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THE CAUCHY PROBLEM FOR ONE-DIMENSIONAL WAVE EQUATIONS WITH A NONLINEAR DISSIPATIVE TERM

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Abstract. The Cauchy problem for one-dimensional wave equations with a nonlinear dissipative term is investigated. Under consideration are the problems of uniqueness and existence of local, global and blow-up solutions. The paper's originality is the coalescence of the two standard methods: a priori estimate of solutions in the class of continuous functions is given by energetic methods; basing on this result a priori estimate in the class of continuously differentiable functions using classical method of characteristics is obtained.

1 The statement of the problem

For one-dimensional wave equations with a nonlinear dissipative term [14], [13, p. 57],

$$Lu := u_{tt} - u_{xx} + g(x,t,u)u_t = f(x,t), \tag{1.1}$$

in the half-plane $\Omega := \{(x,t) : x \in \mathbb{R},\ t > 0\}$, let us consider the Cauchy problem with the following initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where f, g, φ, ψ are given real-valued functions, and u is the unknown real-valued function.

It should be noted that for nonlinear hyperbolic equations the questions of uniqueness and existence of local, global and blow-up solutions for initial and other problems are considered in numerous literature (see e.g., [1, 6-19, 21, 23, 25]).

Below we show that under certain requirements on the nonlinear function $g(x,t,s)$ with respect to the variable s problem (1.1), (1.2) is locally solvable. There are also obtained conditions of global solvability, violation of which, generally speaking, may cause the blow-up of the solution within finite interval of time.

Let $P_0 := P_0(x_0,t_0)$ be an arbitrary point of the domain Ω and $D_{P_0} := \{(x,t) : t + x_0 - t_0 < x < -t + x_0 + t_0,\ t > 0\}$ be the triangular domain bounded by the characteristic segments $\gamma_{1,P_0} : x = t + x_0 - t_0,\ 0 \le t \le t_0$ and $\gamma_{2,P_0} : x = -t + x_0 + t_0,\ 0 \le t \le t_0$ of equation (1.1), and the segment $\gamma_{P_0} : t = 0,\ x_0 - t_0 \le x \le x_0 + t_0$.

First we consider the Cauchy problem for equation (1.1) in the bounded domain D_{P_0}: find a solution $u = u(x,t),\ (x,t) \in D_{P_0}$, of equation (1.1) with the initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in \gamma_{P_0}, \tag{1.3}$$

where φ, ψ are given real-valued functions on γ_{P_0}.

Definition 1.1. Let $f \in C(\overline{D}_{P_0}),\ g \in C(\overline{D}_{P_0} \times \mathbb{R}),\ \varphi \in C^1(\gamma_{P_0})$ and $\psi \in C(\gamma_{P_0})$. We say that a function u is a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0}, if $u \in C^1(\overline{D}_{P_0})$ and there exists a sequence of the functions $u_n \in C^2(\overline{D}_{P_0})$, such that $u_n \to u,\ Lu_n \to f,\ u_n(\cdot,0) \to \varphi$ and $u_{nt}(\cdot,0) \to \psi$ as $n \to \infty$ in the spaces $C^1(\overline{D}_{P_0}),\ C(\overline{D}_{P_0}),\ C^1(\gamma_{P_0}),\ C(\gamma_{P_0})$, respectively.

Remark 1.1. It is obvious, that a classical solution of problem (1.1), (1.3) of the class $C^2(\overline{D}_{P_0})$ is a strong generalized solution of this problem of the class C^1 in the domain D_{P_0}. Inversely, if a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0} belongs to the space $C^2(\overline{D}_{P_0})$, then it will also be a classical solution of this problem.

Definition 1.2. Let $f \in C(\overline{\Omega}),\ g \in C(\overline{\Omega} \times \mathbb{R}),\ \varphi \in C^1(\mathbb{R}),\ \psi \in C(\mathbb{R})$. We say that problem (1.1), (1.3) is globally solvable in the class C^1, if for any point $P_0 \in \Omega$ the problem has a strong generalized solution of class C^1 in the domain D_{P_0} in the sense of Definition 1.1.

Definition 1.3. Let $f \in C(\overline{\Omega}),\ g \in C(\overline{\Omega} \times \mathbb{R}),\ \varphi \in C^1(\mathbb{R}),\ \psi \in C(\mathbb{R})$. We say that a function $u \in C^1(\overline{\Omega})$ is a global strong generalized solution of problem (1.1), (1.2) of class C^1, if for any point $P_0 \in \Omega$ it is a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0} in the sense of Definition 1.1.

