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OPTIMAL BANACH FUNCTION SPACE FOR A GIVEN CONE OF DECREASING FUNCTIONS IN A WEIGHTED L_p - SPACE

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Abstract. The problem is considered of constructing optimal (i.e. minimal) generalized Banach function space or optimal Banach function space, containing the given cone of nonnegative, decreasing functions in a weighted Lebesgue space.

1 Introduction

In the paper the problem is considered of constructing optimal (i.e. minimal) generalized Banach function space (briefly: GBFS) or optimal (i.e. minimal) Banach function space (briefly: BFS), containing a given cone of nonnegative, decreasing functions (i.e. $0 \le h \downarrow$) in the weighted space

$$L_{p,u}(0,T) = \left\{ f \in M : \|f\|_{L_{p,u}(0,T)} = \left(\int_0^T |f|^p u \, dt \right)^{\frac{1}{p}} < \infty \right\}.$$

Here M is the set of all measurable functions, $0 < p < \infty$, $T \in \mathbb{R}_+ = (0, \infty), u$ is a positive, measurable function:

$$K_0 = \{ h \in L_{p,u}(0,T) : \ 0 \le h \downarrow \} \tag{1.1}$$

equipped with the functional

$$\rho_{K_0}(h) = \|h\|_{L_{p,u}(0,T)} = \left(\int_0^T h^p u \, dt \right)^{\frac{1}{p}}. \tag{1.2}$$

Exact description is obtained of the optimal GBFS, containing cone (1.1).
The structure of this paper is the following. In the introduction we shortly describe the basic concepts and facts of BFS and GBFS theory necessary for the further. In Section 2 we consider basic notation and formulate the results of the paper. In Section 3 proofs of main results are contained.
Throughout the paper we will use the following concepts and facts of GBFS theory.

Let us define as $L_0(0,T)$ the set of measurable, finite almost everywhere (shortly: a. e.) functions. Set $L_0^+(0,T) = \{g \in L_0(0,T),\ g \geq 0\}$. For cone (1.1) we define embedding into BFS (GBFS) $X = X(0,T)$.
We recall the definition of a function norm (shortly: FN) introduced by C. Bennett and R. Sharpley [2].

Definition 1. A mapping $\rho : L_0^+ \to [0,\infty]$ is called a function norm (shortly: FN) if, for all $f, g, f_n (n \in \mathbb{N})$ from L_0^+, for all constants $\alpha \geq 0$ and for all μ-measurable subsets $E \subset A$ the following properties hold:
$(P1)\quad \rho(f) = 0 \Leftrightarrow f = 0 \quad \mu$ -a.e.; $\rho(\alpha f) = \alpha\rho(f); \rho(f+g) \leq \rho(f) + \rho(g)$;

$$(P2) \quad 0 \leq g \leq f \quad \mu\text{-}a.e. \Rightarrow \rho(g) \leq \rho(f)$$

(property of monotonicity);

$$(P3) \quad 0 \leq f_n \uparrow f \quad \mu\text{-}a.e. \Rightarrow \rho(f_n) \uparrow \rho(f)$$

(the Fatou property);

$$(P4) \quad \mu(E) < \infty \Rightarrow \int_E f d\mu \leq C_E \rho(f)$$

(local integrability) for some $C_E \in \mathbb{R}$, depending on E and ρ, but independent of f;

$$(P5) \quad \mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty.$$

Definition 2. Let ρ be a FN. The collection $X = X(\rho)$ of all functions f in L_0, for which $\rho(|f|) < \infty$ is called BFS, generated by FN ρ; for each f in X define

$$\|f\|_X = \rho(|f|).$$

Now we generalize these definitions in the following way.

Definition 3. A mapping $\rho : L_0^+ \to [0,\infty]$ is called a generalized function norm (shortly: GFN), if it satisfies the conditions (P1)-(P3) from the Definition 1.1, also the following conditions are fulfilled:

$$(P4)' \quad \mu(E) < \infty \Rightarrow \exists h_E \in L_0^+, h_E > 0 \quad \mu\text{-}a.e.\ on \quad E, \text{ so that}$$

$$\int_E f h_E d\mu \leq \rho(f).$$

Here, h_E depends on E and ρ, but is independent of $f \in L_0^+$.

$$(P5)' \quad \mu(E) < \infty \Rightarrow \exists f_E \in L_0^+, f_E > 0 \quad \mu\text{-}a.e.\ on \quad E : \rho(f_E) < \infty.$$

Definition 4. Let ρ be a GFN. The collection $X = X(\rho)$ of all functions f in L_0, for which $\rho(|f|) < \infty$, is called GBFS, generated by GFN ρ; for each f in X define

$$\|f\|_X = \rho(|f|).$$

Next, we formulate some results concerning general properties of BFS (GBFS) (see [1], [2] for proofs).

Theorem 1.1. *Let X be a BFS (GBFS).Then, X is complete.*

Definition 5. For a FN (GFN) ρ we determine an associate norm ρ' on L_0^+ by the formula: for $g \in L_0^+$

$$\rho'(g) = \sup\left\{ \int_A fg d\mu : f \in L^+, \rho(f) \le 1 \right\}.$$

Theorem 1.2. *Let ρ be a FN (GFN). Then the associate norm ρ' is a FN (GFN) itself, and the space $X' = X(\rho')$, generated by this norm, is a BFS (GBFS).*

Remark 1. Detailed proofs of the above theorems are presented respectively in [2,Ch.1-2] and [1,Ch.1].

Finally, we mention a well known lemma which is useful in consideration of discrete norms.

Lemma 1.1. *Let $0 < p \le \infty$, $\quad 1 \le q \le \infty$; $\quad \beta_m > 0$, $\quad \frac{\beta_{m+1}}{\beta_m} \ge B > 1, m \in \mathbb{Z} = 0, \pm 1, \pm 2, \pm 3, \ldots$ (or $m \in \mathbb{N} = 0, 1, 2, 3, \ldots$). Then for all $a_m \ge 0$ the following estimates hold:*

$$\left(\sum_m \left[\beta_m (\sum_{l \ge m} a_l^q)^{\frac{1}{q}} \right]^p \right)^{\frac{1}{p}} \le c(B,p) \left(\sum_m [\beta_m a_m]^p \right)^{\frac{1}{p}}; \tag{1.3}$$

$$\left(\sum_m \left[\beta_m^{-1} (\sum_{l \le m} a_l^q)^{\frac{1}{q}} \right]^p \right)^{\frac{1}{p}} \le c(B,p) \left(\sum_m [\beta_m^{-1} a_m]^p \right)^{\frac{1}{p}}. \tag{1.4}$$

Here, $c(B,p) \in \mathbb{R}_+$ (with natural modification at $p = \infty$ and/or $q = \infty$).
Notice that the inverse inequalities are right with the constant equal to unit, so that we have the two-sided estimates in (1.3) and (1.4).

2 Notation and results

Let cone (1.1) be given, and let $X = X(0,T)$ be a GBFS.

Definition 6. The embedding

$$K_0 \mapsto X \tag{2.1}$$

means that $K_0 \subset X$ and there exists constant $c_{K_0} \in \mathbb{R}$ such that

$$\|h\|_X \le c_{K_0}\, \rho_{K_0}(h), \qquad h \in K_0. \tag{2.2}$$

Definition 7. BFS (GBFS) $X_0 = X_0(0,T)$ is called optimal (minimal) for the embedding (2.1), if

1) $K_0 \mapsto X_0$,
2) if (2.1) holds for some GBFS $X = X(0,T)$ then it follows that $X_0 \subset X$.

