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SOLVABILITY OF QUASI-LINEAR MULTI-POINT BOUNDARY VALUE PROBLEM AT RESONANCE

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Abstract. In this paper, we consider the following second order quasi-linear differential equation:

$$(\Phi_p(x'))' + f(t, x) = 0, \qquad 0 < t < 1,$$

where $\Phi_p(s) = |s|^{p-2}s$, $p \geq 2$, subject to certain boundary conditions. The criteria of solvability of these boundary value problems are given by employing the recent generalization of coincidence degree method. We also give an example to illustrate our conclusions.

1 Introduction

In this paper, we consider the following second order quasi-linear differential equation:

$$(\Phi_p(x'))' + f(t, x) = 0, \qquad 0 < t < 1, \tag{1.1}$$

subject to one of the following boundary conditions:

$$x(0) = 0, \quad \Phi_p(x'(1)) = \sum_{i=1}^{m-2} \alpha_i \Phi_p(x'(\eta_i)), \tag{1.2}$$

$$x'(0) = 0, \quad \Phi_p(x'(1)) = \sum_{i=1}^{m-2} \alpha_i \Phi_p(x'(\eta_i)), \tag{1.3}$$

$$x(0) = x(\xi), \quad \Phi_p(x'(1)) = \sum_{i=1}^{m-2} \alpha_i \Phi_p(x'(\eta_i)), \tag{1.4}$$

where $\Phi_p(s) = |s|^{p-2}s$, is the p-Laplacian, $p \geq 2$; $\eta_i (1 \leq i \leq m-2)$ are fixed points with $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$; $0 < \xi < 1$; $\alpha_i (1 \leq i \leq m-2)$ are nonnegative

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constants and $\sum_{i=1}^{m-2} \alpha_i = 1$(resonance condition), $\sum_{i=1}^{m-2} \alpha_i \eta_i \neq 1$.

By using the coincidence degree method, various existence results of the solutions of boundary value problems (BVPs) at resonance have been established in the literature, for example, see [4,7-10] and the references cited therein. These results are, however, confined to BVPs with linear leading term x'', i.e., to the case $p = 2$ in equation (1.1), mainly because the traditional coincidence degree method only applies to linear operators. As the p-Laplacian of a function comes frequently into play in many practical situations (for example, in the description of fluid dynamical and nonlinear elastic mechanical phenomena), very recently increasing attention has been drawn to the study of BVPs with the p-Laplacian. For example, one is referred to Cheung and Ren [1-3] and the references cited there. One useful technique used by Cheung and Ren is to translate the p-Laplacian equation into a 2-dimensional system for which Mawhin's Continuation Theorem [5] applies. In this paper, we shall follow the line of this method and by using a newly developed coincidence degree method by Ge and Ren in [6], we obtain the solvability of second order quasi-linear multi-point equation (1.1) with boundary conditions (1.2), (1.3) or (1.4) at resonance for $p \geq 2$.

2 Preliminary results

Let X and Z be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively. A continuous operator

$$M : X \cap \text{dom}M \to Z \tag{2.1}$$

is said to be **quasi-linear** if

(a) $$\text{Im}M := M(X \cap \text{dom}M) \text{ is a closed subset of } Z, \tag{2.2}$$

(b) $\ker M := \{x \in X \cap \text{dom}M : Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$. (2.3)

Let $X_1 = \ker M$ and X_2 be the complement space of X_1 in X, then $X = X_1 \oplus X_2$. On the other hand, suppose Z_1 is a subspace of Z and Z_2 is the complement of Z_1 in Z so that $Z = Z_1 \oplus Z_2$. Let $P : X \to X_1$ and $Q : Z \to Z_1$ be two projectors and $\Omega \subset X$ an open and bounded set with origin $\theta \in \Omega$. Throughout the paper we use θ to denote the origin of a linear space.

Suppose $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0,1]$ is a continuous operator. Denote N_1 by N. Let $\Sigma_\lambda = \{x \in \overline{\Omega} : Mx = N_\lambda x\}$. N_λ is said to be $M-$**compact** in $\overline{\Omega}$ if

(c) there is a vector subspace Z_1 of Z with $\dim Z_1 = \dim X_1$ and an operator $R : \overline{\Omega} \times [0,1] \to X_2$ being continuous and compact such that for $\lambda \in [0,1]$,

$$(I - Q)N_\lambda(\overline{\Omega}) \subset \text{Im}M \subset (I - Q)Z, \tag{2.4}$$

$$QN_\lambda x = 0, \quad \lambda \in (0,1), \Leftrightarrow QNx = 0, \tag{2.5}$$

$$R(\cdot, 0) \text{ is the zero operator and } R(\cdot,\lambda)|_{\Sigma_\lambda} = (I - P)|_{\Sigma_\lambda}, \tag{2.6}$$

$$M[P + R(\cdot,\lambda)] = (I - Q)N_\lambda. \tag{2.7}$$

Let $J : Z_1 \to X_1$ be a homeomorphism with $J(\theta) = \theta$. Define $S_\lambda : \overline{\Omega} \cap \text{dom}M \to X, 0 \le \lambda \le 1$ by

$$S_\lambda = P + R(\cdot, \lambda) + JQN. \tag{2.8}$$

Then S_λ is a completely continuous mapping.