Remark 1.2. Note that in the case when the theorem of existence and uniqueness of a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0} is valid for any $P_0 \in \Omega$, then the existence follows of a unique global strong generalized solution of problem (1.1), (1.2) of class C^1 in the sense of Definition 1.3.

The paper is organized in the following way. In the second section under a certain constraint on the function g an a priori estimate is obtained for a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0} in the sense of Definition 1.1 in the space $C(\overline{D}_{P_0})$, and then, basing on this, it is obtained in the space $C^1(\overline{D}_{P_0})$. In the third section the equivalency is proved of the posed problem to the problem for a system of nonlinear integral equations of Volterra type. In the fourth section the local and global solvability is proved of problems (1.1), (1.3) and (1.1), (1.2). In the fifth section the uniqueness of the solution is proved . In the sixth section, for the case when condition (2.1) is violated, the question of nonexistence is studied of a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0} in the sense of Definition 1.1.

2 A priori estimates for a strong generalized solution of problem (1.1), (1.3) in the classes $\mathbf{C}(\overline{\mathbf{D}}_{\mathbf{P_0}})$ and $\mathbf{C}^1(\overline{\mathbf{D}}_{\mathbf{P_0}})$

Consider the conditions

$$g(x,t,s) \geq -M,\ (x,t,s) \in \overline{\Omega} \times \mathbb{R},\ M := const > 0 \tag{2.1}$$

and

$$f \in C(\overline{\Omega}), \quad g \in C(\overline{\Omega} \times \mathbb{R}), \quad \varphi \in C^1(\mathbb{R}), \quad \psi \in C(\mathbb{R}). \tag{2.2}$$

Lemma 2.1. *Let conditions (2.1), (2.2) be satisfied and P_0 be an arbitrary point of the domain Ω. Then if u is a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0}, then the following a priori estimate holds*

$$\|u\|_{C(\overline{D}_{P_0})} \leq c_0\big(\|f\|_{C(\overline{D}_{P_0})} + \|\varphi\|_{C^1(\gamma_{P_0})} + \|\psi\|_{C(\gamma_{P_0})}\big), \tag{2.3}$$

with a positive constant $c_0 = c_0(t_0, M)$ independent of u and f, φ, ψ, where

$$\|\varphi\|_{C^1(\gamma_{P_0})} := \max\big\{\|\varphi\|_{C(\gamma_{P_0})}, \|\varphi'\|_{C(\gamma_{P_0})}\big\}.$$

Proof. Let u be a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0}. Then due to Definition 1.1 there exists a sequence of functions $u_n \in C^2(\overline{D}_{P_0})$, such that

$$\begin{gathered}\lim_{n\to\infty} \|u_n - u\|_{C^1(\overline{D}_{P_0})} = 0, \quad \lim_{n\to\infty} \|Lu_n - f\|_{C(\overline{D}_{P_0})} = 0, \\ \lim_{n\to\infty} \|u_n(\cdot,0) - \varphi\|_{C^1(\gamma_{P_0})} = 0, \quad \lim_{n\to\infty} \|u_{nt}(\cdot,0) - \psi\|_{C(\gamma_{P_0})} = 0,\end{gathered} \tag{2.4}$$

and, therefore

$$\lim_{n\to\infty} \|g(x,t,u_n)u_{nt} - g(x,t,u)u_t\|_{C(\overline{D}_{P_0})} = 0. \tag{2.5}$$

Here

$$\|u\|_{C^1(\overline{D}_{P_0})} := \max\big\{\|u\|_{C(\overline{D}_{P_0})},\ \|u_x\|_{C(\overline{D}_{P_0})},\ \|u_t\|_{C(\overline{D}_{P_0})}\big\}.$$

Consider the function $u_n \in C^2(\overline{D}_{P_0})$ as a solution to the following problem

$$Lu_n = f_n, \tag{2.6}$$

$$u_n|_{\gamma_{P_0}} = \varphi_n, \quad u_{nt}|_{\gamma_{P_0}} = \psi_n. \tag{2.7}$$

Here

$$f_n := Lu_n, \quad \varphi_n := u_n(\cdot,0), \quad \psi_n := u_{nt}(\cdot,0). \tag{2.8}$$

