Theorem 6. *Let, in addition to the hypotheses of Theorem 5, the following conditions hold:*

(i) functional (3.3) *possesses a K-extremum at a point $y(\cdot) \in H^1[0;T]$;*

(ii) the function $\frac{\partial f}{\partial z}(x, y, y')$ is absolutely continuous on $[0;T]$.

Then the generalized Euler-Lagrange equation:

$$L(f)(y) = \frac{\partial f}{\partial y}(x, y, y') - \frac{d}{dx}\left(\frac{\partial f}{\partial z}(x, y, y')\right) = 0 \quad \text{a.e. on } [0;T] \tag{3.5}$$

holds. In particular, the condition (ii) is fulfilled in the case

$$(\partial f/\partial z) \in C^1([0;T] \times \mathbb{R}^2), \quad y(\cdot) \in W^{2,2}[0;T].$$

The solutions of equation (3.5), satisfying condition (ii) of Theorem 6, are called *the $K-$ extremals* of functional (3.3) in the space $H^1[0;T]$.

Let us also formulate the *generalized sufficient Legendre-Jacobi condition* [3] of a strong K-extremum in the case of Sobolev space $H^1[0;T]$.

Theorem 7. *Let $f : [0;T] \times \mathbb{R}^2 \to \mathbb{R}$, $f \in W^2K_2(z)$, $y(\cdot)$ be a K-extremal of functional* (3.3) *in $H_0^1[0;T]$ and the functions $\frac{\partial f}{\partial z}(x, y(x), y'(x))$ and $\frac{\partial^2 f}{\partial y \partial z}(x, y(x), y'(x))$ be absolutely continuous on K-extremal $y(\cdot)$. Suppose that:*

1) the stregthened Legendre condition, i.e.

$$\frac{\partial^2 f}{\partial z^2}(x, y(x), y'(x)) > 0$$

is fulfilled everywhere on $[0;T]$*;*

2) the generalized Jacobi condition is fulfilled, i.e. every solution of the Jacobi equation

$$-\frac{d}{dx}\left(\frac{\partial^2 f}{\partial z^2}(x, y(x), y'(x))u'\right) + \left[-\frac{d}{dx}\left(\frac{\partial^2 f}{\partial y \partial z}(x, y(x), y'(x))\right) + \right.$$

$$\left. + \frac{\partial^2 f}{\partial y^2}(x, y(x), y'(x))\right] u \overset{a.e}{=} 0 \tag{3.6}$$

in the class $H^1[0;T]$, satisfying the initial conditions $u(0) = 0$, $u'(0) = 1$, does not vanishes for $0 < x \leq T$.

Then Euler-Lagrange functional (3.3) *possesses a strong K-minimum at the point $y(\cdot)$.*

This theorem allows to extend the results of Section 1 to the case of a K-minimum in $H^1[a;b]$.

Theorem 8. *Let variational functional* (1.1) *at a $W^{2,2}$–smooth point $y_0(\cdot) \in H^1[a;b]$ satisfy the Euler-Lagrange equation, and, in addition, $R(x, y, z) \in W_K^2(z)$.*

Then, under the conditions and notation of Theorem 1,

1) *for $p > 0$, $q \geq 0$, $\Phi(y)$ attains a strong K-minimum at $y_0(\cdot)$ (without any restriction on the length of $[a;b]$);*

2) *for $p > 0$, $q < 0$, and under restriction* (1.3) *on the length of $[a;b]$, $\Phi(y)$ attains a strong K-minimum at $y_0(\cdot)$ as well.*

Theorem 9. *Let, under the conditions and notation of Theorem 8, variational functional* (1.1) *attain a K-minimum at a $W^{2,2}$–smooth point $y_0(\cdot)$ from $H^1[a;b]$ satisfying the boundary conditions $y(a) = y_0(a)$, $y(b) = y_0(b)$ and the strengthened Legendre condition.*

Then the integrand f takes form (2.8)*, where $C \in \mathbb{R}$; P, q, $p > 0$, in C^2 and ρ in$W_K^2(z)$ can be chosen arbitrarily.*

This results implies formula (2.9) giving the general form of functional (1.1) having a K-minimum at a $W^{2,2}$-smooth point $y_0(\cdot)$ in $H^1[a;b]$ satisfying the strengthened Legendre condition.

4 Minimal pseudoquadratic representation for integrands of variarional functionals from the class $W^2K_2(z)$

This Section is devoted to description of a suitable class of integrands for which the corresponding variational functionals in $H^1[a;b]$ possess the compact–analytical properties allowing analytical investigation of a compact extremum.

Let us consider an integrand $f \in W^2K_2(z)$, i.e.

$$f(x,y,z) = A(x,y,z) + B(x,y,z)\cdot z + C(x,y,z)\cdot z^2,$$

where A, B, $C \in W_K(z)$, with analogous representations for the gradient $\nabla_{yz}f$ and the Hessian $H_{yz}f$.