Theorem 1 ([6]). *Let X and Z be two Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively, and $\Omega \subset X$ an open and bounded nonempty set. Suppose*

$$M : X \cap domM \to Z$$

is a quasi-linear operator and

$$N_\lambda : \overline{\Omega} \to Z, \qquad \lambda \in [0,1]$$

are $M-$compact. In addition, if
(H1) $Mx \ne N_\lambda x, \quad \lambda \in (0,1), \quad x \in \partial\Omega,$
(H2) $\deg\{JQN, \Omega \cap \ker M, 0\} \ne 0,$
where $N = N_1$, then the abstract equation $Mx = Nx$ has at least one solution in $\overline{\Omega}$.

3 Solvability of BVP (1.1) – (1.2)

Now we discuss the existence of solution for BVP (1.1) – (1.2) by applying Theorem 1.

Here a function u defined on $[0,1]$ is said to be a solution to BVP (1.1) – (1.2) if $u \in V = \{v \in C^1[0,1] : \ \Phi_p(v') \in C^1[0,1]\}$ satisfying BVP (1.1)-(1.2).

In this section, we let $X = \{x \in C[0,1] : x(0) = 0, \ \ \Phi_p(u'(1)) = \sum_{i=1}^{m-2} \alpha_i \Phi_p(x'(\eta_i))\}$ and $Z = C[0,1]$ with sup norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively. Clearly, X, Z are Banach spaces.

Define $M : X \cap \text{dom}M \to Z$ by

$$(Mx)(t) = (\Phi_p(x'(t)))'. \tag{3.1}$$

Then

$$\ker M = \{x = at : a \in \mathbb{R}\}, \qquad \text{dom}M = V,$$

$$\begin{aligned}
\text{Im}M &= \{y \in Z, \ (\Phi_p(x'))' = y(t), \ \ \text{for some} \ \ x(t) \in X \cap \text{dom}M\} \\
&= \left\{ y \in Z, \ \Phi_p(x'(t)) = B + \int_0^t y(t)dt, \ \ x(0) = 0, \ \ \Phi_p(x'(1)) = \sum_{i=1}^{m-2} \alpha_i \Phi_p(x'(\eta_i)) \right\} \\
&= \left\{ y \in Z : \ \ \int_0^1 y(t)dt = \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} y(t)dt \right\} \\
&= \left\{ y \in Z, \ \ \sum_{i=1}^{m-2} \alpha_i \int_{\eta_i}^1 y(s)ds = 0 \right\}.
\end{aligned}$$

Let

$$X_1 = \ker M, \qquad X_2 = \{x \in X : x(1) = 0\},$$

$$Z_1 = \mathbb{R}, \qquad Z_2 = \mathrm{Im} M.$$

Obviously $\dim X_1 = \dim Z_1 = 1$ and $X = X_1 \oplus X_2$. Define $P : X \to X_1$, $Q : Z \to Z_1$ by

$$Px = x(1)t, \qquad Qy = \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \int_{\eta_i}^{1} y(s)ds. \tag{3.2}$$

Then for any $y \in Z$, we have $y_1 \in \mathrm{Im}M$ if $y_1 = y - Q(y)$. In fact,

$$\begin{aligned}\sum_{i=1}^{m-2} \alpha_i \int_{\eta_i}^{1} y_1(s)ds &= \sum_{i=1}^{m-2} \alpha_i \int_{\eta_i}^{1} y(s)ds - Q(y) \sum_{i=1}^{m-2} \alpha_i \int_{\eta_i}^{1} ds \\ &= \sum_{i=1}^{m-2} \alpha_i \int_{\eta_i}^{1} y(s)ds - \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \int_{\eta_i}^{1} y(s)ds \sum_{i=1}^{m-2} \alpha_i (1 - \eta_i) \\ &= 0.\end{aligned}$$

So $y_1 \in \mathrm{Im}M$. That is to say, $Z = Z_1 \oplus Z_2$.

For any $\overline{\Omega} \subset X$ define $N_\lambda : \overline{\Omega} \to Z$ by

$$(N_\lambda x)(t) = -\lambda f(t, x(t)). \tag{3.3}$$

Clearly, $(I - Q)N_0$ is a zero operator, and

$$(I - Q)N_\lambda(\overline{\Omega}) \subset \mathrm{Im}M \subset (I - Q)Z,$$

i.e., (2.4) holds. Obviously (2.5) holds, too.

Let the homeomorphism $J : Z_1 \to X_1$ be defined by

$$J(a) = at, \qquad a \in \mathbb{R}, \quad t \in [0, 1]. \tag{3.4}$$

Define $R : \overline{\Omega} \times [0, 1] \to X_2$ by

$$R(x, \lambda)(t) = \int_0^t \Phi_p^{-1} \left[\Phi_p(x(1)) + c - \int_0^s \lambda f(\tau, x(\tau))d\tau \right] ds - x(1)t, \quad 0 \le t \le 1 \tag{3.5}$$

where c is a constant depending on (x, λ) and satisfying

$$\int_0^1 \Phi_p^{-1} \left[\Phi_p(x(1)) + c - \int_0^s \lambda f(\tau, x(\tau))d\tau \right] ds - x(1) = 0. \tag{3.6}$$

We now show that for given $x \in \overline{\Omega}$, $\lambda \in [0, 1]$, (3.6) has a unique solution $c = c(x, \lambda)$. Let

$$F(c) = \int_0^1 \Phi_p^{-1} \left[\Phi_p(x(1)) + c - \int_0^s \lambda f(\tau, x(\tau))d\tau \right] ds - x(1)$$

and

$$c_1 = \min_{0\le t\le 1}\int_0^t \lambda f(\tau, x(\tau))d\tau, \qquad c_2 = \max_{0\le t\le 1}\int_0^t \lambda f(\tau, x(\tau))d\tau.$$

Clearly $F(c)$ is continuous and increasing with respect to c on $[c_1, c_2]$ and $F(c_1) \le 0 \le F(c_2)$. Therefore there is a unique $c \in [c_1, c_2]$ satisfying (3.6).