Let us fix the point $P_0' := P_0'(x_0', t_0') \in D_{P_0}$. Then, it is obvious, that $D_{P_0'} \subset D_{P_0}$ and $\gamma_{P_0'} \subset \gamma_{P_0}$. Multiplying both sides of equality (2.6) by u_{nt} and integrating the obtained equality over the domain $D_{P_0',\tau} := \big\{(x,t) \in D_{P_0'} : 0 < t < \tau\big\},\ 0 < \tau < t_0'$, we have

$$\int\limits_{D_{P_0',\tau}} (u_{nt}^2)_t dxdt - 2\int\limits_{D_{P_0',\tau}} (u_{nx}u_{nt})_x dxdt + \int\limits_{D_{P_0',\tau}} (u_{nx}^2)_t dxdt$$

$$+2\int\limits_{D_{P_0',\tau}} g(x,t,u_n)u_{nt}^2dxdt = 2\int\limits_{D_{P_0',\tau}} f_nu_{nt}dxdt.$$

Let $\omega_{P_0',\tau} := \overline{D}_{P_0'} \cap \{t=\tau\}$. Then according to (2.7), integrating by parts the left-hand side of the last equality we get

$$\begin{aligned}
&2\int\limits_{D_{P_0',\tau}} f_nu_{nt}dxdt - 2\int\limits_{D_{P_0',\tau}} g(x,t,u_n)u_{nt}^2dxdt\\
&=\sum_{i=1}^{2}\int\limits_{\gamma_{i,P_0',\tau}} \nu_t^{-1}\big[\big(\nu_tu_{nx}-\nu_xu_{nt}\big)^2+u_{nt}^2\big(\nu_t^2-\nu_x^2\big)\big]ds\\
&-\int_{x_0'-t_0'}^{x_0'+t_0'}\big[\varphi_n'^2(x)+\psi_n^2(x)\big]dx+\int\limits_{\omega_{P_0',\tau}}\big(u_{nx}^2+u_{nt}^2\big)dx,
\end{aligned}\tag{2.9}$$

where $\nu := (\nu_x,\nu_t)$ is the unit vector of the outer normal to $\partial D_{P_0',\tau}$ and $\gamma_{i,P_0',\tau} := \gamma_{i,P_0'} \cap \{t\le\tau\},\ i=1,2.$

Taking into account that the relations $(\nu_t^2-\nu_x^2)\big|_{\gamma_{i,P_0'}} = 0,\ i=1,2,$ which take place everywhere on the characteristics $\gamma_{i,P_0'},\ i=1,2,$ of equation (1.1), due to (2.1) by (2.9) we obtain

$$\begin{aligned}
w_n(\tau) \le\ & 2\int\limits_{D_{P_0',\tau}} f_nu_{nt}dxdt + 2M\int\limits_{D_{P_0',\tau}} u_{nt}^2dxdt\\
&+\int_{x_0'-t_0'}^{x_0'+t_0'}\big[\varphi_n'^2(x)+\psi_n^2(x)\big]dx.
\end{aligned}\tag{2.10}$$

Here

$$w_n(\tau) := \int\limits_{\omega_{P_0',\tau}}\big(u_{nx}^2+u_{nt}^2\big)dx+\sum_{i=1}^{2}\int\limits_{\gamma_{i,P_0',\tau}}\nu_t^{-1}\big(\nu_tu_{nx}-\nu_xu_{nt}\big)^2ds.\tag{2.11}$$

Taking into account the obvious inequality

$$2\int\limits_{D_{P_0',\tau}} f_nu_{nt}dxdt \le \int\limits_{D_{P_0',\tau}} u_{nt}^2dxdt + \|f_n\|^2_{L_2(D_{P_0,\tau})}$$

by (2.10) it follows that

$$w_n(\tau) \le \big(1+2M\big)\int\limits_{D_{P_0',\tau}} u_{nt}^2dxdt + \|f_n\|^2_{L_2(D_{P_0,\tau})} + \int_{x_0'-t_0'}^{x_0'+t_0'}\big[\varphi_n'^2(x)+\psi_n^2(x)\big]dx.$$

Whence, due to (2.11), we have

$$w_n(\tau) \le \beta\int\limits_0^\tau w_n(\sigma)d\sigma + \|f_n\|^2_{L_2(D_{P_0,\tau})} + \alpha_n,\quad 0<\tau<t_0',$$

where

$$\beta := 1 + 2M,\ \alpha_n := \int_{x_0-t_0}^{x_0+t_0} \left[\varphi_n'^2(x) + \psi_n^2(x)\right] dx. \tag{2.12}$$

