EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 6, Number 1 (2015), 76 – 84

BOCHKAREV INEQUALITY FOR THE FOURIER TRANSFORM OF FUNCTIONS IN THE LORENTZ SPACES $L_{2,r}(\mathbb{R})$

G.K. Mussabayeva, N.T. Tleukhanova

Communicated by N.A. Bokayev

Key words: Lorentz spaces, Fourier transform.

AMS Mathematics Subject Classification: 42B10.

Abstract. In this article we prove an analogue of Hardy -Littlewood- Stein inequality for the Fourier transform in Lorentz space $L_{2,r}(\mathbb{R})$.

1 Introduction

Let $f(x)$ be a scalar-valued μ measurable function which is finite almost everywhere, we introduce the distribution function $m(\sigma, f)$ defined by

$$m(\sigma, f) = \mu(\{x : |f(x)| > \sigma\})$$

and f^* its non-increasing rearrangement of the function f.

$$f^*(t) = \inf\{\sigma : m(\sigma, f) \leqslant t\}. \tag{1.1}$$

In [1], the Lorentz space $L_{p,q}$ is defined that $f \in L_{p,q}$, $1 \leqslant p \leqslant \infty$, if and only if

$$\|f\|_{L_{p,q}} = \left(\int\limits_{\mathbb{R}} \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty$$

when $1 \leqslant q < \infty$,

$$\|f\|_{L_{p,\infty}} = \sup_t t^{\frac{1}{p}} f^*(t)$$

when $q = \infty$.

Well-known classical theorem Hausdorff-Young [11], whereby if $f(x)$ is a 1- periodic function and satisfying

$$\int\limits_0^1 |f(x)|^p dx < \infty, \quad 1 < p \leqslant 2,$$

then following relation holds

$$\left(\sum_{n=-\infty}^{+\infty} |c_n|^{p'} \right)^{\frac{1}{p'}} \leqslant \left(\int\limits_0^1 |f(x)|^p dx \right)^{\frac{1}{p}}, \tag{1.2}$$

where

$$c_n = \int_0^1 f(x)e^{-2\pi inx}dx, \qquad n \in \mathbb{Z}$$

are the Fourier coefficients of $f(x)$ and $p' = \frac{p}{p-1}$.
An analogue of theorem Hausdorff-Young for the Fourier transform was proved by Titchmarsh [9],[10] and the inequality of Titchmarsh for Fourier integrals

$$\left(\int_{-\infty}^{\infty} \left|\widehat{f}(\xi)\right|^{p'} d\xi\right)^{\frac{1}{p'}} \leqslant \left(\int_{-\infty}^{\infty} |f(x)|^p\, dx\right)^{\frac{1}{p}}, \tag{1.3}$$

where

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx$$

is Fourier transform of $f(x)$.

The Hardy-Littlewood inequalities for the Lorentz spaces $L_{p,q}(\mathbb{R})$ was proved by Stein [3]-[8]:

Let $1 < p < 2$, $1 \leqslant q \leqslant \infty$ and $p' = \frac{p}{p-1}$, then

$$\left(\sum_{k=1}^{\infty} \left(k^{\frac{1}{p'}} c_k^*\right)^q \frac{1}{k}\right)^{\frac{1}{q}} \leqslant C\|f\|_{L_{p,q}[0,1]}, \tag{1.4}$$

where c_k^* is a non-increasing rearrangement of the Fourier coefficients.

$$\|\widehat{f}\|_{L_{p',q}(\mathbb{R})} \leqslant C\|f\|_{L_{p,q}(\mathbb{R})}, \tag{1.5}$$

Unlike inequalities (1.2) and (1.3), relations (1.4) and (1.5) don't cover the case of $p = 2$.

S.V.Bochkarev in [2] for the Fourier transform on the torus has shown that the Hardy -Littlewood- Stein inequality is not the case.