In [4] the formula is obtain for norm, associated to the norm of optimal BFS (GBFS):

$$\|g\|_{X_0'} = \sup\left\{ \int\limits_0^T |g|h\,dt : \ h \in K_0, \rho_{K_0}(h) \le 1 \right\}. \tag{2.3}$$

Here, $g \in L_0(0,T)$. Recall that BFS (GBFS) $X'(0,T)$, associated to BFS (GBFS) $X(0,T)$, has the norm (see [2; Ch. 1,2])

$$\|g\|_{X'} = \sup\left\{ \int\limits_0^T |g|hdt : \ h \in X(0,T), \|h\|_X \le 1 \right\}. \tag{2.4}$$

So, if the cone K_0 coincided with BFS (GBFS) X and $\rho_{K_0}(h) = \|h\|_X$, then formula (2.3), according to principal of duality, would arrived at equality $X_0 = X_0'' = X$. In our case it is not true, and formula (2.3) allows to construct the space, associated to the optimal BFS (GBFS) $X_0(0,T)$, containing K_0.

The aim of this work is to construct optimal BFS (GBFS) $X_0(0,T)$ for cone K_0, relying on (2.3). The main result is following.

Theorem 2.1. *Let the cone K_0 (1.1), equipped with functional ρ_{K_0} (1.2), be given. Define $U(t) = \int_0^t u d\tau, 0 < U(t) < \infty, t \in (0,T)$. Then, optimal GBFS X_0, containing cone K_0 (1.1), has the norm*

$$\|f\|_{X_0(0,T)} = \left(\int_0^T \|f\|^p_{L_\infty(t,T)}\, u(t)dt \right)^{\frac{1}{p}}, \quad 1 \le p < \infty; \tag{2.5}$$

$$\|f\|_{X_0(0,T)} = \int_0^T \|f\|_{L_\infty(t,T)}\, \tilde{u}(t)dt, \quad 0 < p < 1, \quad \tilde{u}(t) = \frac{1}{p} U(t)^{\frac{1}{p}-1} u(t). \tag{2.6}$$

Remark 2. In the case $U(T-0) := U(T) < \infty$ X_0 is a BFS if the following condition is fulfilled: $u^{-\frac{p'}{p}} \in L_1^{loc}(0,T)$. In the case $U(T) = \infty$ X_0 is GBFS, but not BFS.

Remark 3. Note that nonweight case u is considered in [5].

3 Proofs of results formulated in Section 2

3.1. Proof of Theorem 2.1 in the case $1 \le p < \infty$.

Proof. Consider two cases.

1. $U(T) = \infty$. Let us construct optimal GBFS. We use the following partition: $U(t_m) = 2^m, m \in \mathbb{Z}$. As $0 < U(t) \uparrow$,
$U(+0) = 0, U(T) = \infty$, so $0 < t_m < t_{m+1} < \ldots < T, m \in \mathbb{R}$, $\lim\limits_{m\to-\infty} t_m = 0$, $\lim\limits_{m\to+\infty} t_m = T$.

At first, consider the case $p = 1$. For $g \in L_0^+(0,T)$, according to (2.3),

$$\|g\|_{X_0'(0,T)} = \sup\left\{ \int\limits_0^T gh\,dt : \ h \in L_{1,u}(0,T), \ 0 \le h \downarrow, \ \|h\|_{L_{1,u}(0,T)} \le 1 \right\}.$$

This value is obtained in [3]:

$$\|g\|_{X_0'(0,T)} = A_g := \sup \left\{ U(t)^{-1} \int_0^t g\, d\tau : \ t \in (0,T) \right\}. \tag{3.1}$$

Our aim is to show that the norm, associated with norm (3.1), is equivalent to the norm on right-hand side of (2.4) for $p = 1$. For this purpose, in view of $\mathbb{Z} = 0, \pm 1, \pm 2, \ldots$, denote

$$B_g = \sup_{l \in \mathbb{Z}} \ 2^{-l} \int_{t_{l-1}}^{t_l} g d\tau. \tag{3.2}$$

It is obvious that $B_g \le A_g$. Conversely,

$$A_g = \sup_{k \in \mathbb{Z}} \ \sup_{t \in [t_{k-1}, t_k)} \ U(t)^{-1} \int_0^t g\, d\tau \le \sup_{k \in \mathbb{Z}} \left[2^{-(k-1)} \sum_{l \le k} \int_{t_{l-1}}^{t_l} g\, d\tau \right].$$

But $\int_{t_{l-1}}^{t_l} g\, d\tau \le 2^l B_g$, $l \in \mathbb{Z}$. So,

$$A_g \le B_g \ \sup_{k \in \mathbb{Z}} \left[2^{-(k-1)} \sum_{l \le k} 2^l \right] = B_g \ \sup_{k \in \mathbb{Z}} \left[2^{-(k-1)} 2^{k+1} \right] = 4 B_g. \tag{3.3}$$

Next, for $f, g \in L_0^+(0,T)$ we have

$$\int_0^T f g\, d\tau = \sum_{k \in \mathbb{Z}} \int_{t_{k-1}}^{t_k} f g\, d\tau \le \sum_{k \in \mathbb{Z}} \|f\|_{L_\infty(t_{k-1}, t_k)} \int_{t_{k-1}}^{t_k} g\, d\tau \le$$

$$\le B_g \sum_{k \in \mathbb{Z}} 2^k \|f\|_{L_\infty(t_{k-1}, t_k)} \le A_g \sum_{k \in \mathbb{Z}} 2^k \|f\|_{L_\infty(t_{k-1}, t_k)}.$$

Thus, according to (3.1), for $f \in L_0^+(0,T)$

$$\|f\|_{X_0(0,T)} = \sup \left\{ \int_0^T f g\, d\tau : \ g \in L_0^+(0,T), \ A_g \le 1 \right\} \le \sum_{k \in \mathbb{Z}} 2^k \|f\|_{L_\infty(t_{k-1}, t_k)}. \tag{3.4}$$

At the same time, for every $\epsilon \in (0,1)$, $f \in L_0^+(0,T)$, $k \in \mathbb{Z}$, there exists g_k, such that

$$0 \le g_k \in L_1(t_{k-1}, t_k), \qquad \int_{t_{k-1}}^{t_k} g_k\, d\tau = 1; \tag{3.5}$$

$$\int_{t_{k-1}}^{t_k} f g_k\, d\tau \ge (1 - \epsilon) \|f\|_{L_\infty(t_{k-1}, t_k)}. \tag{3.6}$$

Now let us define the function $\widetilde{g} \in L_0^+(0,T)$ by the following formula

$$\widetilde{g} = 2^k g_k(\tau), \quad \tau \in [t_{k-1}, t_k) \quad k \in \mathbb{Z}. \tag{3.7}$$

Then, according to (3.2), (3.5),

$$B_{\widetilde{g}} = \sup_{k\in\mathbb{Z}} \, 2^{-k} \int\limits_{t_{k-1}}^{t_k} \widetilde{g}\, d\tau = 1.$$

So, taking into account (3.3), we obtain $A_{\widetilde{g}} \le 4$. Therefore,

$$\|f\|_{X_0(0,T)} = \frac{1}{4} \sup \left\{ \int\limits_0^T fg\, d\tau : \; g \in L_0^+(0,T), \; A_g \le 4 \right\} \ge$$

$$\frac{1}{4} \int\limits_0^T f\widetilde{g}\, d\tau = \frac{1}{4} \sum_{k\in\mathbb{Z}} 2^k \int\limits_{t_{k-1}}^{t_k} f g_k\, d\tau.$$

Taking into account inequality (3.6), we have

$$\|f\|_{X_0(0,T)} \ge \frac{1-\epsilon}{4} \sum_{k\in\mathbb{Z}} 2^k \|f\|_{L_\infty(t_{k-1},t_k)} \ge \frac{1}{4} \|f\|_{X_0(0,T)}$$

for every $\epsilon \in (0,1)$ and f, not depending on ϵ.
Together with (3.4) we obtain from here

$$\|f\|_{X_0(0,T)} \le \sum_{k\in\mathbb{Z}} 2^k \|f\|_{L_\infty(t_{k-1},t_k)} \le 4\|f\|_{X_0(0,T)}. \tag{3.8}$$