Since the function f is twice continuously differentiable on $\Omega\times\mathbb{R}\times\mathbb{R}$ in (y,z), application of the second order Taylor formula in z at a point $(x,y,0)$ leads to

$$f(x,y,z) = f(x,y,0) + \frac{\partial f}{\partial z}(x,y,0)\cdot z + \frac{\partial^2 f}{\partial z^2}(x,y,0)\cdot\frac{z^2}{2} + \varphi(x,y;z), \tag{4.1}$$

where $\varphi(x,y;z) = o(z^2)$ as $z\to 0$ locally uniformly in x, y. Set

$$R(x,y,z) = \frac{\partial^2 f}{\partial z^2}(x,y,0) + \frac{\varphi(x,y;z)}{z^2} \quad \text{as} \quad z\neq 0; \qquad R(x,y,0) = 0.$$

Then $R\in W_K(z)$ and, denoting by $P(x,y) = f(x,y,0)$, $Q(x,y) = \frac{\partial f}{\partial z}(x,y,0)$, we obtains from (4.1)

$$f(x,y,z) = P(x,y) + Q(x,y)\cdot z + R(x,y,z)\cdot\frac{z^2}{2}, \tag{4.2}$$

where P, $Q\in C^2$; $R\in W_K(z)$.

Now, using the equalities

$$\frac{\partial f}{\partial y} = \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial y}\cdot z + \frac{\partial R}{\partial y}\cdot\frac{z^2}{2}, \quad \frac{\partial f}{\partial z} = Q + R\cdot z + \frac{\partial R}{\partial z}\cdot\frac{z^2}{2},$$

$$\frac{\partial^2 f}{\partial y\partial z} = \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial y}\cdot z + \frac{\partial^2 R}{\partial y\partial z}\cdot\frac{z^2}{2}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 Q}{\partial y^2}\cdot z + \frac{\partial^2 R}{\partial y^2}\cdot\frac{z^2}{2},$$

$$\frac{\partial^2 f}{\partial z^2} = R + 2\frac{\partial R}{\partial z}\cdot z + \frac{\partial^2 R}{\partial z^2}\cdot\frac{z^2}{2}, \tag{4.3}$$

it follows by the conditions

$$\nabla_{yz}f \in WK_2(z), \quad H_{yz}f\in WK_2(z)$$

that

$$\nabla_{yz}R\in W_K(z), \quad H_{yz}R\in W_K(z).$$

Thus, in the representation (4.2), $R\in W_K^2(z)$.

Conversely, if representation (4.2) holds with P, $Q\in C^2$; $R\in W_K^2(z)$, then the condition $f\in W^2K_2(z)$ holds automatically. Thus, the following theorem is proved.

Theorem 10. *The representability of f in form (4.2) with P, $Q\in C^2$; $R\in W_K^2(z)$, is necessary and sufficient for f to be in the class $W^2K_2(z)$.*

5 Stationary form of the Legendre-Jacobi conditions

A simple example of the "harmonic oscillator" in which case

$$\Phi(y) = \int_0^T (y'^2 - y^2)dx$$

shows that restriction (1.3) on the length of $[a; b]$ in Theorem 8(2) can be too strong. Now we consider some types of integrands for which an essential weakening of the restriction (3) is possible.

Let us consider the variational functional

$$\Phi(y) = \int_0^T \left(R(x, y, y') \cdot \frac{y'^2}{2} + Q(x, y) \cdot y' + P(x, y) \right) dx, \quad y(\cdot) \in H_0^1[0; T], \tag{5.1}$$

where P, $Q \in C^2$; $R \in W_K^2(z)$.

Note that, according to Theorem 10, the integrand

$$f(x, y, z) = R(x, y, z) \cdot \frac{z^2}{2} + Q(x, y) \cdot z + P(x, y)$$

is of the class $W^2K_2(z)$. Hence, by Theorem 5, this functional is well-posed and twice K-differentiable in $H_0^1[0; T]$.

Now, consider conditions ensuring existence of a K-extremum at zero for variational functional (5.1).

1) From equalities (4.3) we obtain

$$\frac{\partial f}{\partial y}(x, 0, 0) = \frac{\partial P}{\partial y}(x, 0), \quad \frac{\partial f}{\partial z}(x, 0, 0) = Q(x, 0). \tag{5.2}$$

Then, substituting (5.2) in the Euler-Lagrange variational equation along K-extremal $y_0(x) \equiv 0$ $(0 \le x \le T)$ for functional (5.1) and taking into account, that $f \in C^2$, we obtain

$$(d/dx)[f(x, 0, 0)] = (d/dx)[Q(x, 0)],$$

hence

$$\frac{\partial P}{\partial y}(x, 0) - \frac{\partial Q}{\partial x}(x, 0) \equiv 0. \tag{5.3}$$

2) Next, let us study conditions of fulfilment of the generalized Legendre-Jacobi sufficient condition for a strong K-extremum of the functional $\Phi(y)$ in Sobolev space H_0^1 (Theorem 7) along a K-extremal $y_0(\cdot)$. Note first that it is necessary to impose the additional requirement of the absolute continuity along the K–extremal $y_0(x) \equiv 0$ of the functions

$$\frac{\partial f}{\partial z}(x, y_0(x), y_0'(x)) = Q(x, 0) \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial z}(x, y_0(x), y_0'(x)) = \frac{\partial Q}{\partial y}(x, 0).$$

i) *The stregthened Legendre condition* in the case of a K-minimum for functional (5.1) takes the form

$$\frac{\partial^2 f}{\partial z^2}(x,0,0) = R(x,0,0) > 0 \tag{5.4}$$

everywhere along $[0;T]$.

