EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 6, Number 1 (2015), 65 – 75

HOMOTOPY ANALYSIS METHOD AND HOMOTOPY PADÉ APPROXIMANTS FOR SOLVING THE FORNBERG-WHITHAM EQUATION

H.R. Marasi, A. Pourmostafa Aqdam

Communicated by E.D. Nursultanov

Key words: homotopy analysis method, homotopy Padé technique, Fornberg-Whitham equation.

AMS Mathematics Subject Classification: 35A15, 35A25.

Abstract. In this paper, the homotopy analysis method (HAM) is sharpened to solve the Fornberg-Whitham equation. Homotopy-Padé technique, and the use of proper initial gauss and auxiliary linear operator are employed to accelerate the convergence of approximations. Results demonstrate the power of the HAM equipped with these techniques in increasing the convergence rate and enlarging the region of convergence.

1 Introduction

It is obvious that the exact solutions of the nonlinear partial differential equations can help us to know the described process. So an important issue of the nonlinear partial differential equations is to find their new exact solutions. Various methods for obtaining exact solutions to nonlinear partial differential equations have been proposed, such as: truncated expansion method [13, 14, 25, 26], the simplest equation method [15], an automated tanh-function method [24], the polygons methods [12] and the Clarkson-Kruskal direct method [4]. Traveling wave solution is an important type of solution for the partial differential equations and many nonlinear partial differential equations have been found to have a variety of traveling wave solutions. The Fornberg-Whitham equation, given as

$$u_t - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx},$$

has a type of traveling wave solution called a Kink-like wave solution and anti Kink-like solutions. For most differential equations, no exact solutions is known and, in some cases, it is not even clear whether a unique solution exist. On the other hand often the obtained exact solution is not proper to use. So, approximation methods have been developed. Degaspersi and Procesi [6], studied the following family of third order dispersive partial differential equation

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = (c_1 u^2 + c_2 u_x^2 + c_3 u_{xx})_x, \tag{1.1}$$

where $\alpha, \gamma, c_i, \quad i = 0, 1, 2, 3$ are real constants. They found that there are only three equations that satisfy the asymptotic integrablity condition within the family, the

Korteweg-de Vries equation, the Camassa-Holm equation and the Degasperis-Procesi equation. For $c_1 = \frac{-3c_3}{2\alpha^2}, c_2 = \frac{c_3}{2}$ Eq. (1.1) becomes Fornberg-Whitham equation. In 1967, in order to discuss wave-breaking's qualitative behavior, Fornberg and Whitham gave the Fornberg-Whitham equation. In comparison with the Camassa-Holm equation and Degasperis equation, the Fornberg-Whitham equation is not integrable, so research on the solution of Fornberg-Whitham equation is very difficult, but it is important for studying wave-breaking, analysis property and this equation is qualitative behavior of wave-breaking. In the last two decades with the rapid development of nonlinear science there has appeared ever increasing interest of scientists, physicists and engineers in the analytical techniques for nonlinear problems. Analytic techniques are based on either perturbation techniques or traditional non-perturbation methods. Perturbation method is one of the well-known methods for solving nonlinear problems analytically. It is based on the existence of small/large parameters, the so-called perturbation quantities [22]. However, many nonlinear problems do not contain such kind of perturbation quantities. In general, the perturbation method is valid only for weakly nonlinear problems. To overcome the restrictions of perturbation techniques, some non-perturbation techniques are proposed, such as the Lyapunov's artificial small parameter method [18], the δ-expansion method [10], the Adomian's decomposition method [2], the homotopy perturbation method [7] and the variational iteration method (VIM) [8, 19]. These schemes generate an infinite series solution and do not have the problem of rounding errors. The approximate analytical methods, unlike the implicit finite difference methods, do not require the numerical solution of system of equation. Using these non-perturbation methods, one can indeed obtain approximations even if there are no small/large physical parameters. However, the convergence of solution series is not guaranteed. Liao [17] took the lead to apply the homotopy [9], a basic concept in topology, to gain analytic approximations of nonlinear differential equations. Different from perturbation techniques, the HAM is valid no matter if a nonlinear problem contains small/large physical parameters. More importantly, unlike all other analytic techniques, the HAM provides us with a simple way to adjust and control the convergence radius of solution series. Thus, one can always get accurate approximations by means of the HAM. In recent years, the HAM has been successfully applied for solving various nonlinear problems in many branches of sciences. It is worth to point out that the homotopy analysis method and the Adomian decomposition method for solving the Fornberg-Whitham equation were applied in [1].