Theorem A. *Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal system of complex-valued functions on $[0,1]$, such that*

$$\|\varphi_n\|_\infty \leqslant M, \ n = 1, 2, ...$$

let $f \in L_{2,r}[0,1]$, $2 < r \leqslant \infty$. Then the inequality

$$\sup_{n\in N} \frac{1}{|n|^{\frac{1}{2}}(\log(n+1))^{\frac{1}{2}-\frac{1}{r}}} \sum_{m=1}^{n} a_m^* \leqslant C\|f\|_{L_{2,r}[0,1]} \tag{1.6}$$

holds, where a_n are the Fourier coefficients of the system $\{\varphi_n\}_{n=1}^{\infty}$.

2 Main result

The aim of this paper is to obtain an analogue of Bochkarev theorem for the Fourier transform. The main result of this paper is the following.

Theorem 2.1. *Let* $\Re_N = \{A = \bigcup_{i=1}^{N} A_i$, *where* A_i *are segments in* $\mathbb{R}\}$. *Then for any function* $f \in L_{2,r}(\mathbb{R})$ *and* $2 < r < \infty$ *we have the inequality*

$$\sup_{N \geqslant 8} \sup_{A \subset \Re_N} \frac{1}{|A|^{\frac{1}{2}} \log_2(1+N)^{\frac{1}{2}-\frac{1}{r}}} \left| \int_A \hat{f}(\xi) d\xi \right| \leqslant 23 \|f\|_{L_{2,r}}. \tag{2.1}$$

To prove Theorem 2.1, we establish the following lemmas.

Lemma 2.1. *Let* $\frac{4}{3} < q < 2 < r < 4$ *and* $f \in L_{q,r}(\mathbb{R})$. *Then for any measurable set* A *of finite measure of* $\Re_N$, *the inequality*

$$\frac{1}{|A|^{\frac{1}{q}}} \left| \int_A \hat{f}(\xi) d\xi \right| \leqslant 2 \left(\frac{q}{2-q} \right)^{\frac{1}{q'}\left(\frac{r-2}{r}\right)} \|f\|_{L_{q,r}(\mathbb{R})} \tag{2.2}$$

holds.

Proof.

$$\left| \int_A \hat{f}(\xi) d\xi \right| \leqslant \left| \int_A \int_{-\infty}^{\infty} f(t) e^{-i\xi t} dt d\xi \right| \leqslant |A| \|f\|_{L_1}$$

and from the Plancherel's theorem, we have

$$\left| \int_A \hat{f}(\xi) d\xi \right| \leqslant |A|^{\frac{1}{2}} \|f\|_{L_2}.$$

Let $\tau \in (0, +\infty)$. Define

$$f_1(x) = \begin{cases} f(x), & \text{if } |f(x)| \leqslant f^*(\tau), \\ 0, & \text{otherwise,} \end{cases}$$

$$f_0(x) = f(x) - f_1(x).$$

Then, for any $0 < \tau < \infty$, we have

$$\left| \int_A \hat{f}(\xi) d\xi \right| \leqslant \left| \int_A \hat{f}_0(\xi) d\xi \right| + \left| \int_A \hat{f}_1(\xi) d\xi \right|$$

$$\leqslant |A| \int_0^{\tau} f^*(s) ds + |A|^{\frac{1}{2}} \left(\int_{\tau}^{\infty} (f^*(s))^2 ds \right)^{\frac{1}{2}}.$$