ii) *The Jacobi condition*: the generalized Jacobi equation for functional (5.1) along K-extremal $y_0(\cdot)$

$$-\frac{d}{dx}\left[\frac{\partial^2 f}{\partial z^2}(x,0,0)u'\right] + \left[-\frac{d}{dx}\left(\frac{\partial^2 f}{\partial y \partial z}(x,0,0)\right) + \frac{\partial^2 f}{\partial y^2}(x,0,0)\right] u \overset{\text{a.e.}}{=} 0$$

in view of equalities (5.1) takes the form

$$-\frac{d}{dx}\left[R(x,0,0)u'\right] + \left[-\frac{d}{dx}\left(\frac{\partial Q}{\partial y}(x,0)\right) + \frac{\partial^2 P}{\partial y^2}(x,0)\right] u \overset{\text{a.e.}}{=} 0.$$

Or, taking into account $f \in C^2$, it follows that

$$-\frac{d}{dx}\left[R(x,0,0)u'\right] + \left[\frac{\partial^2 P}{\partial y^2}(x,0) - \frac{\partial^2 Q}{\partial x \partial y}(x,0)\right] u \overset{\text{a.e.}}{=} 0, \tag{5.5}$$

under the initial conditions $u(0) = 0$, $u'(0) = 1$.

Now, let us consider sufficient conditions of fulfillment of the Jacobi condition.

Assume that the following additional conditions are satisfied

$$R(x,0,0) \equiv r > 0, \tag{5.6}$$

$$\left(\frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 Q}{\partial x \partial y}\right)(x,0) \equiv p. \tag{5.7}$$

Then equation (5.5) takes the form

$$ru'' - pu = 0, \quad u(0) = 0, \;\; u'(0) = 1. \tag{5.8}$$

Consider all the possible cases: $p = 0$, $p > 0$, $p < 0$.

a) $p = 0$. Equation (5.8) takes the form: $u'' = 0$, whence the solution $u(x) = x$ satisfies the Jacobi condition:

$$u(x) \neq 0, \quad 0 < x \leq T, \quad \text{for all} \quad T > 0.$$

b) $p > 0$. Equation (5.8) takes the form: $u'' = \frac{p}{r}u$ with $\frac{p}{r} > 0$, whence the solution $u(x) = \sqrt{\frac{r}{p}}\operatorname{sh}\sqrt{\frac{p}{r}}x$ satisfies the Jacobi condition:

$$u(x) \neq 0, \quad 0 < x \leq T, \quad \text{for all} \quad T > 0.$$

c) $p < 0$. Equation (5.8) takes the form: $u'' = -\left|\frac{p}{r}\right| u$ with $\frac{p}{r} < 0$, whence the solution $u(x) = \sqrt{\left|\frac{r}{p}\right|} \sin \sqrt{\left|\frac{p}{r}\right|} x$ satisfies the Jacobi condition:

$$u(x) \neq 0, \quad 0 < x \leq T, \qquad \text{only for the case}$$

$$T < \pi \sqrt{\left|\frac{r}{p}\right|}.$$

We say that conditions (5.3), (5.6), (5.7) (providing fulfilment of the Legendre-Jacobi sufficient conditions at zero) are the *stationary form of the Legendre-Jacobi conditions at zero*, or *(SLJ)-conditions.*

Now, let consider some generalization of the conditions (SLJ). Namely, replace the conditions (5.6)–(5.7) by the following ones:

$$R(x, 0, 0) = r \cdot e^{\alpha x} \quad (r > 0, \ \alpha \in \mathbb{R}), \tag{5.9}$$

$$\left(\frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 Q}{\partial x \partial y}\right)(x, 0) = p \cdot e^{\alpha x}. \tag{5.10}$$

Then equation (5.5) takes the form

$$r \cdot u'' + (r\alpha) \cdot u' - p \cdot u = 0, \quad u(0) = 0, \ u'(0) = 1, \tag{5.11}$$

with the discriminant

$$D = r(r\alpha^2 + 4p).$$

Consider all the possible cases: $r\alpha^2 + 4p = 0$, $r\alpha^2 + 4p > 0$, $r\alpha^2 + 4p < 0$.

a) $r\alpha^2 + 4p = 0$. The characteristic equation for equation (5.11) takes the form: $\lambda = -\alpha/2$, whence the solution $u(x) = x \cdot e^{-\alpha x/2}$ satisfies the Jacobi condition:

$$u(x) \neq 0, \quad 0 < x \leq T, \qquad \text{for all} \quad T > 0.$$

b) $r\alpha^2 + 4p > 0$. The characteristic equation for equation (5.11) takes the form:

$$r \cdot \lambda^2 + (r\alpha) \cdot \lambda - p = 0 \quad (D > 0),$$

whence the solution

$$u(x) = 2e^{-\frac{\alpha}{2}x} \cdot \sqrt{\frac{r}{r\alpha^2 + 4p}} \cdot \operatorname{sh}\left(\frac{1}{2}\sqrt{\frac{r\alpha^2 + 4p}{r}} x\right)$$

satisfies the Jacobi condition:

$$u(x) \neq 0, \quad 0 < x \leq T, \qquad \text{for all} \quad T > 0.$$

c) $r\alpha^2 + 4p < 0$. The characteristic equation for equation (5.11) takes the form:

$$r \cdot \lambda^2 + (r\alpha) \cdot \lambda - p = 0 \quad (D < 0),$$

whence the solution

$$u(x) = 2e^{-\frac{\alpha}{2}x} \cdot \sqrt{\frac{r}{|r\alpha^2 + 4p|}} \cdot \sin\left(\frac{1}{2}\sqrt{\frac{|r\alpha^2 + 4p|}{r}}x\right)$$

satisfies the Jacobi condition:

$$u(x) \neq 0, \quad 0 < x \leq T, \quad \text{only for the case}$$

$$T < 2\pi\sqrt{\frac{r}{|r\alpha^2 + 4p|}}.$$

We call conditions (5.9)–(5.10) *the exponential* (*SLJ*)*-conditions.* Of course, the "usual" (SLJ)-conditions (5.6)–(5.7) are a particular case of conditions (5.9)–(5.10) corresponding to $\alpha = 0$.