We claim that the c is continuous dependence on (x, λ) by uniqueness of c.

If not, there is a point $(x_0, \lambda_0) \in \overline{\Omega} \times [0, 1]$ and a sequence $(x_n, \lambda_n) \to (x_0, \lambda_0)$ such that $c_n = c(x_n, \lambda_n) \not\to c(x_0, \lambda_0) = c_0$. Let

$$r = \max\left\{\|x\| : x \in \overline{\Omega}\right\}$$

and

$$d = \max_{|x|\le r, 0\le t\le 1} |f(t, x)|.$$

Then $-d \le c_1 \le c_2 \le d$ for the c_1 and c_2 given above. It yields that $-d \le c_n \le d$. So there is a subsequence of (x_n, λ_n), say, the sequence (x_n, λ_n) itself, such that

$$c_n = c(x_n, \lambda_n) \to \widetilde{c} \neq c_0.$$

However,

$$F(c_n) = \int_0^1 \Phi_p^{-1}\left[\Phi_p(x_n(1) + c_n - \int_0^s \lambda_n f(\tau, x_n(\tau))d\tau\right] - x_n(1) = 0,$$

and Lebesgue's theorem yields

$$F(\widetilde{c}) = \int_0^1 \Phi_p^{-1}\left[\Phi_p(x_0(1) + \widetilde{c} - \int_0^s \lambda_0 f(\tau, x_0(\tau))d\tau\right] - x_0(1) = 0,$$

which contradicts the uniqueness of $c = c(x_0, \lambda_0)$.

For any bounded set $\Omega \neq \phi$, $\lambda \in [0, 1]$, it is easy to see that $R : \overline{\Omega} \times [0, 1] \to X_2 \subset X$ is relatively compact and continuous. By (3.5), we have for

$$x \in \Sigma_\lambda = \{x \in \overline{\Omega} :\ Mx = N_\lambda x\} = \{x \in \overline{\Omega} :\ (\Phi_p(x'))' = -\lambda f(t, x)\}$$

that

$$\begin{aligned} R(x, \lambda)(t) &= \int_0^t \Phi_p^{-1}\left[\Phi_p(x(1)) + c - \int_0^s \lambda f(\tau, x(\tau))d\tau\right] ds - x(1)t \\ &= \int_0^t \Phi_p^{-1}\left[\Phi_p(x(1)) + c + \int_0^s (\Phi_p(x'(\tau))'d\tau\right] ds - x(1)t \qquad (3.7) \\ &= \int_0^t \Phi_p^{-1}\left[\Phi_p(x(1)) + c + \Phi_p(x'(s) - \Phi_p(x'(0))\right] ds - x(1)t. \end{aligned}$$

If we choose $c = -\Phi_p(x(1)) + \Phi_p(x'(0))$, then

$$R(x, \lambda)(1) = \int_0^1 \Phi_p^{-1}\left[\Phi_p(x'(s))\right] ds - x(1) = x(1) - x(1) = 0.$$

As proved above, c is unique. This implies that $c = -\Phi_p(x(1)) + \Phi_p(x'(0))$ and hence

$$\begin{aligned} R(x,\lambda)(t) &= \int_0^t \Phi_p^{-1}\left[\Phi_p(x(1)) - \Phi_p(x(1)) + \Phi_p(x'(0)) + \int_0^s (\Phi_p(x'(\tau)))' d\tau\right] ds - x(1)t \\ &= x(t) - x(1)t, \end{aligned} \tag{3.8}$$

which yields the second part of (2.6).

At the same time, we have

$$R(x,0)(t) = \int_0^t \Phi_p^{-1}\left[\Phi_p(x(1)) + c\right] ds - x(1)t$$

and (3.6) implies $c = 0$. So $R(x,0)(t) \equiv 0$ for each $x \in \overline{\Omega}$. Then the first part of (2.6) holds.

Besides, it is easy to verify that (2.7) also holds.

Therefore N_λ is $M-$compact in $\overline{\Omega}$.

Now we prove

Theorem 2. *Suppose $f \in C^0([0,1] \times R, R)$. Under the following two conditions*
(A1) There is a constant $M_0 > 0$ such that

$$xf(t,x) < 0, \qquad t \in [0,1], \quad x \in R \quad \textit{with} \quad |x| > M_0;$$

(A2) There is a constant $M_1 > 0$ with $M_1 > \frac{M_0}{\eta_1}$ such that

$$f_{M_1} < \Phi_p(M_1) - \Phi_p\left(\frac{M_0}{\eta_1}\right) \qquad \textit{where} \quad f_{M_1} = \max_{t\in[0,1],\ |x|\le M_1} |f(t,x)|;$$

BVP (1.1) – (1.2) has at least one solution x with $\|x\|_X < M_1$.

Proof. Consider

$$\begin{cases} (\Phi_p(x'))' + \lambda f(t,x) = 0, \quad 0 < t < 1, \\ x(0) = 0, \quad \Phi_p(x'(1)) = \sum\limits_{i=1}^{m-2} \alpha_i \Phi_p(x'(\eta_i)), \end{cases} \tag{3.9}$$

which is equivalent to

$$Mx = N_\lambda x, \qquad \lambda \in [0,1] \tag{3.10}$$

in X where M and N_λ are defined as above.