Let us estimate right-hand side value of (2.5) in the case $p=1$. We have

$$\int\limits_0^T \|f\|_{L_\infty(\tau,T)} u(\tau)\, d\tau = \sum_{k\in\mathbb{Z}} \int\limits_{t_{k-1}}^{t_k} \|f\|_{L_\infty(\tau,T)} u(\tau)\, d\tau \le \sum_{k\in\mathbb{Z}} \|f\|_{L_\infty(t_{k-1},T)} \int\limits_{t_{k-1}}^{t_k} u(\tau) d\tau =$$

$$= \sum_{k\in\mathbb{Z}} \|f\|_{L_\infty(t_{k-1},T)} 2^{k-1} \le \sum_{k\in\mathbb{Z}} 2^{k-1} \sum_{l\ge k} \|f\|_{L_\infty(t_{l-1},t_l)} =$$

$$= \sum_{l\in\mathbb{Z}} \|f\|_{L_\infty(t_{l-1},t_l)} \sum_{k\le l} 2^{k-1} = \sum_{l\in\mathbb{Z}} \|f\|_{L_\infty(t_{l-1},t_l)} 2^l. \tag{3.9}$$

At the same time,

$$\int\limits_0^T \|f\|_{L_\infty(\tau,T)} u(\tau)\, d\tau = \sum_{k\in\mathbb{Z}} \int\limits_{t_{k-1}}^{t_k} \|f\|_{L_\infty(\tau,T)} u(\tau)\, d\tau \ge \sum_{k\in\mathbb{Z}} \|f\|_{L_\infty(t_k,T)} \int\limits_{t_{k-1}}^{t_k} u(\tau)\, d\tau \ge$$

$$\geq \frac{1}{4}\sum_{k\in\mathbb{Z}}\|f\|_{L_\infty(t_k,t_{k+1})}2^{k+1}=\frac{1}{4}\sum_{l\in\mathbb{Z}}\|f\|_{L_\infty(t_{l-1},t_l)}2^{l}. \tag{3.10}$$

As a result, from (3.9) and (3.10) we obtain

$$\int_0^T\|f\|_{L_\infty(\tau,T)}u(\tau)\,d\tau\leq\sum_{l\in\mathbb{Z}}\|f\|_{L_\infty(t_{l-1},t_l)}2^{l}\leq 4\int_0^T\|f\|_{L_\infty(\tau,T)}u(\tau)\,d\tau.$$

It follows from here and from (3.8) that

$$\frac{1}{4}\|f\|_{X_0(0,T)}\leq\int_0^T\|f\|_{L_\infty(\tau,T)}u(\tau)\,d\tau\leq 4\|f\|_{X_0(0,T)}.$$

It proves equivalence (2.5) in the case $p=1$.

Now, we consider the case $1<p<\infty$. For $g\in L_0^+(0,T)$, according to (2.3), we have

$$\|g\|_{X_0'(0,T)}=\sup\left\{\int_0^T gh\,dt:\ 0\leq h\downarrow;\ h(t+0)=h(t),\ t\in(0,T);\ \|h\|_{L_{p,u}(0,T)}\leq 1\right\}.$$

The following two-sided estimate takes place for this value (see [8]): let $g\in L_0^+(0,T)$, then

$$\|g\|_{X_0'(0,T)}\cong\widetilde{\rho}_0(g), \tag{3.11}$$

here

$$\widetilde{\rho}_0(g)=\left(\int_0^T\left[\int_0^t g\,d\tau\right]^{p'}U^{-p'}(t)u(t)\,dt\right)^{\frac{1}{p'}}. \tag{3.12}$$

Let us show that for $g\in L_0^+(0,T)$

$$\widetilde{\rho}_0(g)\cong\rho_1(g):=\left(\sum_{k\in\mathbb{Z}}2^{-\frac{kp'}{p}}\left(\int_{t_{k-1}}^{t_k}g\,d\tau\right)^{p'}\right)^{\frac{1}{p'}}. \tag{3.13}$$

We have

$$\widetilde{\rho}_0(g)=\left(\sum_{k\in\mathbb{Z}}\int_{t_{k-1}}^{t_k}\left[\int_0^t g\,dt\right]^{p'}U^{-p'}(t)u(t)\,dt\right)^{\frac{1}{p'}}. \tag{3.14}$$

So,

$$\widetilde{\rho}_0(g)\leq\left(\sum_{k\in\mathbb{Z}}\left(\int_0^{t_k}g\,d\tau\right)^{p'}\int_{t_{k-1}}^{t_k}U^{-p'}(t)u(t)\,dt\right)^{\frac{1}{p'}}\cong\left(\sum_{k\in\mathbb{Z}}\left(\sum_{l\leq k}\int_{t_{l-1}}^{t_l}g\,d\tau\right)^{p'}2^{-\frac{kp'}{p}}\right)^{\frac{1}{p'}}.$$

Since $2^{-\frac{(k+1)p'}{p}} = 2^{-\frac{p'}{p}}2^{-\frac{kp'}{p}}$ and $0 < 2^{-\frac{p'}{p}} < 1$, then according to Lemma 1.1,

$$\left(\sum_{k\in\mathbb{Z}}\left(\sum_{l\le k}\int_{t_{l-1}}^{t_l} g\,d\tau\right)^{p'}2^{-\frac{kp'}{p}}\right)^{\frac{1}{p'}} \cong \left(\sum_{k\in\mathbb{Z}}\left(\int_{t_{k-1}}^{t_k} g\,d\tau\right)^{p'}2^{-\frac{kp'}{p}}\right)^{\frac{1}{p'}}. \tag{3.15}$$

We obtain from these estimates that:

$$\widetilde{\rho}_0(g) \le c\rho_1(g). \tag{3.16}$$

Let us obtain inverse estimate. According to (3.14), we have for $g \in L_0^+(0,T)$

$$\widetilde{\rho}_0(g) \ge \left(\sum_{k\in\mathbb{Z}}\left(\int_0^{t_{k-1}} g\,d\tau\right)^{p'}\int_{t_{k-1}}^{t_k} U^{-\frac{p'}{p}-1}u\,dt\right)^{\frac{1}{p'}} \cong \left(\sum_{k\in\mathbb{Z}}\left(\int_0^{t_{k-1}} g\,d\tau\right)^{p'}2^{-\frac{(k-1)p'}{p}}\right)^{\frac{1}{p'}} \ge$$

$$\ge \left(\sum_{k\in\mathbb{Z}}\left(\int_{t_{k-2}}^{t_{k-1}} g\,d\tau\right)^{p'}2^{-\frac{(k-1)p'}{p}}\right)^{\frac{1}{p'}} = \left(\sum_{l\in\mathbb{Z}}\left(\int_{t_{l-1}}^{t_l} g\,d\tau\right)^{p'}2^{-\frac{lp'}{p}}\right)^{\frac{1}{p'}}.$$

Estimate (3.13) follows from here and from (3.16).