Using Holder's inequality, we obtain

$$\left|\int_A \hat{f}(\xi)d\xi\right| \leqslant \left(|A|\left(\int_0^\tau \left(t^{\frac{1}{q}}f^*(t)\right)^r \frac{dt}{t}\right)^{\frac{1}{r}} \left(\int_0^\tau t^{\frac{r'}{q'}}\frac{dt}{t}\right)^{\frac{1}{r'}}\right.$$

$$\left.+|A|^{\frac{1}{2}}\left(\int_\tau^\infty \left(t^{\frac{1}{q}}f^*(t)\right)^r \frac{dt}{t}\right)^{\frac{1}{r}} \left(\int_\tau^\infty \left(t^{1-\frac{2}{q}}\right)^{\frac{r}{r-2}}\frac{dt}{t}\right)^{\frac{r-2}{2r}}\right)$$

$$\leqslant \|f\|_{L_{q,r}}\left(|A|\tau^{1-\frac{1}{q}}\left(1-\frac{1}{q}\right)^{\frac{1}{r}-1} + |A|^{\frac{1}{2}}\tau^{\frac{1}{2}-\frac{1}{q}}\left(\frac{r(2-q)}{q(r-2)}\right)^{\frac{1}{r}-\frac{1}{2}}\right).$$

Now let $\tau = \left(\frac{q(r-2)}{r(2-q)}\right)^{\frac{r-2}{r}}\left(\frac{q}{q-1}\right)^{\frac{2(1-r)}{r}}/|A|$ and using the $\frac{4}{3} < q < 2 < r < 4$, we can estimate

$$\left|\int_A \hat{f}(\xi)d\xi\right| \leqslant 2|A|^{\frac{1}{q}}\left(\frac{q}{2-q}\right)^{\frac{1}{q'}\left(\frac{r-2}{r}\right)}\|f\|_{L_{q,r}}.$$

The method of the proof in this paper is based on a study of the constants in inequalities of the form (2.2), namely their dependence on the relevant parameters. □

Lemma 2.2. *Let $2 < p < 4$, $p' = \frac{p}{p-1}$, $2 < r < \infty$, $f \in L_{p,r}$ and $A = \bigcup_{i=1}^N A_i$, where A_i are segments in $\mathbb{R}$. Then we have*

$$\frac{1}{|A|^{\frac{1}{p}}N^{\frac{1}{p'}-\frac{1}{p}}}\left|\int_A \hat{f}(\xi)d\xi\right| \leqslant C\|f\|_{L_{p,r}},$$

where

$$C = 4\sqrt{2}\left(\frac{r(p-2)}{p(r-2)}\right)^{\frac{1}{p}\left(\frac{2-r}{r}\right)}.$$

Proof. Let $f \in C_0^\infty$.

$$\left|\int_A \hat{f}(\xi)d\xi\right| = \left|\int_A\int_{-\infty}^\infty f(t)e^{-i\xi t}dtd\xi\right| \leqslant \sum_{i=1}^N\int_{A_i}\int_{-\infty}^\infty |f(t)e^{-i\xi t}dtd\xi|$$

$$\leqslant \sum_{i=1}^N\int_{-\infty}^\infty |f(t)|\left|\int_{A_i} e^{-i\xi t}d\xi\right|dt = \sum_{i=1}^N\int_{-\infty}^\infty |f(t)|\left|\int_{a_i}^{b_i} e^{-i\xi t}d\xi\right|dt = \sum_{i=1}^N\int_{-\infty}^\infty |f(t)|\left|\frac{2\sin\frac{(b_i-a_i)}{2}t}{t}\right|dt$$

$$\leqslant \sum_{i=1}^N\int_{-\infty}^\infty |f(t)|\min\left((b_i-a_i),\frac{2}{t}\right)dt \leqslant 2\sum_{i=1}^N\int_{-\infty}^\infty |f(t)|\min\left(|A_i|,\frac{1}{t}\right)dt. \qquad (2.3)$$

From the Plancherel's theorem we get

$$\left|\int\limits_A \hat{f}(\xi)d\xi\right| \leqslant |A|^{\frac{1}{2}}\|f\|_{L_2}. \tag{2.4}$$