There also exist some techniques to accelerate the convergence of a series solution, such as padé technique which is widely applied. The homotopy-padé technique was proposed by means of combing the padé technique with the homotopy analysis method. In this paper using homotopy Padé technique HAM is improved for solving Fornberg-Whitham equation. It is found that the convergence rate is increased and the region of convergence can be greatly enlarged.

2 Homotopy analysis method

In this section we describe the main points of HAM method. Consider the following equation

$$\mathcal{N}[u(x,t)] = 0 \tag{2.1}$$

where $\mathcal{N}$ is a nonlinear operator and x, t are spatial and temporal independent variables and $u(x,t)$ is unknown function. By means of generalizing the traditional homotopy method, the zero order deformation equation is constructed as

$$(1-q)\mathcal{L}[\phi(x,t;q) - u_0(x,t)] = q\hbar\mathcal{N}[\phi(x,t;q)], \tag{2.2}$$

where $\mathcal{L}$ is a linear operator, $q \in [0,1]$ is the embedding parameter, $\hbar$ is a nonzero auxiliary parameter and $u_0(x,t)$ is an initial guess of $u(x,t)$. When $q = 0$, the zero order deformation equation becomes

$$\mathcal{L}[\phi(x,t;0) - u_0(x,t)] = 0,$$

so

$$\phi(x,t;0) = u_0(x,t),$$

and when $q = 1$ we have

$$\hbar\mathcal{N}[\phi(x,t;1)] = 0,$$

so, since $\hbar \neq 0$, we get

$$\mathcal{N}[\phi(x,t;1)] = 0,$$

equivalently $\phi(x,t;1)$is the solution of (1.1). Thus as the embedding parameter q increase from 0 to 1, the solution $\phi(x,t;q)$ of (2.1) varies continuously from the initial approximation $u_0(x,t)$to the exact solution $u(x,t)$. Such kind of continues variation is called deformation in topology, and this is the reason why we call (2.1) the zero order deformation equation. Since $\phi(x,q)$ is also dependent upon the embedding parameter $q \in [0,1]$, we expand it into the Maclaurin series with respect to q:

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{2.3}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}|_{q=0}, \tag{2.4}$$

called the homotopy-Maclaurin series. Note that we have extremely large freedom to choose auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$. Assuming that, the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$ are so properly chosen that the above homotopy-Maclaurin series converges at $q = 1$, we have the so-called homotopy series solution

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{2.5}$$

which satisfies the original equation $\mathcal{N}[u(x) = 0]$. The governing equation of $u_m(x,t)$ can be derived from the zero order deformation equation(2.1). To this end, define the vector

$$\overrightarrow{u_n} = \{u_0(x,t), u_1(x,t), u_2(x,t), \ldots, u_n(x,t)\}.$$

Differentiating the zero order deformation equation m times with respect to q and then setting $q = 0$ and finally dividing by $m!$, we have the so-called m-th order deformation equation

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \Re_m(\overrightarrow{u_{m-1}}), \quad (2.6)$$

where

$$\Re_m(\overrightarrow{u_{m-1}(x,t)}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}|_{q=0}, \quad (2.7)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

The high-order deformation equation (2.6) is always linear with the known term on the right-hand side, therefore it is easy to solve, as long as we choose the auxiliary linear operator $\mathcal{L}$ properly.