Now we show that for all $f, g \in L_0^+(0,T)$

$$\int_0^T fg\,d\tau \le \rho_1(g)\left(\sum_{k\in\mathbb{Z}} 2^k\|f\|^p_{L_\infty(t_{k-1},t_k)}\right)^{\frac{1}{p}}. \tag{3.17}$$

Actually, using consequently Hölder's inequality for integrals and for sums, we obtain

$$\int_0^T fg\,d\tau = \sum_{k\in\mathbb{Z}}\int_{t_{k-1}}^{t_k} fg\,d\tau \le \sum_{k\in\mathbb{Z}}\|f\|_{L_\infty(t_{k-1},t_k)}\int_{t_{k-1}}^{t_k} g\,d\tau \le$$

$$\le \left(\sum_{k\in\mathbb{Z}} 2^k\|f\|^p_{L_\infty(t_{k-1},t_k)}\right)^{\frac{1}{p}}\left(\sum_{k\in\mathbb{Z}}\left(\int_{t_{k-1}}^{t_k} g\,d\tau\right)^{p'}2^{-\frac{kp'}{p}}\right)^{\frac{1}{p'}},$$

which arrives at (3.17). Denote

$$|||f||| := \left(\sum_{k\in\mathbb{Z}} 2^k\|f\|^p_{L_\infty(t_{k-1},t_k)}\right)^{\frac{1}{p}}, \tag{3.18}$$

and rewrite the inequality:

$$\int_0^T fg\,d\tau \le |||f|||\,\rho_1(g). \tag{3.19}$$

For $f \in L_0^+(0,T)$ we use the following equality for the norm in optimal GBFS:

$$\|f\|_{X_0(0,T)} = \sup \left\{ \int_0^T fg\,d\tau : \ g \in L_0^+(0,T), \ \|g\|_{X_0'(0,T)} \le 1 \right\}$$

and take into account equivalences (3.11) and (3.13). Then, for $f \in L_0^+(0,T)$ we get

$$\|f\|_{X_0(0,T)} \cong \sup \left\{ \int_0^T fg\,d\tau : \ g \in L_0^+(0,T); \ \rho_1(g) \le 1 \right\}. \tag{3.20}$$

It follows immediately from here and from (3.19) that there exists $c_3 \in \mathbb{R}_+$, such that

$$\|f\|_{X_0(0,T)} \le c_3 |||f|||, \qquad f \in L_0^+(0,T). \tag{3.21}$$

Our aim is to obtain inverse estimate. It follows from the accuracy of Hölder's inequality for sequences that there exists sequence $\{\alpha_k\}_{k\in\mathbb{Z}}$ with the properties:

$$\alpha_k \ge 0, \quad k \in \mathbb{Z}, \quad \|\{\alpha_k\}\|_{l_{p'}} = \left(\sum_{k\in\mathbb{Z}} \alpha_k^{p'} \right)^{\frac{1}{p'}} = 1;$$

$$\sum_{k\in\mathbb{Z}} 2^{\frac{k}{p}} \|f\|_{L_\infty(t_{k-1},t_k)}\, \alpha_k = \left(\sum_{k\in\mathbb{Z}} 2^k \|f\|^p_{L_\infty(t_{k-1},t_k)} \right)^{\frac{1}{p}}. \tag{3.22}$$

Next, it follows from the accuracy of Hölder's inequality for integrals that for every $\epsilon \in (0,1)$, $f \in L_0^+(0,T)$, $k \in \mathbb{Z}$ there exists $g_k \in L_0^+(t_{k-1},t_k)$, such that

$$\int_{t_{k-1}}^{t_k} g_k\,d\tau = 2^{\frac{k}{p}} \alpha_k; \qquad \int_{t_{k-1}}^{t_k} fg_k\,d\tau \ge (1-\epsilon) 2^{\frac{k}{p}} \alpha_k \|f\|_{L_\infty(t_{k-1},t_k)}. \tag{3.23}$$

Consider that

$$\widetilde{g}(\tau) = g_k(\tau), \qquad \tau \in [t_{k-1},t_k), \qquad k \in \mathbb{Z}.$$

Then, $\widetilde{g} \in L_0^+(0,T)$ and

$$\rho_1(\widetilde{g}) = \left(\sum_{k\in\mathbb{Z}} \left(\int_{t_{k-1}}^{t_k} g_k\,d\tau \right)^{p'} 2^{-\frac{kp'}{p}} \right)^{\frac{1}{p'}} = \left(\sum_{k\in\mathbb{Z}} \alpha_k^{p'} \right)^{\frac{1}{p'}} = 1.$$

It follows from here and from (3.20) that

$$\|f\|_{X_0(0,T)} \ge c_4 \int_0^T f\widetilde{g}\,d\tau = c_4 \sum_{k\in\mathbb{Z}} \int_{t_{k-1}}^{t_k} fg_k\,d\tau.$$

Substituting here (3.23) and (3.22), we obtain

$$\|f\|_{X_0(0,T)} \ge (1-\epsilon) c_4 \left(\sum_{k\in\mathbb{Z}} 2^k \|f\|^p_{L_\infty(t_{k-1},t_k)} \right)^{\frac{1}{p}} = (1-\epsilon) c_4 |||f|||$$

for every $\epsilon \in (0,1)$, f, not depending on ϵ. Thus,

$$\|f\|_{X_0(0,T)} \geq c_4 |||f|||.$$

Combined with (3.21), this estimate yields

$$\|f\|_{X_0(0,T)} \cong |||f|||. \tag{3.24}$$

Now we have to prove the equivalence

$$|||f||| \cong \left(\int_0^T \|f\|^p_{L_\infty(t,T)} u(t)\, dt \right)^{\frac{1}{p}}, \tag{3.25}$$

and relation (2.5) will be obtained for $1 < p < \infty$. We have

$$\int_0^T \|f\|^p_{L_\infty(t,T)} u\, dt = \sum_{k\in\mathbb{Z}} \int_{t_{k-1}}^{t_k} \|f\|^p_{L_\infty(t,T)} u(t)\, dt \leq \sum_{k\in\mathbb{Z}} \|f\|^p_{L_\infty(t_{k-1},T)} \int_{t_{k-1}}^{t_k} u(t)\, dt =$$

$$= \sum_{k\in\mathbb{Z}} \|f\|^p_{L_\infty(t_{k-1},T)} 2^{k-1} \leq \sum_{k\in\mathbb{Z}} \left(\sum_{l\geq k} \|f\|_{L_\infty(t_{l-1},t_l)} \right)^p 2^{k-1}. \tag{3.26}$$

Now we use Lemma 1.1, according to which

$$\sum_{k\in\mathbb{Z}} \left(\sum_{l\geq k} \|f\|_{L_\infty(t_{l-1},t_l)} \right)^p 2^{k-1} \cong \sum_{k\in\mathbb{Z}} \|f\|^p_{L_\infty(t_{k-1},t_k)} 2^{k-1}. \tag{3.27}$$

Using estimate (3.26), we obtain

$$\left(\int_0^T \|f\|^p_{L_\infty(t,T)} u(t)\, dt \right)^{\frac{1}{p}} \leq c_5 \left(\sum_{k\in\mathbb{Z}} \|f\|^p_{L_\infty(t_{k-1},t_k)} 2^{k-1} \right)^{\frac{1}{p}} = c_5 2^{-\frac{1}{p}} |||f|||. \tag{3.28}$$

Thus, we obtain upper estimate of the right side of (3.25) by the left side. At the same time,

$$\left(\int_0^T \|f\|^p_{L_\infty(t,T)}\, dt \right)^{\frac{1}{p}} = \left(\sum_{k\in\mathbb{Z}} \int_{t_{k-1}}^{t_k} \|f\|^p_{L_\infty(t,T)}\, dt \right)^{\frac{1}{p}} \geq \left(\sum_{k\in\mathbb{Z}} \|f\|^p_{L_\infty(t_k,T)} \int_{t_{k-1}}^{t_k} dt \right)^{\frac{1}{p}} \geq$$

$$\geq \left(\sum_{k\in\mathbb{Z}} \|f\|_{L_\infty(t_k,t_{k+1})} 2^{k-1} \right)^{\frac{1}{p}} = \left(\sum_{k\in\mathbb{Z}} \|f\|_{L_\infty(t_{l-1},t_l)} 2^{l-2} \right)^{\frac{1}{p}} = \frac{1}{4} |||f|||. \tag{3.29}$$

So, (3.25) follows from here and from (3.28).
It remains to prove that $X_0'(0,T)$ is GBFS. For this purpose we will check fulfilment of properties $(P1)-(P3)$ and $(P4)', (P5)'$. In $X_0'(0,T)$ an equivalent norm has the form:

$$\rho_1(g) := \left(\sum_{k\in\mathbb{Z}} 2^{-\frac{kp'}{p}} \left(\int_{t_{k-1}}^{t_k} g\, d\tau \right)^{p'} \right)^{\frac{1}{p'}}, g \in L_0^+(0,T).$$

It is obvious that this norm satisfies properties $(P1)-(P3)$.