Take $\Omega = \{x \in X : \|x\|_X < M_1\}$. We show that

$$Mx \neq N_\lambda x, \quad \lambda \in (0,1), \quad x \in \partial\Omega. \tag{3.11}$$

If not, there are $\lambda_0 \in (0,1)$ and $u \in \partial\Omega$ such that

$$Mu = N_{\lambda_0} u,$$

then there is $t_0 \in [0,1]$ such that

$$|u(t_0)| = M_1, \quad |u(t)| \le M_1, \quad t \in [0,1].$$

Without loss of generality, suppose $u(t_0) = M_1$.

Clearly $t_0 \neq 0$ since $u(0) = 0$.

If $t_0 \in (0, 1)$, then

$$u'(t_0) = 0$$

and there is $\delta \in (0, t_0)$ such that

$$u'(t) \geq 0, \quad t \in (t_0 - \delta, t_0). \tag{3.12}$$

However, $(\Phi_p(u'(t_0)))' = -\lambda_0 f(t, u(t_0)) = -\lambda_0 f(t, M_1) > 0$ implies

$$\Phi_p(u'(t)) < \Phi_p(u'(t_0)) = 0, \qquad t \in (t_0 - \delta, t_0)$$

and then

$$u'(t) < 0, \quad t \in (t_0 - \delta, t_0),$$

a contradiction to (3.12).

If $t_0 = 1$, then

$$|u(1)| = M_1 \ \text{ and } \ |u(t)| \leq M_1, \quad t \in [0, 1). \tag{3.13}$$

By boundary condition $\Phi_p(u'(1)) = \sum_{i=1}^{m-2} \alpha_i \Phi_p(u'(\eta_i))$, we know there is a $\eta \in [\eta_1, 1)$ such that

$$u'(\eta) = u'(1)$$

which yields there is $\xi \in (\eta, 1) \subseteq (\eta_1, 1)$ such that

$$u''(\xi) = 0.$$

Since $p \geq 2$ from equation (3.3) we find

$$\Phi_p'(u'(\xi))u''(\xi) + \lambda_0 f(\xi, u(\xi)) = 0.$$

So we have

$$f(\xi, u(\xi)) = 0$$

which together with assumption (A1) yields

$$|u(\xi)| \leq M_0$$

and then there is $\theta \in (0, \xi)$ such that

$$|u'(\theta)| = \left| \frac{u(\xi) - u(0)}{\xi - 0} \right| = \frac{|u(\xi)|}{\xi} \leq \frac{M_0}{\eta_1}.$$

Thus by $Mu = N_{\lambda_0} u$ we have

$$\Phi_p(u'(t)) = \Phi_p(u'(\theta)) - \int_\theta^t \lambda_0 f(s, u(s)) ds, \quad t \in [0, 1],$$

i.e.,

$$\Phi_p(|u'(t)|) = |\Phi_p(u'(t))| \leq \Phi_p\left(\frac{M_0}{\eta_1}\right) + f_{M_1}, \quad t \in [0, 1].$$

That is

$$\max_{t\in[0,1]} |u'(t)| \leq \Phi_p^{-1}\left[\Phi_p\left(\frac{M_0}{\eta_1}\right) + f_{M_1}\right].$$

From $u(1) = u(0) + \int_0^1 u'(s)ds$, we find

$$\begin{aligned} M_1 = |u(1)| = \left|\int_0^1 u'(s)ds\right| &\leq \int_0^1 |u'(s)|\, ds \\ &\leq \max_{t\in[0,1]} |u'(t)| \\ &\leq \Phi_p^{-1}\left[\Phi_p\left(\frac{M_0}{\eta_1}\right) + f_{M_1}\right]. \end{aligned}$$

So

$$\Phi_p(M_1) \leq \Phi_p\left(\frac{M_0}{\eta_1}\right) + f_{M_1},$$

i.e.,

$$f_{M_1} \geq \Phi_p(M_1) - \Phi_p\left(\frac{M_0}{\eta_1}\right)$$

which contradicts assumption (A2).

Then (3.5) holds.

As for the degree, we have

$$\deg\{JQN, \Omega\cap X_1, 0\} = \deg\{QNJ, J^{-1}(\Omega\cap X_1), J^{-1}(0)\} = \deg\{QNJ, (-M_1, M_1), 0\}.$$

As $M_1 t > M_1\eta_1 > M_0$ for $t \in [\eta_1, 1]$, it follows that

$$QNy|_{y=M_1} = \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i\eta_i} \int_{\eta_i}^1 f(t, M_1 t)dt < 0,$$

$$QNy|_{y=-M_1} = \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i\eta_i} \int_{\eta_i}^1 f(t, -M_1 t)dt > 0,$$

hence we have

$$\deg\{JQN, \Omega\cap X_1, 0\} = \deg\{QNJ, (-R, R), 0\} \neq 0.$$

Applying Theorem 1 we reach the conclusion. □

4 Solvability of BVP (1.1) – (1.3)

A function u defined on $[0,1]$ is said to be a solution to BVP (1.1) – (1.3) if $u \in V = \{v \in C^1[0,1] : \ \Phi_p(v') \in C^1[0,1]\}$ satisfying BVP (1.1) – (1.3).

In this section, we let $X = \{x \in C[0,1] : x'(0) = 0, \ \ \Phi_p(x'(1)) = \sum_{i=1}^{m-2} \alpha_i \Phi_p(x'(\eta_i))\}$ and $Z = C[0,1]$ with sup norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively. Clearly, X, Z are Banach spaces.