3 Homotopy-Padé technique

A Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $u(x)$. The [L/M] Padé approximant to a function $u(x)$ is given by

$$[\frac{L}{M}] = \frac{P_L(x)}{Q_M(x)}, \quad (3.1)$$

where $P_L(x)$ is polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. The formal power series

$$u(x) = \sum_{i=1}^{\infty} a_i x^i,$$

$$u(x) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1}),$$

determine the coefficients of $P_L(x)$ and $Q_M(x)$ by comparing the coefficient of like powers.

Also, the [m,n] Padé for $u(x,t)$ based on top explanation is of the form

$$\frac{\sum_{k=0}^{m} F_{m,k}(x) t^k}{\sum_{k=0}^{n} G_{m,k}(x) t^k},$$

or

$$\frac{\sum_{k=0}^{m} F_{m,k}(t) x^k}{\sum_{k=0}^{n} G_{m,k}(t) x^k},$$

where $F_{m,k}(x)$ and $G_{m,k}(t)$ are functions to be determined. The homotopy-Padé technique was proposed by means of combining the Padé technique with the homotopy analysis method [16]. The corresponding [m, n] Padé approximant for a series (2.3) about the embedding parameter q, is expressed by

$$\frac{\sum_{k=0}^{m} A_{m,k}(x,t)q^k}{\sum_{k=0}^{n} B_{m,k}(x,t)q^k}, \tag{3.2}$$

where $A_{m,k}(x,t)$ and $B_{m,k}(x,t)$are determined by following approximations

$$u_0(x,t), u_1(x,t), ..., u_{m+n}(x,t). \tag{3.3}$$

By setting $q = 1$ the [m, n] homotopy-Padé approximant is obtained as

$$\frac{\sum_{k=0}^{m} A_{m,k}(x,t)}{\sum_{k=0}^{n} B_{m,k}(x,t)}. \tag{3.4}$$

It has been found that the [m,n] Homotopy-Padé approximation often converges faster than the corresponding traditional [m,n] Padé approximation, besides, homotopy-Padé approximants often converges faster than solutions calculated by homotopy analysis method. In many cases the [m,m] Homotopy-Padé approximation is independent of the auxiliary parameter $\hbar$.

4 Solutions

Consider the Fornberg-Whitham equation

$$u_t - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx}, \tag{4.1}$$

with the initial condition

$$u(x,0) = exp(\frac{1}{2}x). \tag{4.2}$$

The exact solution of the problem is

$$u(x,t) = exp(\frac{1}{2}x - \frac{2}{3}t).$$

From (4.1) we define the nonlinear operator

$$N[\phi(x,t;q)] = \frac{\partial\phi(x,t;q)}{\partial t} - \frac{\partial^3\phi(x,t;q)}{\partial x^2\partial t} + \frac{\partial\phi(x,t;q)}{\partial x} - \phi(x,t;q)\frac{\partial^3\phi(x,t;q)}{\partial x^3}$$
$$+\phi(x,t;q)\frac{\partial\phi(x,t;q)}{\partial x} - 3\frac{\partial\phi(x,t;q)}{\partial x}\frac{\partial^2\phi(x,t;q)}{\partial x^2}, \tag{4.3}$$

and choose the linear operator

$$L[\phi(x,t;q) = \frac{\partial\phi(x,t;q)}{\partial t}, \tag{4.4}$$

with the property

$$L(c_1) = 0,$$

where c_1 is the integration constant. According to condition (4.2) we can choose the initial approximation $u_0(x,t) = e^{\frac{1}{2}x}$. By the analysis in section 3, we construct the zero order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = q\hbar N[\phi(x,t;q)]. \tag{4.5}$$

Obviously, when $q = 0$ and $q = 1$ we can write

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t).$$

Therefore, as embedding parameter q increases from 0 to 1, $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. We obtain the m-th-order deformation equation