6 Inverse extremal problem for variational functionals in $H^1[a; b]$ under the (SLJ)–condition

Our aim in this Section is to describe *all* integrands of the class $W^2K_2(z)$ satisfying conditions (5.3), (5.6), (5.7):

$$\frac{\partial P}{\partial y}(x,0) - \frac{\partial Q}{\partial x}(x,0) \equiv 0, \quad \left(\frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 Q}{\partial x \partial y}\right)(x,0) \equiv p, \quad R(x,0,0) \equiv r > 0, \quad (6.1)$$

where P, Q, R are coefficients of minimal representation (5.1) of the integrand f; P, $Q \in C^2$; $R \in W^2_K(z)$.

1) Choose $P(x,y) \in C^2$ arbitrarily. Then the first of equations (6.1) leads to:

$$\left(\frac{\partial Q}{\partial x}(x,0) = \frac{\partial P}{\partial y}(x,0)\right) \Leftrightarrow \left(Q(x,0) = \int\limits_0^x \frac{\partial P}{\partial y}(t,0)dt + C_1\right),$$

hence it follows

$$Q(x,y) = \int\limits_0^x \frac{\partial P}{\partial y}(t,0)dt + \widetilde{Q}(x,y), \qquad \text{where} \quad \widetilde{Q}(x,0) \equiv C_1. \qquad (6.2)$$

2) The second of equations (6.1) leads to:

$$\frac{\partial^2 Q}{\partial x \partial y}(x,0) = \frac{\partial^2 P}{\partial y^2}(x,0) - p. \qquad (6.3)$$

Substituting of (6.2) in (6.3) leads to:

$$\left(\frac{\partial^2 \widetilde{Q}}{\partial x \partial y}(x,0) = \frac{\partial^2 Q}{\partial x \partial y}(x,0) = \frac{\partial^2 P}{\partial y^2}(x,0) - p\right) \Rightarrow$$

$$\Rightarrow \left(\frac{\partial \widetilde{Q}}{\partial y}(x,0) = \int\limits_0^x \left[\frac{\partial^2 P}{\partial y^2}(t,0) - p \right] dt + C_2 \right),$$

hence

$$\frac{\partial \widetilde{Q}}{\partial y}(x,y) = \int\limits_0^x \left[\frac{\partial^2 P}{\partial y^2}(t,0) - p \right] dt + \widetilde{q}(x,y), \quad \text{where } \widetilde{q}(x,0) \equiv C_2.$$

Now, by integrating in y we obtain:

$$\widetilde{Q}(x,y) = \int\limits_0^y ds \int\limits_0^x \left[\frac{\partial^2 P}{\partial y^2}(t,0) - p \right] dt + \int\limits_0^y \widetilde{q}(x,s)ds + C_1, \quad \text{where } \widetilde{q}(x,0) \equiv C_2. \quad (6.4)$$

Next, by substituting (6.4) in (6.2) we obtain:

$$Q(x,y) = \int\limits_0^x \frac{\partial P}{\partial y}(t,0)dt + \int\limits_0^y ds \int\limits_0^x \left[\frac{\partial^2 P}{\partial y^2}(t,0) - p \right] dt + \int\limits_0^y \widetilde{q}(x,s)ds + C_1,$$

where $\widetilde{q}(x,0) \equiv C_2$. From here, denoting $\widetilde{q}(x,y) = q(x,y) - q(x,0) + C_2$, after not complicated transformations we obtain:

$$Q(x,y) = \int\limits_0^x \left[\frac{\partial P}{\partial y}(t,0) + y \cdot \frac{\partial^2 P}{\partial y^2}(t,0) \right] dt +$$

$$+ \int\limits_0^y [q(x,s) - q(x,0)]ds + (C_1 + C_2 y - p \cdot xy), \quad (6.5)$$

where the function $q \in C^2$ and the constants C_1 and C_2 can be chosen arbitrarily.

3) By setting

$$R(x,y,z) = r(x,y,z) - r(x,0,0) + r, \quad (6.6)$$

we obtain the last of conditions (6.1): $R(x,0,0) \equiv r$ as well. At last, substituting of (6.5)–(6.6) in (4.2) leads to the desired result:

$$f(x,y,z) = P(x,y) + \left\{ \int\limits_0^x \left[\frac{\partial P}{\partial y}(t,0) + y \cdot \frac{\partial^2 P}{\partial y^2}(t,0) \right] dt + \int\limits_0^y [q(x,s) - q(x,0)]ds + \right.$$

$$\left. + (C_1 + C_2 y - p \cdot xy) \right\} \cdot z + [r(x,y,z) - r(x,0,0) + r] \cdot \frac{z^2}{2}, \quad (6.7)$$

where the functions $P(x,y) \in C^2$, $q(x,y) \in C^2$, $r(x,y,z) \in W_K^2(z)$ and the constants C_1 and C_2 can be chosen arbitrarily.