Define $M : X \cap \mathrm{dom} M \to Z$ by

$$(Mx)(t) = (\Phi_p(x'(t)))'. \tag{4.1}$$

Then

$$\ker M = \{x = a : a \in \mathbb{R}\}, \qquad \mathrm{dom} M = V,$$

$$\begin{aligned}\mathrm{Im} M &= \{y \in Z, \ (\Phi_p(x'))' = y(t), \ \text{ for some } \ x(t) \in X \cap \mathrm{dom} M\} \\ &= \left\{ y \in Z, \ \Phi_p(x'(t)) = B + \int_0^t y(s)ds, \ \ x'(0) = 0, \ \ \Phi_p(x'(1)) = \sum_{i=1}^{m-2} \alpha_i \Phi_p(x'(\eta_i)) \right\} \\ &= \left\{ y \in Z, \ \ \sum_{i=1}^{m-2} \alpha_i \int_{\eta_i}^1 y(s)ds = 0 \right\}.\end{aligned}$$

Let

$$X_1 = \ker M, \qquad X_2 = \{x \in X : x(0) = 0\},$$

$$Z_1 = \mathbb{R}, \qquad Z_2 = \mathrm{Im} M.$$

Obviously $\dim X_1 = \dim Z_1 = 1$ and $X = X_1 \oplus X_2$. Define $P : X \to X_1$, $Q : Z \to Z_1$ by

$$Px = x(0), \qquad Qy = \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \int_{\eta_i}^1 y(s)ds. \tag{4.2}$$

Then from Section 3, we know that $Z = Z_1 \oplus Z_2$.

For all $\overline{\Omega} \subset X$, define $N_\lambda : \overline{\Omega} \to Z$ by

$$(N_\lambda x)(t) = -\lambda f(t, x(t)). \tag{4.3}$$

Clearly, $(I - Q)N_0$ is a zero operator, and

$$(I - Q)N_\lambda(\overline{\Omega}) \subset \mathrm{Im} M \subset (I - Q)Z,$$

i.e., (2.4) holds. Obviously (2.5) holds, too.

Let the homeomorphism $J : Z_1 \to X_1$ be defined by

$$J(a) = a, \qquad a \in \mathbb{R}, \ \ t \in [0,1]. \tag{4.4}$$

Define $R : \overline{\Omega} \times [0,1] \to X_2$ by

$$R(x,\lambda)(t) = \int_0^t \Phi_p^{-1}\left[\Phi_p(x(0)) + c + \int_s^1 \lambda f(\tau, x(\tau))d\tau\right] ds, \quad 0 \le t \le 1 \tag{4.5}$$

where c is a constant depending on (x, λ) and satisfying

$$\Phi_p^{-1}\left[\Phi_p(x(0)) + c + \int_0^1 \lambda f(\tau, x(\tau))d\tau\right] = 0, \tag{4.6}$$

that is $c = -\Phi_p(x(0)) - \int_0^1 \lambda f(\tau, x(\tau))d\tau$. It's easily to see that c is unique and continuous dependence on (x, λ).

For any bounded set $\Omega \neq \phi$, $\lambda \in [0,1]$, it is easy to see that $R : \overline{\Omega} \times [0,1] \to X_2 \subset X$ is relatively compact and continuous.

From (4.5) and (4.6), for

$$x \in \Sigma_\lambda = \{x \in \overline{\Omega} : \ Mx = N_\lambda x\} = \{x \in \overline{\Omega} : \ (\Phi_p(x'))' = -\lambda f(t,x)\},$$

we have

$$\begin{aligned} c &= -\Phi_p(x(0)) - \int_0^1 \lambda f(\tau, x(\tau))d\tau = -\Phi_p(x(0)) + \int_0^1 (\Phi_p(x'(\tau))'d\tau \\ &= -\Phi_p(x(0)) + \Phi_p(x'(1)), \ \lambda \neq 0 \end{aligned}$$

and

$$\begin{aligned} R(x,\lambda)(t) &= \int_0^t \Phi_p^{-1}\left[\Phi_p(x(0)) + c + \int_s^1 \lambda f(\tau, x(\tau))d\tau\right] ds \\ &= \int_0^t \Phi_p^{-1}\left[\Phi_p(x(0)) - \Phi_p(x(0)) + \Phi_p(x'(1)) - \int_s^1 (\Phi_p(x'(\tau))'d\tau\right] ds \\ &= \int_0^t \Phi_p^{-1}\left[\Phi_p(x(0)) - \Phi_p(x(0)) + \Phi_p(x'(1)) + \Phi_p(x'(s)) - \Phi_p(x'(1))\right] ds \\ &= x(t) - x(0), \end{aligned} \tag{4.7}$$

which yields the second part of (2.6).

At the same time, for $\lambda = 0$, we have $c = -\Phi_p(x(0)) - \int_s^1 \lambda f(\tau, x(\tau))d\tau = -\Phi_p(x(0))$. Then we have

$$R(x,0)(t) = \int_0^t \Phi_p^{-1}[\Phi_p(x(0)) - \Phi_p(x(0))]ds = 0.$$

So $R(x,0)(t) \equiv 0$ for each $x \in \overline{\Omega}$. Then the first part of (2.6) holds.

Besides, it is easy to verify that (2.7) also holds.

Therefore N_λ is $M-$compact in $\overline{\Omega}$.