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar\Re_m(\overrightarrow{u_{m-1}}), \tag{4.6}$$

subject to the initial condition

$$u_m(x,t) = 0,$$

where

$$\Re_m(\overrightarrow{u_{m-1}}) = \frac{\partial u_{m-1}(x,t)}{\partial t} - \frac{\partial^3 u_{m-1}(x,t)}{\partial x^2 \partial t} + \frac{\partial u_{m-1}(x,t)}{\partial x} \tag{4.7}$$
$$+\sum_{k=0}^{m-1}[-u_k(x,t)\frac{\partial^3 u_{m-1-k}(x,t)}{\partial x^3} + u_k(x,t)\frac{\partial u_{m-1-k}(x,t)}{\partial x} - 3\frac{\partial u_k(x,t)}{\partial x}\frac{\partial^2 u_{m-1-k}(x,t)}{\partial x^2}].$$

Applying the operator L^{-1} on both sides of (4.6) the solution of mth-order deformation equation for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[\Re_m(\overrightarrow{u_{m-1}})]. \tag{4.8}$$

We now successively obtain

$$\begin{aligned}
u_0(x,t) &= exp(\frac{1}{2}x),\\
u_1(x,t) &= exp(\frac{1}{2}x)[\frac{\hbar t}{2}],\\
u_2(x,t) &= exp(\frac{1}{2}x)[\frac{4\hbar + 3\hbar^2}{8}t + \frac{\hbar^2 t^2}{8}],\\
u_3(x,t) &= exp(\frac{1}{2}x)[t(\frac{48}{96}\hbar + \frac{72}{96}\hbar^2 + \frac{27}{96}\hbar^3) + t^2(\frac{24}{96}\hbar^2 + \frac{18}{96}\hbar^3) + t^3(\frac{1}{48}\hbar^3)],\\
&\vdots
\end{aligned} \tag{4.9}$$

We use nine terms in evaluating the approximate solution as

$$u_{app} = \sum_{i=0}^{8} u_i.$$

The $[4,4]$ homotopy-Padé approximant for the obtained approximate solution is

$$\frac{e^{\frac{x}{2}}(8505 - 2835t + 405t^2 - 30t^3 + t^4)}{8505 + 2835t + 405t^2 + 30t^3 + t^4}. \tag{4.10}$$

5 Conclusions

Fig.1 shows the $\hbar$-curves for $u_t(0, 0.5)$, $u_{tt}(0, 0.5)$ and $u_{ttt}(0, 0.5)$ obtained by the 8th-order HAM approximate solution. The horizontal line segment that denotes the valid region of the $\hbar$ and guarantees convergence extends approximately from $-1.7 < \hbar < -0.9$. Fig.2 shows the exact solution and the approximate solution obtained by the [4,4] homotopy-padé technique. As shown in Table.1, Table.2, Table.3, Table.4, Fig.3, Fig.4 and Fig.5 the approximation given by the [4,4] homotopy-padé technique converges much faster than those given by 8th-order HAM.

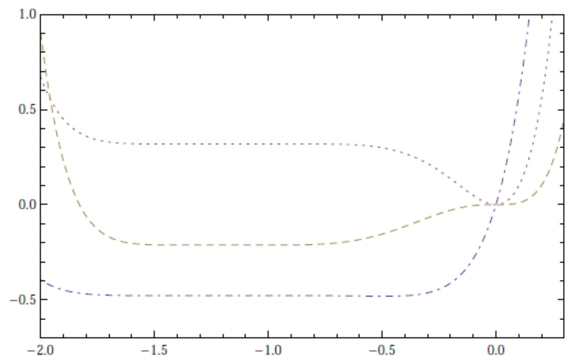

Fig. 1. The $\hbar-$ curves of $u_t(0, 0.5)$ (DotDashed) and $u_{tt}(0, 0.5)$ (Dotted) and $u_{ttt}(0, 0.5)$ (Dashed) given by $8th-$ order HAM approximate solution.