Let us pass now to the case of an arbitrary K–extremal $y_0(x)$ of the class $W^{2,2}[a;b]$. In this case we, use the auxiliary variational functional (see Introduction):

$$\widetilde{\Phi}(y) = \Phi(y + y_0) = \int\limits_a^b f(x, y + y_0(x), y' + y_0'(x))\, dx =: \int\limits_a^b \widetilde{f}(x,y,y')\, dx.$$

Applying to $\widetilde{f}$ representation (6.7) together with the consequent shifts $y+y_0 \longmapsto y$ and $z+y_0'(x) \longmapsto z$ leads to the general form of the integrand for the case under consideration:

$$f(x,y,z) = P(x,y-y_0(x)) + \{\int\limits_0^x \left[\frac{\partial P}{\partial y}(t,0) + (y-y_0(x))\cdot\frac{\partial^2 P}{\partial y^2}(t,0)\right] dt +$$

$$+\int\limits_0^{y-y_0(x)} [q(x,s)-q(x,0)]\,ds + (C_1 + C_2\cdot(y-y_0(x)) - p\cdot x\cdot(y-y_0(x)))\}\cdot(z-y_0'(x)) +$$

$$+[r(x,y-y_0(x),z-y_0'(x)) - r(x,0,0)+r]\cdot\frac{(z-y_0'(x))^2}{2},$$

where the functions

$$P(x,y)\in C^2, \quad q(x,y)\in C^2, \quad r(x,y,z)\in W_K^2(z)$$

and the constants C_1 and C_2 can be chosen arbitrarily.

Finally, let us obtain an analogue of representation (6.7) for the case of exponential (SLJ)-conditions:

$$\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)(x,0)\equiv 0; \quad \left(\frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 Q}{\partial x\partial y}\right)(x,0) = p\cdot e^{\alpha x}; \quad R(x,0,0) = r\cdot e^{\alpha x} \ (r>0).$$

Choosing $P(x,y)\in C^2$ arbitrarily and acting analogously with 1)–2) above, we can obtain

$$Q(x,y) = \int\limits_0^x \left[\frac{\partial P}{\partial y}(t,0) + y\cdot\frac{\partial^2 P}{\partial y^2}(t,0)\right] dt +$$

$$+\int\limits_0^y [q(x,s)-q(x,0)]\,ds + \left(C_1 + C_2 y - p\cdot\frac{e^{\alpha x}-1}{\alpha}\cdot y\right),$$

where the functions $P(x,y)\in C^2$, $q\in C^2$ and the constants C_1 and C_2 can be chosen arbitrarily.

Next, by setting

$$R(x,y,z) = r(x,y,z) - r(x,0,0) + r\cdot e^{\alpha x}$$

and acting analogously to 3) above, we can obtain the desired result:

$$f(x,y,z) = P(x,y) + \{\int\limits_0^x \left[\frac{\partial P}{\partial y}(t,0) + y\cdot\frac{\partial^2 P}{\partial y^2}(t,0)\right] dt +$$

$$+\int\limits_0^y [q(x,s)-q(x,0)]\,ds + \left(C_1 + C_2 y - p\cdot\frac{e^{\alpha x}-1}{\alpha}\cdot y\right)\}\cdot z +$$

$$+ [r(x,y,z) - r(x,0,0) + r \cdot e^{\alpha x}] \cdot \frac{z^2}{2},$$

where the functions

$$P(x,y) \in C^2, \quad q(x,y) \in C^2, \quad r(x,y,z) \in W_K^2(z)$$

and the constants C_1 and C_2 can be chosen arbitrarily.

7 Properties of mappings of class $W_K^2(z)$

Aiming at applying formulas (6.7) where the mappings $R(x,y,z) \in W_K^2(z)$, we consider some properties of the such mappings.

First, denote by $W^2(z)$ the class of the mappings $\varphi(z)$ having $\varphi(z)$, $\varphi'(z)$ and $\varphi''(z)$ possessing uniform continuity and boundedness for $-\infty < z < \infty$. Then the following statements are valid.

Proposition 3. *If*

$$R(x,y,z) = \sum_{k=1}^{n} \alpha_k(x,y) \cdot \beta_k(z), \quad \textit{where} \quad \alpha_k \in C^2, \quad \beta_k \in W^2(z),$$

then $R \in W_K^2(z)$.

Proof. Let us fix $k = \overline{1,n}$. As $\alpha_k(x,y)$ is uniformly continuous and bounded in x, y by the Weierstrass theorem, and $\beta_k(z)$ is uniformly continuous globally in z by the assumption, then obviously $\alpha_k(x,y) \cdot \beta_k(z) \in W_K(z)$.

Next, the gradient

$$\nabla_{yz}[\alpha_k(x,y) \cdot \beta_k(z)] = \left(\frac{\partial \alpha_k}{\partial y}(x,y) \cdot \beta_k(z), \alpha_k(x,y) \cdot \beta_k'(z) \right)$$

has the coefficients from $W_K(z)$ and therefore is in that class. Finally, the Hessian

$$H_{yz}[\alpha_k(x,y) \cdot \beta_k(z)] = \begin{pmatrix} \frac{\partial^2 \alpha_k}{\partial y^2}(x,y) \cdot \beta_k(z), & \frac{\partial \alpha_k}{\partial y} \cdot \beta_k'(z) \\ \frac{\partial \alpha_k}{\partial y}(x,y) \cdot \beta_k'(z), & \alpha_k(x,y) \cdot \beta_k''(z) \end{pmatrix}$$

has the coefficients in $W_K(z)$ as well and therefore is in that class. So, $\alpha_k(x,y) \cdot \beta_k(z) \in W_K^2(z)$, hence $R(x,y,z) \in W_K^2(z)$ evidently follows. □

Proposition 4. *If* $R_1, \ldots, R_m \in W_K^2(z)$; $\varphi(u_1, u_2, \ldots, u_m) \in C^2$, *then* $\varphi(R_1(x,y,z), \ldots, R_m(x,y,z)) \in W_K^2(z)$.