Now we prove

Theorem 3. *Suppose $f \in C^0([0,1] \times R, R)$. Under the following two conditions*
(A1) There is a constant $M_0 > 0$ such that

$$xf(t,x) < 0, \qquad t \in [0,1], \quad x \in R \quad \textit{with} \quad |x| > M_0;$$

(A2) There is a constant $M_1 > M_0$ such that

$$f_{M_1} < \Phi_p(M_1 - M_0), \qquad \textit{where} \quad f_{M_1} = \max_{t\in[0,1],\ |x|\le M_1} |f(t,x)|;$$

BVP (1.1) – (1.3) has at least one solution x with $\|x\|_X < M_1$.

Proof. Consider

$$\begin{cases} (\Phi_p(x'))' + \lambda f(t,x) = 0, \qquad 0 < t < 1, \\ x'(0) = 0, \quad \Phi_p(x'(1)) = \sum\limits_{i=1}^{m-2} \alpha_i \Phi_p(x'(\eta_i)), \end{cases} \tag{4.8}$$

which is equivalent to

$$Mx = N_\lambda x, \qquad \lambda \in [0,1] \tag{4.9}$$

in X where M and N_λ are defined as above.

Take $\Omega = \{x \in X : \|x\|_X < M_1\}$. We show that

$$Mx \neq N_\lambda x, \quad \lambda \in (0,1), \quad x \in \partial\Omega. \tag{4.10}$$

If not, there exist $\lambda_0 \in (0,1)$ and $u \in \partial\Omega$ such that

$$Mu = N_{\lambda_0} u,$$

then there is a $t_0 \in [0,1]$ such that

$$|u(t_0)| = M_1, \quad |u(t)| \le M_1, \quad t \in [0,1].$$

Without loss of generality, suppose $u(t_0) = M_1$.

If $t_0 = 0$, from $x'(0) = 0$ we know that there exists a $\delta \in (0,1)$ such that

$$u'(t) \le 0, \quad t \in (0,\delta). \tag{4.11}$$

However, $(\Phi_p(u'(0)))' = -\lambda_0 f(t,u(0)) = -\lambda_0 f(t,M_1) > 0$ implies

$$\Phi_p(u'(t)) > \Phi_p(u'(0)) = 0, \qquad t \in (0,\delta)$$

and then

$$u'(t) > 0, \quad t \in (0,\delta),$$

a contradiction to (4.11).

If $t_0 = 1$, then

$$|u(1)| = M_1 \quad \text{and} \quad |u(t)| \le M_1, \quad t \in [0,1). \tag{4.12}$$

By boundary condition $\Phi_p(u'(1)) = \sum_{i=1}^{m-2} \alpha_i \Phi_p(u'(\eta_i))$, we know that there exists an $\eta \in [\eta_1, 1)$ and a $\xi \in (\eta, 1) \subseteq (\eta_1, 1)$ such that

$$|u(\xi)| \leq M_0.$$

By$Mu = N_{\lambda_0} u$ we have

$$\Phi_p(u'(t)) = \Phi_p(u'(0)) - \int_0^t \lambda_0 f(s, u(s)) ds, \quad t \in [0, 1],$$

i.e.,

$$\Phi_p(|u'(t)|) = |\Phi_p(u'(t))| \leq f_{M_1}, \quad t \in [0, 1].$$

That is

$$\max_{t \in [0,1]} |u'(t)| \leq \Phi_p^{-1}[f_{M_1}].$$

From $u(1) = u(\xi) + \int_\xi^1 u'(s) ds$, we find

$$M_1 = |u(1)| = \left| u(\xi) + \int_\xi^1 u'(s) ds \right| \leq |u(\xi)| + \int_\xi^1 |u'(s)| ds$$
$$\leq M_0 + \max_{t \in [0,1]} |u'(t)| \leq M_0 + \Phi_p^{-1}[f_{M_1}].$$

So

$$\Phi_p^{-1}(f_{M_1}) \geq M_1 - M_0,$$

i.e.,

$$f_{M_1} \geq \Phi_p(M_1 - M_0)$$

which contradicts assumption (A2).

Similar to the proof of Theorem 2, we can easily get that t is also not in (0,1). So (4.10) holds.

As for the degree, we have

$$\deg\{JQN, \Omega \cap X_1, 0\} = \deg\{QN, (-M_1, M_1), 0\}.$$

As $M_1 > M_0$, it follows that

$$QNy|_{y=M_1} = \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \int_{\eta_i}^1 f(t, M_1) dt < 0,$$

$$QNy|_{y=-M_1} = \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \int_{\eta_i}^1 f(t, -M_1) dt > 0,$$

we have

$$\deg\{JQN, \Omega \cap X_1, 0\} = \deg\{QN, (-M_1, M_1), 0\} \neq 0.$$

Applying Theorem 1 we reach the conclusion. □

5 Solvability of BVP (1.1) – (1.4)

A function u defined on $[0,1]$ is said to be a solution to BVP (1.1) – (1.4) if $u \in V = \{v \in C^1[0,1] : \Phi_p(v') \in C^1[0,1]\}$ satisfying BVP (1.1) – (1.4).

In this section, we let $X = \{x \in C[0,1] : x(0) = x(\xi), \quad \Phi_p(x'(1)) = \sum_{i=1}^{m-2} \alpha_i \Phi_p(x'(\eta_i))\}$ and $Z = C[0,1]$ with sup norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively. Clearly, X, Z are Banach spaces.