x_i/t_i	0.2	0.4	0.6	0.8	1
-4	1.14542×10^{-9}	1.02543×10^{-7}	5.03369×10^{-8}	1.05801×10^{-7}	2.30377×10^{-7}
-2	3.11538×10^{-9}	2.78742×10^{-7}	1.3683×10^{-7}	2.87597×10^{-7}	6.26231×10^{-7}
0	8.46358×10^{-9}	7.57698×10^{-7}	3.71942×10^{-7}	7.81769×10^{-7}	1.70227×10^{-6}
2	2.30064×10^{-8}	2.05964×10^{-6}	1.01104×10^{-6}	2.12507×10^{-6}	4.62725×10^{-6}
4	6025379×10^{-8}	5.59868×10^{-6}	2.7483×10^{-6}	5.77653×10^{-6}	1.25782×10^{-5}

Table 1: The absolute errors for differences between the exact solution and the approximate solution obtained by the HAM ($\hbar = -1$) at some points.

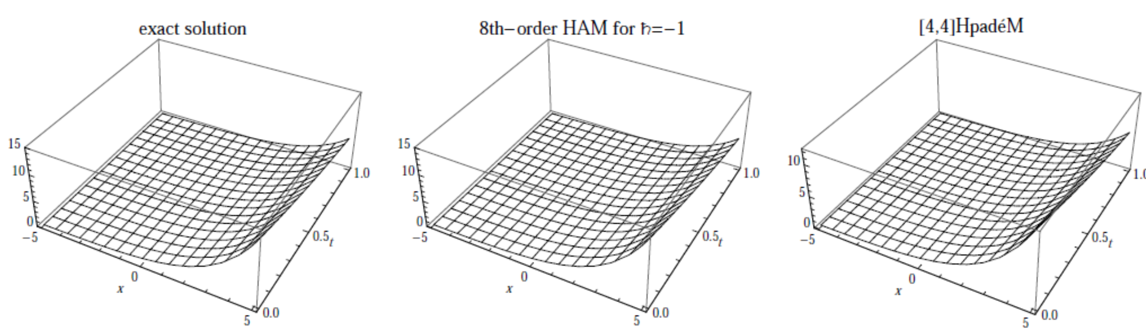

Fig. 2. The behavior of the solution: exact, 8th-order HAM and [4,4] HPT orderly left to right.

x_i/t_i	0.2	0.4	0.6	0.8	1
-4	1.32507×10^{-11}	539397×10^{-11}	1.19404×10^{-10}	1.06198×10^{-10}	4.40263×10^{-10}
-2	3.6019×10^{-11}	1.46624×10^{-10}	3.24572×10^{-10}	2.88676×10^{-10}	1.19676×10^{-9}
0	9.7909×10^{-11}	3.98566×10^{-10}	8.82273×10^{-10}	7.847×10^{-10}	3.25313×10^{-9}
2	2.66149×10^{-10}	1.08341×10^{-9}	2.39828×10^{-9}	2.13305×10^{-9}	8.84291×10^{-9}
4	7.23462×10^{-10}	2.94501×10^{-9}	6.51922×10^{-9}	5.79821×10^{-9}	2.40375×10^{-8}

Table 2: The absolute error for difference between the exact solution and 8th-order HAM solution by $\hbar = -1.2$ at some points.

x_i	u_{exact}	u_{HAM}	u_{HPT}	$u_{exact} - u_{HAM}$	$u_{exact} - u_{HPT}$
-4	0.0048279500	0.0049912100	0.0048412030	1.63262×10^{-4}	1.32530×10^{-5}
-2	0.0131237287	0.0135675000	0.0131597541	4.43791×10^{-4}	3.60253×10^{-5}
0	0.0356739933	0.0368803000	0.0357719204	1.20635×10^{-3}	9.79270×10^{-5}
2	0.0966971967	0.1002510000	0.0972381612	3.27920×10^{-3}	2.66193×10^{-4}
4	0.2635971381	0.2725110000	0.2673207265	8.91378×10^{-3}	7.23588×10^{-4}