Consider the function

$$f_1(x) = \begin{cases} f(x), & |f(x)| \leqslant f^*(\tau),\ x \in \mathbb{R}\setminus\left[-\frac{\tau}{2},\frac{\tau}{2}\right] \\ 0, & \text{otherwise}, \end{cases}$$

$$f_0(x) = f(x) - f_1(x).$$

Then using the (2.3) and (2.4) we get

$$\left|\int\limits_A \hat{f}(\xi)d\xi\right| \leqslant \left|\int\limits_A \hat{f}_0(\xi)d\xi\right| + \left|\int\limits_A \hat{f}_1(\xi)d\xi\right|$$

$$\leqslant |A|^{\frac{1}{2}}\|f_0\|_{L_2} + 2\sum_{i=1}^{N}\int\limits_{-\infty}^{\infty}|f_1(x)|\min\left(|A_i|,\frac{1}{|x|}\right)dx$$

$$= B(I_1 + I_2)$$

We have

$$I_1 = \left(\int\limits_{-\infty}^{\infty}|f_0(x)|^2dx\right)^{\frac{1}{2}} = \left(\int\limits_{-\frac{\tau}{2}}^{\frac{\tau}{2}}|f(x)|^2dx + \int\limits_{\frac{\tau}{2}\leqslant|x|<\infty}|f_0(x)|^2dx\right)^{\frac{1}{2}} \leqslant 2\left(\int\limits_0^{\tau}|f^*(t)|^2dt\right)^{\frac{1}{2}}$$

$$I_2 = 2\sum_{i=1}^{N}\int\limits_{-\infty}^{\infty}|f_1(x)|\min\left(|A_i|,\frac{1}{|x|}\right) = 2\sum_{i=1}^{N}\int\limits_{\frac{\tau}{2}\leqslant|x|<\infty}|f_1(x)|\min\left(|A_i|,\frac{1}{|x|}\right)$$

$$\leqslant 2N\int\limits_0^{\tau} f_1^*(t)\frac{1}{t+\tau}dt =$$

$$= 2N\int\limits_0^{\infty} f^*(t+\tau)\frac{dt}{t+\tau} \leqslant 2N\int\limits_{\tau}^{\infty} f^*(t)\frac{dt}{t}.$$

Therefore,

$$\left|\int\limits_A \hat{f}(\xi)d\xi\right| \leqslant 2|A|^{\frac{1}{2}}\left(\int\limits_0^{\tau}(f^*(t))^2dt\right)^{\frac{1}{2}} + 2N\int\limits_{\tau}^{\infty} f^*(t)\frac{dt}{t}.$$

Applying Hólder's inequality, we obtain

$$\left|\int\limits_A \hat{f}(\xi)d\xi\right| \leqslant 2|A|^{\frac{1}{2}} \left(\int\limits_0^\tau (f^*(t)t^{\frac{1}{p}})^r \frac{dt}{t}\right)^{\frac{1}{r}} \left(\int\limits_0^\tau \left(t^{1-\frac{2}{p}}\right)^{\frac{r}{r-2}} \frac{dt}{t}\right)^{\frac{r-2}{2r}}$$

$$+2N \left(\int\limits_\tau^\infty \left(f^*(t)t^{\frac{1}{p}}\right)^r \frac{dt}{t}\right)^{\frac{1}{r}} \left(\int\limits_\tau^\infty t^{-\frac{r'}{p}} \frac{dt}{t}\right)^{\frac{1}{r'}}$$

$$\leqslant \|f\|_{L_{p,r}} \left(2|A|^{\frac{1}{2}} \left(\frac{r(p-2)}{p(r-2)}\right)^{\frac{2-r}{2r}} \tau^{\frac{1}{2}-\frac{1}{p}} + 2Np^{\frac{r-1}{r}}\tau^{-\frac{1}{p}}\right).$$

Now let $\tau = \frac{N^2}{|A|\left(\frac{p-2}{p}\right)^{\frac{2-r}{r}} p^{2\left(\frac{1-r}{r}\right)}}$ and using that $2 < p < 4$ we have the following estimate:

$$\left|\int\limits_A \hat{f}(\xi)d\xi\right| \leqslant 4\sqrt{2}|A|^{1/p} \left(\frac{p-2}{p}\right)^{\frac{1}{p}\left(\frac{2}{r}-1\right)} N^{\frac{1}{p'}-\frac{1}{p}} \|f\|_{L_{p,r}}.$$