Define $M : X \cap \mathrm{dom} M \to Z$ by

$$(Mx)(t) = (\Phi_p(x'(t)))'. \tag{5.1}$$

Then

$$\ker M = \{x = a : a \in \mathbb{R}\}, \qquad \mathrm{dom} M = V,$$

$$\begin{aligned}\mathrm{Im} M &= \{y \in Z,\ (\Phi_p(x'))' = y(t), \ \text{ for some } \ x(t) \in X \cap \mathrm{dom} M\} \\ &= \left\{ y \in Z,\ \sum_{i=1}^{m-2} \alpha_i \int_{\eta_i}^1 y(s)ds = 0 \right\}.\end{aligned}$$

Let

$$X_1 = \ker M, \qquad X_2 = \{x \in X : x(0) = x(\xi) = 0\},$$
$$Z_1 = \mathbb{R}, \qquad Z_2 = \mathrm{Im} M.$$

Obviously $\dim X_1 = \dim Z_1 = 1$ and $X = X_1 \oplus X_2$. Define $P : X \to X_1$, $Q : Z \to Z_1$ by

$$Px = x(0), \qquad Qy = \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \eta_i} \int_{\eta_i}^1 y(s)ds. \tag{5.2}$$

Then from Section 3, we know that $Z = Z_1 \oplus Z_2$.

For any $\overline{\Omega} \subset X$ define $N_\lambda : \overline{\Omega} \to Z$ by

$$(N_\lambda x)(t) = -\lambda f(t, x(t)). \tag{5.3}$$

Clearly, $(I - Q)N_0$ is a zero operator, and

$$(I - Q)N_\lambda(\overline{\Omega}) \subset \mathrm{Im} M \subset (I - Q)Z,$$

i.e., (2.4) holds. Obviously (2.5) holds, too.

Let the homeomorphism $J : Z_1 \to X_1$ be defined by

$$J(a) = a, \qquad a \in \mathbb{R}, \quad t \in [0,1]. \tag{5.4}$$

Define $R : \overline{\Omega} \times [0,1] \to X_2$ by

$$R(x,\lambda)(t) = \int_0^t \Phi_p^{-1}\left[\Phi_p(x(\xi)) + c - \int_0^s \lambda f(\tau, x(\tau))d\tau\right] ds, \quad 0 \le t \le 1 \tag{5.5}$$

where c is a constant depending on (x, λ) and satisfying

$$\int_0^\xi \Phi_p^{-1}\left[\Phi_p(x(\xi)) + c - \int_0^s \lambda f(\tau, x(\tau))d\tau\right] ds = 0. \tag{5.6}$$

We now show that for given $x \in \overline{\Omega},\ \lambda \in [0,1]$, (5.6) has a unique solution $c = c(x, \lambda)$. Let

$$F(c) = \int_0^\xi \Phi_p^{-1}\left[\Phi_p(x(\xi)) + c - \int_0^s \lambda f(\tau, x(\tau))d\tau\right] ds$$

and

$$c_1 = \min_{0\le t\le \xi} \int_0^t \lambda f(\tau, x(\tau))d\tau - \Phi_p(x(\xi)), \qquad c_2 = \max_{0\le t\le 1} \int_0^t \lambda f(\tau, x(\tau))d\tau - \Phi_p(x(\xi)).$$

Clearly $F(c)$ is continuous and increasing with respect to c on $[c_1, c_2]$ and $F(c_1) \le 0 \le F(c_2)$. Therefore there is a unique $c \in [c_1, c_2]$ satisfying (5.6).

We can easily prove that c is continuous dependence on (x, λ) by uniqueness of c. And for any bounded set $\Omega \ne \phi$, $\lambda \in [0,1]$, it is easy to see that $R : \overline{\Omega} \times [0,1] \to X_2 \subset X$ is relatively compact and continuous.

From (5.5) and (5.6), for

$$x \in \Sigma_\lambda = \{x \in \overline{\Omega} :\ Mx = N_\lambda x\} = \{x \in \overline{\Omega} :\ (\Phi_p(x'))' = -\lambda f(t, x)\},$$

we have

$$\begin{aligned} R(x,\lambda)(t) &= \int_0^t \Phi_p^{-1}\left[\Phi_p(x(\xi)) + c - \int_0^s \lambda f(\tau, x(\tau))d\tau\right] ds \\ &= \int_0^t \Phi_p^{-1}\left[\Phi_p(x(\xi)) + c + \int_0^s (\Phi_p(x'(\tau))'d\tau\right] ds \\ &= \int_0^t \Phi_p^{-1}[\Phi_p(x(\xi)) + c + \Phi_p(x'(s)) - \Phi_p(x'(0))]ds, \end{aligned} \tag{5.7}$$

If we choose $c = -\Phi_p(x(\xi)) + \Phi_p(x'(0))$, then

$$R(x,\lambda)(\xi) = \int_0^\xi \Phi_p^{-1}\left[\Phi_p(x'(s))\right] ds = x(\xi) - x(0) = 0.$$

As proved above, c is unique, this implies that $c = -\Phi_p(x(\xi)) + \Phi_p(x'(0))$ and hence

$$\begin{aligned} R(x,\lambda)(t) = &\int_0^t \Phi_p^{-1}\left[\Phi_p(x(\xi)) - \Phi_p(x(\xi)) + \Phi_p(x'(0)) + \int_0^s (\Phi_p(x'(\tau)))'d\tau\right] ds \\ &= x(t) - x(0), \end{aligned} \tag{5.8}$$

which yields the second part of (2.6).

At the same time, we have

$$R(x,0)(t) = \int_0^t \Phi_p^{-1}\left[\Phi_p(x(\xi)) + c\right] ds$$

and (5.6) implies $c = -\Phi_p(x(\xi))$. So $R(x, 0)(t) \equiv 0$ for each $x \in \overline{\Omega}$. Then the first part of (2.6) holds.

Besides, it is easy to verify that (2.7) also holds.