Proof. For an arbitrary compact $C_y \subset \mathbb{R}$ the mappings R_1, ... R_m are uniformly continuous and bounded on $[0;T] \times C_y \times \mathbb{R}$. Hence φ possesses the same properties on the set

$$\prod_{i=1}^{m} R_i\left([0;T] \times C_y \times \mathbb{R}\right)$$

and therefore the composition $\varphi(R_1,\dots,R_m)$ is in the Weierstrass class $W_K(z)$. The analogous properties of $\nabla_{yz}\varphi(R_1,\dots,R_m)$ and $H_{yz}\varphi(R_1,\dots,R_m)$ follow from the representations

$$\frac{\partial}{\partial y}\varphi(R_1,\dots,R_m)=\sum_{i=1}^{m}\frac{\partial\varphi}{\partial u_i}(R_1,\dots,R_m)\cdot\frac{\partial R_i}{\partial y};$$

$$\frac{\partial}{\partial z}\varphi(R_1,\dots,R_m)=\sum_{i=1}^{m}\frac{\partial\varphi}{\partial u_i}(R_1,\dots,R_m)\cdot\frac{\partial R_i}{\partial z};$$

$$\frac{\partial^2}{\partial y^2}\varphi(R_1,\dots,R_m)=$$
$$=\sum_{i=1}^{m}\left[\sum_{j=1}^{m}\frac{\partial^2\varphi}{\partial u_i\partial u_j}(R_1,\dots,R_m)\cdot\frac{\partial R_j}{\partial y}\cdot\frac{\partial R_i}{\partial y}+\frac{\partial\varphi}{\partial u_i}(R_1,\dots,R_m)\cdot\frac{\partial^2 R_i}{\partial y^2}\right];$$

$$\frac{\partial^2}{\partial z^2}\varphi(R_1,\dots,R_m)=$$
$$=\sum_{i=1}^{m}\left[\sum_{j=1}^{m}\frac{\partial^2\varphi}{\partial u_i\partial u_j}(R_1,\dots,R_m)\cdot\frac{\partial R_j}{\partial z}\cdot\frac{\partial R_i}{\partial z}+\frac{\partial\varphi}{\partial u_i}(R_1,\dots,R_m)\cdot\frac{\partial^2 R_i}{\partial z^2}\right];$$

$$\frac{\partial^2}{\partial y\partial z}\varphi(R_1,\dots,R_m)=$$
$$=\sum_{i=1}^{m}\left[\sum_{j=1}^{m}\frac{\partial^2\varphi}{\partial u_i\partial u_j}(R_1,\dots,R_m)\cdot\frac{\partial R_j}{\partial z}\cdot\frac{\partial R_i}{\partial y}+\frac{\partial\varphi}{\partial u_i}(R_1,\dots,R_m)\cdot\frac{\partial^2 R_i}{\partial y\partial z}\right].$$

□

Corollary 1. *By the conditions $R_1,\dots,R_m\in W_K^2(z)$, $\alpha_k(x,y)\in C^2$ $(k=\overline{1,m})$ it follows that*

$$\sum_{k=1}^{m}\alpha_k(x,y)R_k(x,y,z)\in W_K^2(z).$$

Note that the last Corollary generalizes Proposition 3 because

$$\beta_k\in W^2(z)\Rightarrow\beta_k\in W_K^2(z).$$

Corollary 2. *If $R_1,\dots,R_m\in W_K^2(z)$, then $R_1\cdot R_2\cdot\ldots\cdot R_m\in W_K^2(z)$.*

Proposition 5. *If $R(x,y,z)\in W_K^2(z)$, $\psi(z)\in W^2(z)$ then $R(x,y,\psi(z))\in W_K^2(z)$.*

Proof. Since for an arbitrary compact C_y the mapping R is uniformly continuous and bounded on $[0;T]\times C_y\times\mathbb{R}$ and ψ is uniformly continuous and bounded for $z\in\mathbb{R}$, then $R(x,y,\psi(z))$ is of the Weierstrass class $W_K(z)$. The analogous properties of $\nabla_{yz}R(x,y,\psi(z))$ and $H_{yz}R(x,y,\psi(z))$ follow from the representations

$$\frac{\partial}{\partial y}R(x,y,\psi(z))=\frac{\partial R}{\partial y}(x,y,\psi(z));\quad \frac{\partial}{\partial z}R(x,y,\psi(z))=\frac{\partial R}{\partial z}(x,y,\psi(z))\cdot\psi'(z);$$

$$\frac{\partial^2}{\partial y^2}R(x,y,\psi(z)) = \frac{\partial^2 R}{\partial y^2}(x,y,\psi(z)); \quad \frac{\partial}{\partial y\partial z}R(x,y,\psi(z)) = \frac{\partial^2 R}{\partial y\partial z}(x,y,\psi(z))\cdot\psi'(z);$$

$$\frac{\partial^2}{\partial z^2}R(x,y,\psi(z)) = \frac{\partial^2 R}{\partial z^2}(x,y,\psi(z))\cdot(\psi'(z))^2 + \frac{\partial R}{\partial z}(x,y,\psi(z))\cdot\psi''(z).$$

□

Let us note also some properties of the class $W^2(z)$ being used in the constructions above. Denote by $C_b^n(z)$ the class of all functions $\varphi(z)\in C^n$ having bounded derivatives up to n-th order.