Next, let $\delta_k > 0, k \in \mathbb{Z}$ such that $\sum\limits_{k\in\mathbb{Z}} 2^{-\frac{kp'}{p}} \delta_k^{p'} (t_k - t_{k-1})^{p'} < \infty$ and $g_0(t) := \delta_k, t \in (t_{k-1}, t_k], k \in \mathbb{Z}$. Then, $g_0(t) > 0$ on $(0,T)$, and $\rho_1(g_0) < \infty$. It means that property $(P5)'$ is fulfilled. Now let $\rho_1(g) < \infty$. We will find $\sigma_k > 0, k \in \mathbb{Z}$ such that $\sum\limits_{k\in\mathbb{Z}} \sigma_k \left(\int\limits_{t_{k-1}}^{t_k} g\, d\tau \right) \le \rho_1(g)$, and we consider $h(t) = \sum\limits_{k\in\mathbb{Z}} \sigma_k \chi_{(t_{k-1},t_k]}(t) > 0, t \in (0,T)$, moreover, $\int\limits_0^T hg\, d\tau = \sum\limits_{k\in\mathbb{Z}} \sigma_k \int\limits_{t_{k-1}}^{t_k} g\, d\tau \le \rho_1(g)$. It means that property $(P4)'$ is fulfilled.

2. $U(T) < \infty$. Now we pass to constructing of optimal GBFS. We will consider the following partition: $U(t_m) = 2^m, m \in \mathbb{Z}_-$, where $\mathbb{Z}_- = 0, -1, -2, \ldots$. Then $t_0 = T, 0 < t_{m-1} < t_m, m \in \mathbb{Z}_-, \lim\limits_{m\to-\infty} t_m = 0$.
Reasoning analogously to point **1**, for $p = 1$ and $g \in L_0^+(0,1)$, according to (2.3), we obtain

$$\|g\|_{X_0'(0,T)} = \sup \left\{ \int\limits_0^T gh\, dt : \; h \in L_1(0,T), \; 0 \le h \downarrow, \; \|h\|_{L_1(0,T)} \le 1 \right\}.$$

This value is obtained in [3]:

$$\|g\|_{X_0'(0,T)} = A_g := \sup \left\{ U^{-1}(t) \int\limits_0^t g\, d\tau : \; t \in (0,T) \right\}. \tag{3.30}$$

Our aim is to show that the norm, associated with norm (3.30), is equivalent to the norm on right-hand side of (2.5) in the case $p = 1$. For this purpose we denote

$$B_g = \sup_{l\in\mathbb{Z}_-} U(T)^{-1} 2^{-l} \int\limits_{t_{l-1}}^{t_l} g d\tau. \tag{3.31}$$

It is obvious that $B_g \le A_g$. Conversely,

$$A_g = \sup_{k\in\mathbb{Z}_-} \sup_{t\in[t_{k-1},t_k)} U^{-1}(t) \int\limits_{(0,t)} g\, d\tau \le \sup_{k\in\mathbb{Z}_-} U(T)^{-1} \left[2^{-(k-1)} \sum_{l\le k} \int\limits_{t_{l-1}}^{t_l} g\, d\tau \right].$$

But $\int\limits_{t_{l-1}}^{t_l} g\, d\tau \le U(T) 2^l B_g$, $l \in \mathbb{Z}_-$. So,

$$A_g \le B_g \sup_{k\in\mathbb{Z}_-} U(T)^{-1} U(T) \left[2^{-(k-1)} \sum_{l\le k} 2^l \right] = B_g \sup_{k\in\mathbb{Z}_-} \, [2^{-(k-1)}, 2^{k+1}] = 4B_g. \tag{3.32}$$

Next, for $f, g \in L_0^+(0,T)$ we have

$$\int\limits_0^T fg\, d\tau = \sum_{k\in\mathbb{Z}_-} \int\limits_{t_{k-1}}^{t_k} fg\, d\tau \le \sum_{k\in\mathbb{Z}_-} \|f\|_{L_\infty(t_{k-1},t_k)} \int\limits_{t_{k-1}}^{t_k} g\, d\tau \le$$

$$\le B_g U(T) \sum_{k\in\mathbb{Z}_-} 2^k \|f\|_{L_\infty(t_{k-1},t_k)} \le A_g U(T) \sum_{k\in\mathbb{Z}_-} 2^k \|f\|_{L_\infty(t_{k-1},t_k)}.$$

Thus, according to (3.30), for $f \in L_0^+(0,1)$,

$$\|f\|_{X_0(0,T)} = \sup \left\{ \int_0^T fg\, d\tau :\; g \in L_0^+(0,T),\; A_g \le 1 \right\} \le$$

$$\le \sum_{k\in\mathbb{Z}_-} 2^k U(T) \|f\|_{L_\infty(t_{k-1},t_k)}. \tag{3.33}$$

At the same time, for $f \in L_0^+(0,1)$, $k \in \mathbb{Z}_-$, there exists g_k, such that

$$0 \le g_k \in L_1(t_{k-1},t_k), \qquad \int_{t_{k-1}}^{t_k} g_k\, d\tau = 1; \tag{3.34}$$

$$\int_{t_{k-1}}^{t_k} f g_k\, d\tau = \|f\|_{L_\infty(t_{k-1},t_k)}. \tag{3.35}$$

Now, let us define the function $\widetilde{g} \in L_0^+(0,T)$ by the formula

$$\widetilde{g} = 2^k g_k(\tau), \quad \tau \in [t_{k-1}, t_k) \quad k \in \mathbb{Z}_-. \tag{3.36}$$

Then, in view of (3.31), (3.34),

$$B_{\widetilde{g}} = \sup_{k\in\mathbb{Z}_-} U(T)^{-1} 2^{-k} \int_{t_{k-1}}^{t_k} \widetilde{g}\, d\tau = U(T)^{-1};$$

so, by virtue of (3.32), $A_{\widetilde{g}} \le 4U(T)^{-1}$. Therefore,

$$\|f\|_{X_0(0,T)} = \frac{U(T)}{4} \sup \left\{ \int_0^T fg\, d\tau :\; g \in L_0^+(0,T),\; A_g \le 4U(T)^{-1} \right\} \ge$$

$$\ge \frac{U(T)}{4} \int_0^T f\widetilde{g}\, d\tau = \frac{U(T)}{4} \sum_{k\in\mathbb{Z}_-} 2^k \int_{t_{k-1}}^{t_k} f g_k\, d\tau.$$

Taking into account equality (3.35) and inequality (3.33), we obtain from here

$$\|f\|_{X_0(0,T)} \ge \frac{1}{4} \sum_{k\in\mathbb{Z}_-} U(T) 2^k \|f\|_{L_\infty(t_{k-1},t_k)} \ge \frac{1}{4} \|f\|_{X_0(0,T)}. \tag{3.37}$$