Therefore N_λ is $M-$compact in $\overline{\Omega}$.

Now we prove

Theorem 4. *Suppose $f \in C^0([0,1] \times R, R)$. Under the following two conditions*
(A1) There is a constant $M_0 > 0$ such that

$$xf(t,x) < 0, \quad t \in [0,1], \quad x \in R \quad \textit{with} \quad |x| > M_0;$$

(A2) There is a constant $M_1 > M_0$ such that

$$f_{M_1} < \Phi_p(M_1 - M_0), \quad \textit{where} \quad f_{M_1} = \max_{t\in[0,1],\ |x|\le M_1} |f(t,x)|;$$

BVP (1.1)-(1.4) has at least one solution x with $\|x\|_X < M_1$.

Proof. Consider

$$\begin{cases} (\Phi_p(x'))' + \lambda f(t,x) = 0, \quad 0 < t < 1, \\ x(0) = x(\xi), \quad \Phi_p(x'(1)) = \sum\limits_{i=1}^{m-2} \alpha_i \Phi_p(x'(\eta_i)), \end{cases} \tag{5.9}$$

which is equivalent to

$$Mx = N_\lambda x, \qquad \lambda \in [0,1] \tag{5.10}$$

in X where M and N_λ are defined as above.

Take $\Omega = \{x \in X : \|x\|_X < M_1\}$. We show that

$$Mx \neq N_\lambda x, \quad \lambda \in (0,1), \quad x \in \partial\Omega. \tag{5.11}$$

If not, there are $\lambda_0 \in (0,1)$ and $u \in \partial\Omega$ such that

$$Mu = N_{\lambda_0} u,$$

then there exists $t_0 \in [0,1]$ such that

$$|u(t_0)| = M_1, \quad |u(t)| \le M_1, \quad t \in [0,1].$$

Without loss of generality, suppose $u(t_0) = M_1$.

First, it is easy to prove that t_0 is not in $(0,1)$ by using the same method in the proof of Theorem 2. And from the boundary condition (1.4), we know that if $t_0 = 0$, then we also can choose $t_0 = \xi \in (0,1)$. So t_0 not in $[0,1)$.

If $t_0 = 1$, then

$$|u(1)| = M_1 \ \text{ and } \ |u(t)| \le M_1, \quad t \in [0,1). \tag{5.12}$$

By boundary condition $x(0) = x(\xi)$, $\Phi_p(u'(1)) = \sum\limits_{i=1}^{m-2} \alpha_i \Phi_p(u'(\eta_i))$, we know that there exist $\alpha \in (0,\xi)$, $\eta \in [\eta_1, 1)$ and $\zeta \in (\eta, 1) \subseteq (\eta_1, 1)$ such that

$$u'(\alpha) = 0 \text{ and } |u(\zeta)| \le M_0.$$

By $Mu = N_{\lambda_0}u$, we have

$$\Phi_p(u'(t)) = \Phi_p(u'(\alpha)) - \int_\alpha^t \lambda_0 f(s, u(s))ds, \quad t \in [0,1],$$

i.e.,

$$\Phi_p(|u'(t)|) = |\Phi_p(u'(t))| \leq f_{M_1}, \quad t \in [0,1].$$

That is

$$\max_{t\in[0,1]} |u'(t)| \leq \Phi_p^{-1}[f_{M_1}].$$

From $u(1) = u(\zeta) + \int_\zeta^1 u'(s)ds$, we find

$$\begin{aligned} M_1 = |u(1)| = \left| u(\zeta) + \int_\zeta^1 u'(s)ds \right| &\leq |u(\zeta)| + \int_\zeta^1 |u'(s)|ds \\ &\leq M_0 + \max_{t\in[0,1]} |u'(t)| \leq M_0 + \Phi_p^{-1}[f_{M_1}]. \end{aligned}$$

So

$$\Phi_p^{-1}(f_{M_1}) \geq M_1 - M_0,$$

i.e.,

$$f_{M_1} \geq \Phi_p(M_1 - M_0)$$

which contradicts assumption (A2).

Then (5.10) holds.

Similar to the proof of Theorem 3, we also have

$$\deg\{JQN, \Omega \cap X_1, 0\} = \deg\{QN, (-M_1, M_1), 0\} \neq 0.$$

Applying Theorem 1 we reach the conclusion. □

6 Application

For example, let us consider the following BVP

$$\begin{cases} (\Phi_5(x'))' - x^3 - t^2 = 0, \quad 0 < t < 1, \\ x(0) = 0, \quad x'(1) = \frac{1}{3}x'(\frac{3}{4}) + \frac{2}{3}x'(\frac{4}{5}). \end{cases} \tag{3.14}$$

Corresponding to BVP (1.1), we have $f(t,x) = -x^3 - t^2$, $\eta_1 = \frac{3}{4}$, $\eta_2 = \frac{4}{5}$. So M_0, M_1 can be chosen as $M_0 = \frac{3}{2}$, $M_1 = 3 > 2 = \frac{M_0}{\eta_1}$ such that

(A1) $$xf(t,x) < 0, \quad t \in [0,1], \quad |x| > M_0.$$

On the other hand, from $f_{M_1} = 28$, $\Phi_5(M_1) = 81$, $\Phi_5(\frac{M_0}{\eta_1}) = \Phi_5(2) = 16$, we know

$$f_{M_1} = 28 < 81 - 16 = \Phi_5(M_1) - \Phi_5\left(\frac{M_0}{\eta_1}\right)$$

which implies (A2) holds. By applying Theorem 2, we see BVP (3.14) has at least one solution.

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