Property 1. *The following inclusions are valid:*

$$C_b^3(z)\subset W^2(z)\subset C_b^2(z).$$

Property 2. *If $\varphi\in C^2$ and φ is periodic, then $\varphi\in W^2(z)$.*

Property 3. *If $\varphi\in C^2$, $\varphi^{(k)}$ $(\pm\infty)$ exist and are finite for $k=0,1,2$, then $\varphi\in W^2(z)$.*

8 Non–local K–extrema of variational functionals in $H^1[a;b]$

Here we describe a rather wide class of the variational functionals in $H^1[a;b]$ having a non–local compact extremum at zero.

Note first, that if functional (5.1) attains a strong K-minimum at zero then for every zero neighborhood $U(0)\subset H^1$ there exist such values of y that $\Phi(y)>\Phi(0)$. Thus, Φ cannot attain a local maximum at zero.

Now, let us investigate conditions under which functional (5.1) does not attain a local minimum at zero.

Let the integrand of functional (5.1) satisfy conditions (6.1) i.e., it takes form (6.7) and let Φ attain a strong K-minimum at zero. Suppose for convenience $\Phi(0)=0$. By virtue of (6.7), it means

$$\int_0^T P(x,0)dx = 0.$$

The last condition is obviously satisfied under the assumption

$$P(x,0)\equiv 0. \tag{8.1}$$

Introduce also the supplementary conditions:

$$Q(0,0)=0, \tag{8.2}$$

that is equivalent, by virtue of (6.7), to the condition $C_1=0$, and also the alternating signs condition for R:

$$R(x,0,z_0)\le -r_0<0 \qquad (\forall\ x\in[0;T]) \tag{8.3}$$

for some z_0. Let us show that Φ does not attain a local minimum at zero under conditions (8.1)– (8.3).

Set

$$y^{\varepsilon}(x) = \begin{cases} z_0(x-\varepsilon), & \text{as } \ 0 \le x \le \varepsilon; \\ 0, & \text{as } \ \varepsilon \le x \le T \end{cases}$$

for sufficiently small $\varepsilon > 0$.

Obviously, $y^{\varepsilon} \in H_0^1[0;T]$. Moreover,

$$\|y^{\varepsilon}\|_{H^1}^2 = \int\limits_0^{\varepsilon} \left(z_0^2(x-\varepsilon)^2 + z_0^2\right) dx = z_0^2\left(\varepsilon + \frac{\varepsilon^3}{3}\right) \to 0 \quad \text{ as } \ \varepsilon \to 0.$$

The integrand f along the function y^{ε} takes the form

$$f(x, y^{\varepsilon}, (y^{\varepsilon})') =$$

$$= \begin{cases} R(x, z_0(x-\varepsilon), z_0)\cdot \frac{z_0^2}{2} + Q(x, z_0(x-\varepsilon))\cdot z_0 + P(x, z_0(x-\varepsilon)), & 0 \le x \le \varepsilon; \\ 0, & \varepsilon \le x \le T. \end{cases}$$

From here it follows that

$$\Phi(y^{\varepsilon}) = \frac{z_0^2}{2}\cdot \int\limits_0^{\varepsilon} R(x, z_0(x-\varepsilon), z_0)dx + \\ + z_0 \cdot \int\limits_0^{\varepsilon} Q(x, z_0(x-\varepsilon))dx + \int\limits_0^{\varepsilon} P(x, z_0(x-\varepsilon))dx. \quad (8.4)$$

Moreover,

$$\begin{cases} (8.1) \text{ implies} & P(x, z_0(x-\varepsilon)) = o(1) \\ (8.2) \text{ implies} & Q(x, z_0(x-\varepsilon)) = o(1) \\ (8.3) \text{ implies} & R(x, z_0(x-\varepsilon), z_0) = -r_0 + o(1) \end{cases} \quad \text{as } \ \varepsilon \to 0. \quad (8.5)$$

From (8.4)–(8.5) it follows that

$$\Phi(y^{\varepsilon}) = o(\varepsilon) + z_0\cdot o(\varepsilon) + \frac{z_0^2}{2}\cdot [o(\varepsilon) - r_0\varepsilon] = -\frac{z_0^2 r_0}{2}\varepsilon + o(\varepsilon) < 0$$

for sufficiently small $\varepsilon > 0$.

Thus, functional (5.1) cannot attain a local minimum at zero and therefore it does not attain a local extremum at zero. Hence, an arbitrary variational functional $\Phi(y)$ having an integrand satisfying conditions (6.1) and (8.1)–(8.3), attains a non-local K-minimum at zero. Let us summarize the results of our considerations.

Theorem 11. *Consider a functional of the form*

$$\Phi(y) = \int\limits_0^T \left(R(x, y, y')\cdot \frac{y'^2}{2} + Q(x, y)\cdot y' + P(x, y)\right) dx, \quad y(\cdot) \in H_0^1[0;T],$$

where P, $Q \in C^2$; $R \in W_K^2(z)$.