Now let us estimate right-hand side of (2.5) in the case $p = 1$. We have

$$\int\limits_0^T \|f\|_{L_\infty(\tau,T)} u(\tau)\,d\tau = \sum_{k\in\mathbb{Z}_-} \int\limits_{t_{k-1}}^{t_k} \|f\|_{L_\infty(\tau,T)} u(\tau)\,d\tau \le \sum_{k\in\mathbb{Z}_-} \|f\|_{L_\infty(t_{k-1},T)} \int\limits_{t_{k-1}}^{t_k} u(\tau)d\tau =$$

$$= \sum_{k\in\mathbb{Z}_-} \|f\|_{L_\infty(t_{k-1},T)} U(T) 2^{k-1} \le \sum_{k\in\mathbb{Z}_-} U(T) 2^{k-1} \sum_{l=k}^{0} \|f\|_{L_\infty(t_{l-1},t_l)} =$$

$$= \sum_{l\in\mathbb{Z}_-} U(T) \|f\|_{L_\infty(t_{l-1},t_l)} \sum_{k=\infty}^{l} 2^{k-1} = \sum_{l\in\mathbb{Z}_-} U(T) \|f\|_{L_\infty(t_{l-1},t_l)} 2^l. \tag{3.38}$$

At the same time,

$$\int\limits_0^T \|f\|_{L_\infty(\tau,T)} u(\tau)\,d\tau = \sum_{k\in\mathbb{Z}_-} \int\limits_{t_{k-1}}^{t_k} \|f\|_{L_\infty(\tau,T)} u(\tau)\,d\tau \ge \sum_{k\in\mathbb{Z}_-} \|f\|_{L_\infty(t_k,T)} \int\limits_{t_{k-1}}^{t_k} u(\tau)\,d\tau \ge$$

$$\ge \frac{1}{4} \sum_{k\in\mathbb{Z}_-} U(T) \|f\|_{L_\infty(t_k,t_{k+1})} 2^{k+1} = \frac{1}{4} \sum_{l\in\mathbb{Z}_-} U(T) \|f\|_{L_\infty(t_{l-1},t_l)} 2^l. \tag{3.39}$$

As a result, we obtain from (3.38) and (3.39)

$$\int\limits_0^T \|f\|_{L_\infty(\tau,T)}\,d\tau \le \sum_{l\in\mathbb{Z}_-} U(T) \|f\|_{L_\infty(t_{l-1},t_l)} 2^l \le 4 \int\limits_0^T \|f\|_{L_\infty(\tau,T)}\,d\tau.$$

It follows from here and from (3.37) that the estimate is fulfilled

$$\frac{1}{4} \|f\|_{X_0(0,T)} \le \int\limits_0^T \|f\|_{L_\infty(\tau,T)} u(\tau)\,d\tau \le 4 \|f\|_{X_0(0,T)}.$$

It proves relation (2.5) for $p = 1$.

Let us consider the case $1 < p < \infty$. For $g \in L_0^+(0,T)$, according to (2.3), we have

$$\|g\|_{X_0'(0,T)} = \sup \left\{ \int\limits_0^T gh\,dt : \ 0 \le h \downarrow \ t \in (0,T); \ \|h\|_{L_{p,u}(0,T)} \le 1 \right\}.$$

For this value the two-sided estimate takes place (see [8]): let $g \in L_0^+(0,T)$, then

$$\|g\|_{X_0'(0,T)} \cong \rho_0(g) := U(T)^{\frac{-1}{p}} \int\limits_0^T g\,d\tau + \widetilde{\rho}_0(g), \tag{3.40}$$

here

$$\widetilde{\rho}_0(g) = \left(\int\limits_0^T \left[\int\limits_0^t g\,d\tau \right]^{p'} U^{-\frac{p'}{p}-1}(t) u(t)\,dt \right)^{\frac{1}{p'}}. \tag{3.41}$$

Let us show that for $g \in L_0^+(0,T)$

$$\widetilde{\rho}_0(g) \cong \rho_1(g) := \Bigg(\sum_{k\in\mathbb{Z}_-} U(T)^{-\frac{p'}{p}} 2^{-\frac{kp'}{p}} \Bigg(\int\limits_{t_{k-1}}^{t_k} g\, d\tau \Bigg)^{p'} \Bigg)^{\frac{1}{p'}}. \tag{3.42}$$

We have

$$\widetilde{\rho}_0(g) = \Bigg(\sum_{k\in\mathbb{Z}_-} \int\limits_{t_{k-1}}^{t_k} \Bigg[\int\limits_0^t g\, dt \Bigg]^{p'} U^{-p'}(t) u(t)\, dt \Bigg)^{\frac{1}{p'}}. \tag{3.43}$$

Hence,

$$\widetilde{\rho}_0(g) \le \Bigg(\sum_{k\in\mathbb{Z}_-} \Bigg(\int\limits_0^{t_k} g\, d\tau \Bigg)^{p'} \int\limits_{t_{k-1}}^{t_k} U^{-\frac{p'}{p}-1}(t) u(t)\, dt \Bigg)^{\frac{1}{p'}}$$

$$\cong \Bigg(\sum_{k\in\mathbb{Z}_-} \Bigg(\sum_{l\le k} \int\limits_{t_{l-1}}^{t_l} g\, d\tau \Bigg)^{p'} U(T)^{-\frac{p'}{p}} 2^{-\frac{kp'}{p}} \Bigg)^{\frac{1}{p'}}.$$

As $2^{-\frac{(k+1)p'}{p}} = 2^{-\frac{p'}{p}} 2^{-\frac{kp'}{p}}$ and $0 < 2^{-\frac{p'}{p}} < 1$, then by Lemma 1.1,

$$\Bigg(\sum_{k\in\mathbb{Z}_-} \Bigg(\sum_{l\le k} \int\limits_{t_{l-1}}^{t_l} g\, d\tau \Bigg)^{p'} U(T)^{-\frac{p'}{p}} 2^{-\frac{kp'}{p}} \Bigg)^{\frac{1}{p'}} \cong \Bigg(\sum_{k\in\mathbb{Z}_-} \Bigg(\int\limits_{t_{k-1}}^{t_k} g\, d\tau \Bigg)^{p'} U(T)^{-\frac{p'}{p}} 2^{-\frac{kp'}{p}} \Bigg)^{\frac{1}{p'}}. \tag{3.44}$$

We obtain from these estimates that $\widetilde{\rho}_0(g) \le c\rho_1(g)$. In addition, by Hölder's inequality for sequences

$$\int\limits_0^T g\, d\tau = \sum_{k\in\mathbb{Z}_-} \int\limits_{t_{k-1}}^{t_k} g\, d\tau \le \Bigg(\sum_{k\in\mathbb{Z}_-} U(T) 2^k \Bigg)^{\frac{1}{p}} \Bigg(\sum_{k\in\mathbb{Z}_-} \Bigg(\int\limits_{t_{k-1}}^{t_k} g\, d\tau \Bigg)^{p'} 2^{-\frac{kp'}{p}} U(T)^{-\frac{p'}{p}} \Bigg)^{\frac{1}{p'}}$$

$$\cong c\rho_1(g) U(T)^{\frac{1}{p}}.$$

Thus, by virtue of (3.40),

$$\rho_0(g) \le c_1 \rho_1(g). \tag{3.45}$$

Let us obtain inverse estimate. According to (3.43), for $g \in L_0^+(0,T)$ we have

$$\widetilde{\rho}_0(g) \ge \Bigg(\sum_{k\in\mathbb{Z}_-} \Bigg(\int\limits_0^{t_{k-1}} g\, d\tau \Bigg)^{p'} \int\limits_{t_{k-1}}^{t_k} U^{-p'}(t) u(t)\, dt \Bigg)^{\frac{1}{p'}}$$

$$\cong \Bigg(\sum_{k\in\mathbb{Z}_-} \Bigg(\int\limits_0^{t_{k-1}} g\, d\tau \Bigg)^{p'} 2^{-\frac{(k-1)p'}{p}} U(T)^{-\frac{p'}{p}} \Bigg)^{\frac{1}{p'}} \ge$$

$$\geq \Bigg(\sum_{k\in\mathbb{Z}_-} \Bigg(\int\limits_{t_{k-2}}^{t_{k-1}} g\,d\tau \Bigg)^{p'} 2^{-\frac{(k-1)p'}{p}} U(T)^{-\frac{p'}{p}} \Bigg)^{\frac{1}{p'}} = \Bigg(\sum_{l\leq -1} \Bigg(\int\limits_{t_{l-1}}^{t_l} g\,d\tau \Bigg)^{p'} 2^{-\frac{lp'}{p}} U(T)^{-\frac{p'}{p}} \Bigg)^{\frac{1}{p'}}.$$