□

Lemma 2.3. *Let* $A = \bigcup\limits_{i=1}^N A_i$*, where* A_i *are segments. Then for any* $2 < p < 4$ *and* $2 < r < \infty$ *the inequality*

$$\frac{1}{|A|^{\frac{1}{2}}} \left|\int\limits_A \hat{f}(\xi)d\xi\right| \leqslant C\|f\|_{L_{2,r}} \tag{2.5}$$

holds, where

$$C = 8\left(\frac{p}{p-2}\right)^{\frac{1}{p}\left(1-\frac{2}{r}\right)} \left(\frac{p+2}{p}\right)^{\frac{1}{4}} N^{\frac{1}{2}-\frac{1}{p}}.$$

Proof. Let $\tau > 0$,

$$f_1(x) = \begin{cases} f(x), & |f(x)| \leqslant f^*(\tau) \\ 0, & \text{otherwise,} \end{cases}$$

$$f_0(x) = f(x) - f_1(x).$$

Then, using Lemmas 2.1, 2.2 and $2 < r < \infty$, we obtain

$$\left|\int\limits_A \hat{f}(\xi)d\xi\right| \leqslant \left|\int\limits_A \hat{f}_0(\xi)d\xi\right| + \left|\int\limits_A \hat{f}_1(\xi)d\xi\right| \leqslant 2\|f_0\|_{L_{p',r}} \left(\frac{p'}{2-p'}\right)^{\frac{1}{p}\left(\frac{r-2}{r}\right)} |A|^{\frac{1}{p'}}$$

$$+4\sqrt{2}\|f_1\|_{L_{p,r}} \left(\frac{p-2}{p}\right)^{\frac{1}{p}\left(\frac{2-r}{r}\right)} |A|^{\frac{1}{p}} N^{\frac{1}{p'}-\frac{1}{p}}$$

$$\leqslant 2\left(\int\limits_0^{\tau}\left(t^{\frac{1}{p'}}f^*(t)\right)^r\frac{dt}{t}\right)^{\frac{1}{r}}|A|^{\frac{1}{p'}}\left(\frac{p'}{2-p'}\right)^{\frac{1}{p}\left(\frac{r-2}{r}\right)}$$

$$+4\sqrt{2}\left(\int\limits_0^{\infty}\left(t^{\frac{1}{p}}f^*(t+\tau)\right)^r\frac{dt}{t}\right)^{\frac{1}{r}}|A|^{\frac{1}{p}}\left(\frac{p-2}{p}\right)^{\frac{1}{p}\left(\frac{2-r}{r}\right)}N^{\frac{1}{p'}-\frac{1}{p}}.$$

Further, we estimate each integral separately

$$\left(\int\limits_0^{\tau}\left(t^{\frac{1}{p'}}f^*(t)\right)^r\frac{dt}{t}\right)^{\frac{1}{r}}$$

$$=\left(\int\limits_0^{\tau}\left(t^{1-\frac{1}{p}}f^*(t)\right)^r\frac{dt}{t}\right)^{\frac{1}{r}}\leqslant\left(\int\limits_0^{\tau}\left(t^{\frac{1}{2}}f^*(t)\right)^r\frac{dt}{t}\right)^{\frac{1}{r}}\sup_{0\leqslant t\leqslant\tau}t^{\frac{1}{2}-\frac{1}{p}}\leqslant\tau^{\left(\frac{1}{2}-\frac{1}{p}\right)}\|f\|_{L_{2,r}}.$$

$$\left(\int\limits_0^{\infty}\left(t^{\frac{1}{p}}f^*(t+\tau)\right)^r\frac{dt}{t}\right)^{\frac{1}{r}}$$