Table 3: The absolute error for differences between the exact solution and 8th-order HAM approximate fot $\hbar = -1.2$ when $t = 5$. Also the absolute error for differences between the exact solution and [4,4] homotopy padé technique when $t = 5$.

x_i/t_i	0.2	0.4	0.6	0.8	1
-4	6.93889×10^{-17}	2.79082×10^{-14}	9.40595×10^{-13}	1.10033×10^{-11}	7.2083×10^{-11}
-2	1.66533×10^{-16}	7.58282×10^{-14}	2.55682×10^{-12}	2.991×10^{-11}	1.95942×10^{-10}
0	4.44089×10^{-16}	2.06168×10^{-13}	6.95011×10^{-12}	8.13037×10^{-11}	5.32625×10^{-10}
2	1.33227×10^{-15}	5.60441×10^{-13}	1.88924×10^{-11}	2.21007×10^{-10}	1.44783×10^{-9}
4	3.55271×10^{-15}	1.52323×10^{-12}	5.13554×10^{-11}	6.00759×10^{-10}	3.9356×10^{-9}

Table 4: The absolute error for difference between the exact solution and [4,4] HPT approximant at some points.

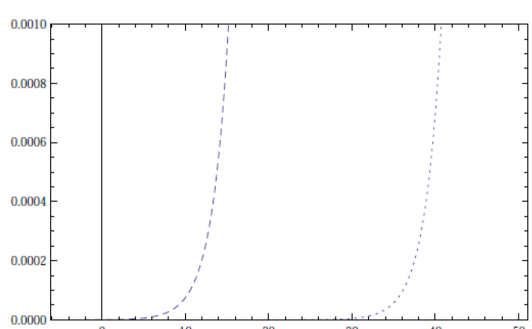

Fig. 3. Error between exact solution and 8th-order HAM(Dashed), also error between exact solution and [4,4] HPT(Dotted) in $t = 0.5$.

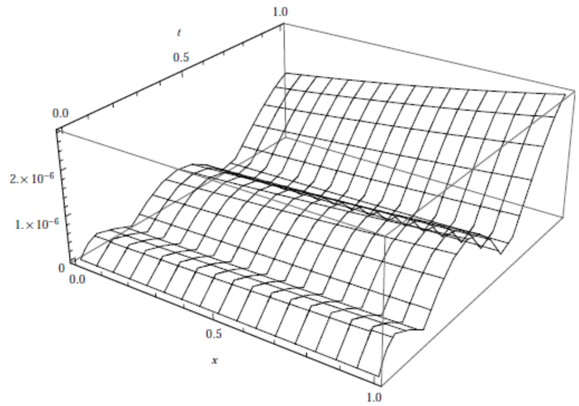

Fig. 4. The behavior of absolute error in the region $0 < x < 1, 0 < t < 1$ by 8th-order HAM.

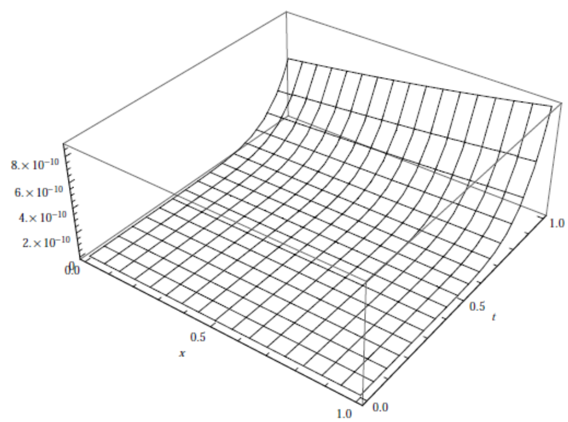

Fig. 5. The behavior of absolute error in the region $0 < x < 1, 0 < t < 1$. by [4,4] HPT.

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H.R. Marasi, A. Pourmostafa Aqdam
Department of Mathematics
University of Bonab
P.O.Box 5551761167, Bonab, Iran
E-mail: marasi@bonabu.ac.ir

Received: 21.09.2014