Furthermore,

$$U(T)^{-\frac{1}{p}} \int\limits_0^T g\,d\tau \geq U(T)^{-\frac{1}{p}} \int\limits_{t_{-1}}^{t_0} g\,d\tau, \qquad g \in L_0^+(0,T).$$

Folding this estimates, we obtain, by virtue of (3.40), that

$$\rho_0(g) \geq c_2 \Bigg(\sum_{k\in\mathbb{Z}_-} \Bigg(\int\limits_{t_{l-1}}^{t_l} g\,d\tau \Bigg)^{p'} 2^{-\frac{lp'}{p}} \Bigg)^{\frac{1}{p'}} = c_2\rho_1(g).$$

Estimate (3.42) follows from here and from (3.45).

Now, using consiquently Hölder's inequality for integrals and sums, for all $f, g \in L_0^+(0,T)$ we have

$$\int\limits_0^T fg\,d\tau \leq \rho_1(g) \Bigg(\sum_{k\in\mathbb{Z}_-} 2^k U(T) \|f\|^p_{L_\infty(t_{k-1},t_k)} \Bigg)^{\frac{1}{p}}. \tag{3.46}$$

Actually, using consiquently Hölder's inequality for integrals and sums we have

$$\int\limits_0^T fg\,d\tau = \sum_{k\in\mathbb{Z}_-} \int\limits_{t_{k-1}}^{t_k} fg\,d\tau \leq \sum_{k\in\mathbb{Z}_-} \|f\|_{L_\infty(t_{k-1},t_k)} \int\limits_{t_{k-1}}^{t_k} g\,d\tau \leq$$

$$\leq \Bigg(\sum_{k\in\mathbb{Z}_-} U(T) 2^k \|f\|^p_{L_\infty(t_{k-1},t_k)} \Bigg)^{\frac{1}{p}} \Bigg(\sum_{k\in\mathbb{Z}_-} \Bigg(\int\limits_{t_{k-1}}^{t_k} g\,d\tau \Bigg)^{p'} 2^{-\frac{kp'}{p}} U(T)^{-\frac{p'}{p}} \Bigg)^{\frac{1}{p'}},$$

that yields (3.46). Denoting

$$|||f||| := \Bigg(\sum_{k\in\mathbb{Z}_-} 2^k U(T) \|f\|^p_{L_\infty(t_{k-1},t_k)} \Bigg)^{\frac{1}{p}}, \tag{3.47}$$

we obtain the inequality:

$$\int\limits_0^T fg\,d\tau \leq |||f|||\, \rho_1(g). \tag{3.48}$$

For $f \in L_0^+(0,T)$ we use the following equality for the norm in optimal GBFS:

$$\|f\|_{X_0(0,T)} = \sup \Bigg\{ \int\limits_0^T fg\,d\tau : \; g \in L_0^+(0,T), \; \|g\|_{X_0'(0,T)} \leq 1 \Bigg\}$$

and take into account (3.40), and (3.42). Then, for $f \in L_0^+(0,T)$ we obtain

$$\|f\|_{X_0(0,T)} \cong \sup \left\{ \int_0^T fg\, d\tau : \ g \in L_0^+(0,T); \ \rho_1(g) \le 1 \right\}. \tag{3.49}$$

It follows immediately from here and from (3.48) that there exists $c_3 \in \mathbb{R}_+$, such that

$$\|f\|_{X_0(0,T)} \le c_3 |||f|||, \qquad f \in L_0^+(0,T). \tag{3.50}$$

Now we will obtain inverse estimate. By accuracy of Hölder's inequality for sequences there exists the sequence $\{\alpha_k\}_{k\in\mathbb{Z}_-}$ with the properties:

$$\alpha_k \ge 0, \quad k \in \mathbb{Z}_-, \quad \|\{\alpha_k\}\|_{l_{p'}} = \Big(\sum_{k\in\mathbb{Z}_-} \alpha_k^{p'}\Big)^{\frac{1}{p'}} = 1;$$

$$\sum_{k\in\mathbb{Z}_-} 2^{\frac{k}{p}} U(T)^{\frac{1}{p}} \|f\|_{L_\infty(t_{k-1},t_k)}\, \alpha_k = \Big(\sum_{k\in\mathbb{Z}_-} 2^k U(T) \|f\|^p_{L_\infty(t_{k-1},t_k)}\Big)^{\frac{1}{p}}. \tag{3.51}$$

Next, by accuracy of Hölder's inequality for integrals for every $\epsilon \in (0,1)$, $f \in L_0^+(0,1)$, $k \in \mathbb{Z}_-$ there exists $g_k \in L_0^+(t_{k-1},t_k)$, such that

$$\int_{t_{k-1}}^{t_k} g_k\, d\tau = 2^{\frac{k}{p}} U(T)^{\frac{1}{p}} \alpha_k; \qquad \int_{t_{k-1}}^{t_k} f g_k\, d\tau \ge 2^{\frac{k}{p}} U(T)^{\frac{1}{p}} \alpha_k \|f\|_{L_\infty(t_{k-1},t_k)} (1-\epsilon). \tag{3.52}$$

Consider the function

$$\widetilde{g}(\tau) = g_k(\tau), \qquad \tau \in [t_{k-1}, t_k), \qquad k \in \mathbb{Z}_-.$$

Then, $\widetilde{g} \in L_0^+(0,1)$, moreover,

$$\rho_1(\widetilde{g}) = \Bigg(\sum_{k\in\mathbb{Z}_-} \Big(\int_{t_{k-1}}^{t_k} g_k\, d\tau\Big)^{p'} 2^{-\frac{kp'}{p}} U(T)^{-\frac{p'}{p}}\Bigg)^{\frac{1}{p'}} = \Big(\sum_{k\in\mathbb{Z}_-} \alpha_k^{p'}\Big)^{\frac{1}{p'}} = 1.$$

It follows from here and from (3.49) that

$$\|f\|_{X_0(0,T)} \ge c_4 \int_0^T f\widetilde{g}\, d\tau = c_4 \sum_{k\in\mathbb{Z}_-} \int_{t_{k-1}}^{t_k} f g_k\, d\tau.$$

Substituting here (3.52) and (3.51), we arrive at

$$\|f\|_{X_0(0,T)} \ge (1-\epsilon) c_4 \Big(\sum_{k\in\mathbb{Z}_-} 2^k U(T) \|f\|^p_{L_\infty(t_{k-1},t_k)}\Big)^{\frac{1}{p}} = (1-\epsilon) c_4 |||f|||.$$