$$\leqslant\left(\int\limits_0^{\tau}\left(t^{\frac{1}{p}}(t+\tau)^{-\frac{1}{2}}\right)^r\frac{dt}{t}\left(\sup_{0<s<\infty}s^{\frac{1}{2}}f^*(s)\right)^r+\int\limits_{\tau}^{\infty}(t^{\frac{1}{p}}f^*(t+\tau))^r\frac{dt}{t}\right)^{\frac{1}{r}}$$

$$\leqslant\left(\int\limits_0^{\tau}\left(t^{\frac{1}{p}}\tau^{-\frac{1}{2}}\right)^r\frac{dt}{t}\sup_{0<s<\infty}\frac{r}{2}\int\limits_0^{s}\xi^{\frac{r}{2}-1}f^*(\xi)^r d\xi+\int\limits_{\tau}^{\infty}\left(t^{\frac{1}{p}}f^*(t)\right)^r\frac{dt}{t}\right)^{\frac{1}{r}}$$

$$\leqslant\left(\tau^{\left(\frac{1}{p}-\frac{1}{2}\right)^r}\frac{p}{2}\|f\|^r_{L_{2,r}}+\tau^{\left(\frac{1}{p}-\frac{1}{2}\right)^r}\int\limits_{\tau}^{\infty}\left(t^{\frac{1}{2}}f^*(t)\right)^r\frac{dt}{t}\right)^{\frac{1}{r}}\leqslant\tau^{\left(\frac{1}{p}-\frac{1}{2}\right)}\left(\frac{p+2}{2}\right)^{\frac{1}{r}}\|f\|_{L_{2,r}}.$$

Therefore given the $2<r<\infty$, we have

$$\left|\int\limits_A\hat{f}(\xi)d\xi\right|\leqslant 2|A|^{1-\frac{1}{p}}\left(\frac{p}{p-2}\right)^{\frac{1}{p}\left(1-\frac{2}{r}\right)}\tau^{\left(\frac{1}{2}-\frac{1}{p}\right)}\|f\|_{L_{2,r}}$$

$$+4\sqrt{2}|A|^{\frac{1}{p}}\left(\frac{p-2}{p}\right)^{\frac{1}{p}\left(\frac{2-r}{r}\right)}\left(\frac{p+2}{2}\right)^{\frac{1}{2}}N^{\frac{1}{p'}-\frac{1}{p}}\tau^{\left(\frac{1}{p}-\frac{1}{2}\right)}\|f\|_{L_{2,r}}.$$

Taking $\tau^{\left(\frac{1}{2}-\frac{1}{p}\right)}=\left(\left(\frac{N}{|A|}\right)^{\frac{1}{p'}-\frac{1}{p}}\left(\frac{p+2}{2}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$, we have

$$\left|\int\limits_A\hat{f}(\xi)d\xi\right|\leqslant 8|A|^{\frac{1}{2}}N^{\frac{1}{2}-\frac{1}{p}}\left(\frac{p}{p-2}\right)^{\frac{1}{p}\left(1-\frac{2}{r}\right)}\left(\frac{p+2}{2}\right)^{\frac{1}{4}}\|f\|_{L_{2,r}}.$$

□

The proof of Theorem 2.1. Let $A \subset \Re_N$ and $|A| = N \leqslant 8$.
In the right part of the constant (2.5) we put $p = 3$. Then

$$8\left(\frac{p}{p-2}\right)^{\frac{1}{p}\left(1-\frac{2}{r}\right)}\left(\frac{p+2}{2}\right)^{\frac{1}{4}} N^{\frac{1}{2}-\frac{1}{p}} \leqslant 20.$$

Therefore, from Lemma 2.3, we get

$$\frac{1}{N^{\frac{1}{2}}\log_2(1+N)^{\frac{1}{2}-\frac{1}{r}}}\left|\int_A \hat{f}(\xi)d\xi\right| \leqslant \frac{1}{N^{\frac{1}{2}}}\left|\int_A \hat{f}(\xi)d\xi\right| \leqslant 20\|f\|_{L_{2,r}}.$$

Now let $A \subset \Re_N$ so that $N > 8$. Put $p = \frac{2\log_2 N}{\log_2 N - 2}$, i.e. $\frac{p-2}{2p} = \frac{1}{\log_2 N}$, then $p \leqslant 6$.