Then under the assumptions:

$$\frac{\partial P}{\partial y}(x,0) - \frac{\partial Q}{\partial x}(x,0) \equiv 0, \quad \left(\frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 Q}{\partial x \partial y}\right)(x,0) \equiv p, \quad R(x,0,0) \equiv r > 0,$$

$$P(x,0) \equiv 0, \quad Q(0,0) = 0,$$

and also under the alternating signs condition for R:

$$R(x,0,z_0) \le -r_0 < 0 \qquad (\forall\ x \in [0;T]);$$

for some z_0, the variational functional $\Phi(y)$ attains a non–local K–minimum at zero for every $T > 0$ in the case of $p \ge 0$ and for $0 < T < \pi\sqrt{\left|\frac{r}{p}\right|}$ in the case of $p < 0$.

Let us consider in the conclusion some concrete examples.

Example 1. Consider the following functional:

$$\Phi(y) = \int_0^{1/3} \left((y')^2 \left(\sin(1+\cos y') - \frac{1}{2}\right) + y' \sin y^2 + y^2\right) dx, \quad y(\cdot) \in H_0^1([0;1/3]).$$

In the case under consideration,

$$P(y) = y^2, \quad Q(y) = \sin y^2, \quad R(z) = 2\sin(1+\cos z) - 1.$$

Direct calculation shows fulfilment of condition (6.1) and (8.1)–(8.3) for the functional $\Phi(y)$. Moreover,

$$R(0) \equiv r = 2\sin 2 - 1 > 0,$$

and for $z = \pi \quad R(\pi) = -1 < 0$.

Thus, since in the case $p \equiv 2 > 0$ and $T = 1/3$, then by virtue of Theorem 11 the functional $\Phi(y)$ attains a non-local K-minimum at zero.

Example 2. Consider the following functional:

$$\Phi(y) = \int_0^1 \left(y^3 \ln(x^2+4) + y' \sin xy + \frac{(y')^2 \cos y'}{2(1+(y')^2)}\right) dx, \quad y(\cdot) \in H_0^1[0;1].$$

In the case under consideration

$$P(x,y) = y^3 \ln(x^2+4), \quad Q(x,y) = \sin xy, \quad R(z) = \frac{\cos z}{1+z^2}.$$

Direct calculation shows fulfilment of conditions (6.1) and (8.1)–(8.3) for the functional $\Phi(y)$. Moreover,

$$R(0) \equiv r = 1 > 0,$$

and for $z = \pi \quad R(\pi) = -1/(1+\pi^2) < 0$.

Thus, since in the case $p \equiv -1 < 0$ and $T = 1 < \pi$, then by virtue of Theorem 11 the functional $\Phi(y)$ attains a non-local K-minimum at zero.

References

[1] E.V. Bozhonok, *On solutions to "almost everywhere"— Euler–Lagrange equation in Sobolev space* H^1. Methods of Funct. Anal. & Top., 13, no. 3 (2007), 262 – 266.

[2] E.V. Bozhonok, *Some existence conditions of compact extrema for variational functionals of several variables in Sobolev space* H^1. Operator Theory: Adv. & Appl., 190 (2009), 141 – 155.

[3] E.V. Bozhonok, I.V. Orlov, *Legendre and Jacobi conditions of compact extrema for variation functionals in Sobolev spaces.* Pratsi Inst. Matem. Nats. Akad. Nauk Ukr., 3 no. 4 (2006), 282 – 293 (in Russian).

[4] B. Dacorogna, *Introduction to the calculus of variations.* Imperial College Press, London, 2004.

[5] M. Giaquinta, S. Hildebrandt, *Calculus of Varitions.* Springer, New York, 1996.

[6] I.V. Orlov, *Compact extrema: general theory and its applications to the variational functionals.* Operator Theory: Adv. & Appl., 190 (2009), 397 – 417.

[7] I.V. Orlov, E.V. Bozhonok, *Conditions of existence, K–continuity and K–differentiability of Euler–Lagrange functional in Sobolev space* W_2^1. Uchenye Zapiski Tavricheskogo Natsynal'nogo Universiteta, 19, no. 2 (2006), 63 – 78 (in Russian).

[8] I. Orlov, *Elimination of Hamilton–Jacobi equation in extreme variational problems.* http://arXiv:1008.2599 (2010), 1 – 10.

[9] I.V. Orlov, *Extreme Problems and Scales of the Operator Spaces.* North–Holland Math. Studies. Funct. Anal & Appl., 197 (2004), 209 – 228.

[10] I.V. Orlov, *K–differentiability and K–extrema*, Ukrainskii Matem. Vestnik, 3, no. 1 (2006), 97 – 115 (in Russian).

[11] I.V. Orlov, *Normal differentiability and extrema of functionals in locally convex space.* Kibernetika i Sistemnyi Analiz, no. 4 (2002), 24 – 35 (in Russian).

[12] E.A.M. Rocha , D.F.M. Torres, *First integrals for problems of calculus of variations on locally convex spaces.* Applied Sciences, 10 (2008), 207 – 218.

[13] H.-J. Schmeisser, W. Sickel, *Spaces of functions of mixed smoothness and approximation from hyperbolic crosses.* J. Approx. Theory, 128, no. 2 (2004), 115 — 150.

[14] I.V. Skrypnik, *Nonlinear elliptic high order equations.* Naukova Dumka, Kiev, 1973 (in Russian).

[15] H. Triebel,*Theory of Function Spaces III. Monographs in Math., 100.* Birkhäuser, Basel, 2006.

Igor Orlov
Faculty of Mathematics and Informatics
Taurida National V. Vernadsky University
4 Vernadsky Avenu
95007 Simferopol, Ukraine
E-mail: old@crimea.edu

Received: 01.11.2010