Thus,

$$\|f\|_{X_0(0,T)} \geq c_4|||f|||.$$

This estimate, combined with (3.50), yields

$$\|f\|_{X_0(0,T)} \cong |||f|||. \tag{3.53}$$

It remains to prove the equivalence

$$|||f||| \cong \left(\int\limits_0^T \|f\|^p_{L_\infty(t,T)} u(t)dt \right)^{\frac{1}{p}}, \tag{3.54}$$

and relation (2.5) will be obtained in the case $1 < p < \infty$. We have

$$\int\limits_0^T \|f\|^p_{L_\infty(t,T)} udt = \sum_{k\in\mathbb{Z}_-} \int\limits_{t_{k-1}}^{t_k} \|f\|^p_{L_\infty(t,T)} u(t)dt \leq \sum_{k\in\mathbb{Z}_-} \|f\|^p_{L_\infty(t_{k-1},T)} \int\limits_{t_{k-1}}^{t_k} u(t)dt =$$

$$= \sum_{k\in\mathbb{Z}_-} \|f\|^p_{L_\infty(t_{k-1},T)} 2^{k-1}U(T) \leq \sum_{k\in\mathbb{Z}_-} \left(\sum_{l=k}^{0} \|f\|_{L_\infty(t_{l-1},t_l)} \right)^p 2^{k-1}U(T).$$

Now, according to Lemma 1.1, we obtain

$$\sum_{k\in\mathbb{Z}_-} \left(\sum_{l=k}^{0} \|f\|_{L_\infty(t_{l-1},t_l)} \right)^p 2^{k-1}U(T) \cong \sum_{k\in\mathbb{Z}_-} \|f\|^p_{L_\infty(t_{k-1},t_k)} 2^{k-1}U(T). \tag{3.55}$$

Applying estimate (3.55), we obtain

$$\left(\int\limits_0^T \|f\|^p_{L_\infty(t,T)} u(t)\,dt \right)^{\frac{1}{p}} \leq c_5 \left(\sum_{k\in\mathbb{Z}_-} \|f\|^p_{L_\infty(t_{k-1},t_k)} 2^{k-1}U(T) \right)^{\frac{1}{p}} = c_5 2^{-\frac{1}{p}} |||f|||. \tag{3.56}$$

Thus, we have upper estimate of right-hand side of (3.52) by left-hand side. At the same time,

$$\left(\int\limits_0^T \|f\|^p_{L_\infty(t,T)} u\,dt \right)^{\frac{1}{p}} = \left(\sum_{k\in\mathbb{Z}_-} \int\limits_{t_{k-1}}^{t_k} \|f\|^p_{L_\infty(t,T)} u\,dt \right)^{\frac{1}{p}} \geq \left(\sum_{k\leq -1} \|f\|^p_{L_\infty(t_k,T)} \int\limits_{t_{k-1}}^{t_k} udt \right)^{\frac{1}{p}} \geq$$

$$\geq \left(\sum_{k\leq -1} \|f\|^p_{L_\infty(t_k,t_{k+1})} 2^{k-1}U(T) \right)^{\frac{1}{p}} = \left(\sum_{l\in\mathbb{Z}_-} \|f\|^p_{L_\infty(t_{l-1},t_l)} 2^{l-2}U(T) \right)^{\frac{1}{p}} = \frac{1}{4}\,|||f|||. \tag{3.57}$$

From here and from (3.56) (3.54) follows. □

3.2. Proof of Theorem 2.1 in the case $0 < p < 1$.

Proof. According to (2.3)

$$\|g\|_{X_0'(0,T)} = \sup\left\{\int_0^T |g|hdt : \quad h \in K_0, \rho_{K_0}(h) \le 1\right\} = \sup_{0<h\downarrow} \frac{\int_0^T |g|hdt}{\left(\int_0^T h^p udt\right)^{\frac{1}{p}}}.$$

According to [3] (for $0 < p < 1$) this value is equal to:

$$\|g\|_{X_0'(0,T)} = \sup_{t\in(0,T)} \frac{\int_0^t |g|d\tau}{U(t)^{\frac{1}{p}}} = \sup_{t\in(0,T)} \frac{\int_0^t |g|d\tau}{\tilde{U}(t)},$$

$$\text{here} \quad \tilde{U}(t) = U(t)^{\frac{1}{p}} = \int_0^t \tilde{u}d\tau.$$

$$\text{Thus,} \quad \|g\|_{X_0'(0,T)} = \|g\|_{\tilde{X}_0'(0,T)}, \quad \text{here}$$

$$\tilde{K}_0 = \{h \in L_{1,\tilde{u}}(0,T) : \quad 0 < h\downarrow\}; \quad \rho_{\tilde{K}_0}(h) = \int_0^T h\tilde{u}dt.$$

Actually,

$$\|g\|_{\tilde{X}_0'(0,T)} = \sup\left\{\int_0^T |g|hdt : \quad h \in \tilde{K}_0, \rho_{\tilde{K}_0}(h) \le 1\right\} =$$

$$= \sup_{0<h\downarrow} \frac{\int_0^T |g|hdt}{\int_0^T h\tilde{u}dt}.$$

According to [3] (for $p = 1$) this value is equal to:

$$\|g\|_{\tilde{X}_0'(0,T)} = \sup_{t\in(0,T)} \frac{\int_0^t |g|d\tau}{\int_0^t \tilde{u}d\tau} = \|g\|_{X_0'(0,T)}.$$

As a result, $X_0'(0,T) = \tilde{X}_0'(0,T) \Rightarrow X_0(0,T) = \tilde{X}_0(0,T)$ is a minimal GBFS for the cone $\tilde{K}_0$. But the norm in space $\tilde{X}_0(0,T)$ is described in (2.4) (for $p = 1$), that means

$$\|f\|_{X_0'(0,T)} = \|f\|_{\tilde{X}_0'(0,T)} = \int_0^T \|f\|_{L_\infty(t,T)} \tilde{u}(t)dt.$$

Here, $\tilde{u}(t) = \frac{d\tilde{U}(t)}{dt} = \frac{1}{p}U(t)^{\frac{1}{p}-1}\frac{dU(t)}{dt}$, that means

$$\tilde{u}(t) = \frac{1}{p}U(t)^{\frac{1}{p}-1}u(t).$$

□

3.3. Proof of Remark 2.1

Proof. For $p = 1$ we demand $U(T) < \infty, \quad u^{-1} \in L_\infty^{loc}(0,T)$. Then, for $B \subset (0,T)$ we have

$$\|\chi_B\|_{X_0(0,T)} = \int_0^T \|\chi_B\|_{L_\infty(t,T)}^p u(t)dt \le \int_0^T u(t)dt = U(T) < \infty.$$

It means that property $(P5)$ is fulfilled.
Moreover, for $f \in X_0(0,T)$ we also have

$$\|f\|_{L_{p,u}(0,T)} = \left(\int_0^T |f|^p u dt \right)^{\frac{1}{p}} \le \left(\int_0^T \|f\|^p_{L_\infty(t,T)} u dt \right)^{\frac{1}{p}} = \|f\|_{X_0(0,T)} .$$

Then, for $B \subset (0,T), \mu(B) < \infty$ in the case $1 < p < \infty$ we have

$$\int_B |f| dt = \int_B |f| u^{\frac{1}{p}} u^{-\frac{1}{p}} dt \le \left(\int_B |f|^p u dt \right)^{\frac{1}{p}} \left(\int_B u^{-\frac{p'}{p}} dt \right)^{\frac{1}{p'}} \le$$

$$\le \|f\|_{X_0(0,T)} \left(\int_B u^{-\frac{p'}{p}} dt \right)^{\frac{1}{p'}} = \|f\|_{X_0(0,T)} C_{B,p,p'}.$$

It means that $(P4)$ is fulfilled.
For $p = 1$

$$\int_B |f| dt = \int_B |f| u u^{-1} dt \le \left(\int_B |f| u dt \right) \left\| u^{-1} \right\|_{L_\infty(B)} \le \|f\|_{X_0(0,T)} \left\| u^{-1} \right\|_{L_\infty(B)} .$$

Then, $X_0(0,T)$ is BFS, because it is obvious that properties $(P1)-(P3)$ are fulfilled. $\square$

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