Note also that $N^{\frac{1}{2}-\frac{1}{p}} = N^{\frac{1}{\log_2 N}} \leqslant 2$.

Using Lemma 2.3, we obtain

$$\frac{1}{N^{\frac{1}{2}}}\left|\int_A \hat{f}(\xi)d\xi\right| \leqslant 23\log_2(1+N)^{\left(\frac{1}{2}-\frac{1}{\log_2 N}\right)\left(1-\frac{2}{r}\right)}\|f\|_{L_{2,r}}$$

$$\leqslant 23\log_2(1+N)^{\frac{1}{2}-\frac{1}{r}}\|f\|_{L_{2,r}}.$$

Taking the top exact bound over all N of $\mathbb{N}$ we obtain

$$\sup_{N\geqslant 8}\sup_{A\subset\Re_N}\frac{1}{N^{\frac{1}{2}}\log_2(1+N)^{\frac{1}{2}-\frac{1}{r}}}\left|\int_A \hat{f}(\xi)d\xi\right| \leqslant 23\|f\|_{L_{2,r}}$$

Theorem is proved. □

Acknowledgments

This work was supported by the grant of the Ministry of Education and Science of the Republic of Kazakhstan (project 3311/GF4)

References

[1] J. Bergh, J. Lófstróm, *Interpolation Spaces*, New York. 1976, 210p.

[2] S.V. Bochkarev, *Hausdorff-Young-Riesz theorem in Lorentz spaces and multiplicative inequalities*, Proc. Steclov Inst. Math. 219 (1997), 96-107.

[3] A.N. Kopezhanova, E.D. Nursultanov, L.-E. Persson,*On inequalities for the Fourier transform of functions from Lorentz spaces*, Mathematical notes. 90 (2011), no. 5, 785-788.

[4] E.D. Nursultanov, *Net spaces and the Fourier transform*, (Russian) Dokl. Akad. Nauk 361 (1998), no 5, 597–599.

[5] E.D Nursultanov, T.U. Aubakirov, *Hardy-Littlewood theorem for Fourier series*, Mathematical notes. 73 (2003), 340-347.

[6] E.D. Nursultanov, *Interpolation properties of some anisotropic spaces and Hardy–Littlewood type inequalities,* East J. Approx. 4 (1998), no. 2, 277-290.

[7] E.D. Nursultanov, *Application of interpolational methods to the study of properties of functions of several variables*, Mathematical Notes. 75 (2004), no.3, 341-351.

[8] M. Stein Elias, *Interpolation of linear operators*, Trans. Amer. Math. Soc. 83 (1956), 482-492.

[9] E.C. Titchmarsh, *A contribution to the theory of Fourier transforms*, Proc. London Math. Soc. 23 (1924), no. 2, 279-289.

[10] E.C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Clarendon Press, Oxford 1937; Russian transl. Gostekhizdat, Moscow-Leningrad 1948.

[11] A. Zygmund, *Trigonometrical series*, 2nd ed. Vols. I, II, Cambridge Univ. Press, New York 1959; Russian transl. Mir, Moscow 1965.

Guliya Mussabayeva
L.N. Gumilyov Eurasian National University
2 Satpayev st,
010010 Astana, Kazakhstan
E-mail: musabaevaguliya@mail.ru

Nazerke Tleukhanova
L.N. Gumilyov Eurasian National University
2 Satpayev st,
010010 Astana, Kazakhstan
E-mail: tleukhanova@rambler.ru

Received: 03.02.2015