From the last inequality, since the value $\|f_n\|^2_{L_2(D_{P_0,\tau})}$, as a function of τ is nondecreasing, and $w_n \geq 0$ due to $\nu_t|_{\gamma_{i,P_0}} = \frac{1}{\sqrt{2}} > 0,\ i = 1, 2$, by the Gronwall lemma (see e.g., [3], p. 13) we obtain

$$w_n(\tau) \leq e^{\beta\tau}\big(\|f_n\|^2_{L_2(D_{P_0,\tau})} + \alpha_n\big). \tag{2.13}$$

Taking into account, that the operator $\nu_t\frac{\partial}{\partial x} - \nu_x\frac{\partial}{\partial t}$ is an interior differential operator on $\gamma_{1,P_0'}$, the integration on over segment $\gamma_{1,P_0'}$, gives

$$u_n(P_0') = \varphi_n(x_0' - t_0') + \int\limits_{\gamma_{1,P_0'}} (\nu_t u_{nx} - \nu_x u_{nt}) ds.$$

Squaring both sides of this equality and applying the Cauchy-Schwartz inequality we have

$$|u_n(P_0')|^2 \leq 2\varphi_n^2(x_0' - t_0') + 2\int\limits_{\gamma_{1,P_0'}} ds \int\limits_{\gamma_{1,P_0'}} (\nu_t u_{nx} - \nu_x u_{nt})^2 ds$$

$$\leq 2\varphi_n^2(x_0' - t_0') + 2\sqrt{2}t_0 \int\limits_{\gamma_{1,P_0'}} (\nu_t u_{nx} - \nu_x u_{nt})^2 ds,$$

whence due to (2.11)-(2.13) we get

$$|u_n(P_0')|^2 \leq 2\varphi_n^2(x_0' - t_0') + 4t_0 e^{\beta t_0}\big(\|f_n\|^2_{L_2(D_{P_0})} + \alpha_n\big) \leq 2\varphi_n^2(x_0' - t_0')$$

$$+4t_0 e^{\beta t_0}\big(\|f_n\|^2_{C(\overline{D}_{P_0})}\text{meas } D_{P_0} + 2t_0\|\varphi_n'\|^2_{C(\gamma_{P_0})} + 2t_0\|\psi_n\|^2_{C(\gamma_{P_0})}\big)$$

$$= 2\varphi_n^2(x_0' - t_0') + 4t_0^2 e^{\beta t_0}\big(t_0\|f_n\|^2_{C(\overline{D}_{P_0})} + 2\|\varphi_n'\|^2_{C(\gamma_{P_0})} + 2\|\psi_n\|^2_{C(\gamma_{P_0})}\big).$$

Here we used the obvious inequalities

$$\|\cdot\|^2_{L_2(D_{P_0})} \leq \|\cdot\|^2_{C(\overline{D}_{P_0})}\text{meas } D_{P_0} = t_0^2\|\cdot\|^2_{C(\overline{D}_{P_0})};\quad \|\cdot\|^2_{L_2(\gamma_{P_0})} \leq 2t_0\|\cdot\|^2_{C(\gamma_{P_0})}.$$

Therefore, using the inequality $\sqrt{a^2 + b^2 + c^2} \leq |a| + |b| + |c|$, we get

$$|u_n(P_0')| \leq c_0\big(\|f_n\|_{C(\overline{D}_{P_0})} + \|\varphi_n\|_{C^1(\gamma_{P_0})} + \|\psi_n\|_{C(\gamma_{P_0})}\big),$$

where $c_0^2 := \max\left\{4t_0^3 e^{\beta t_0},\ 2 + 8t_0^2 e^{\beta t_0},\ 8t_0^2 e^{\beta t_0}\right\}$.

Passing in the last inequality to the limit as $n \to \infty$, due to (2.4), (2.8) we have

$$|u(P_0')| \leq c_0\big(\|f\|_{C(\overline{D}_{P_0})} + \|\varphi\|_{C^1(\gamma_{P_0})} + \|\psi\|_{C(\gamma_{P_0})}\big). \tag{2.14}$$

Since P_0' is an arbitrary point of the domain D_{P_0}, then we obtain estimate (2.3). □

Below, using the classical method of characteristics and taking into account (2.3), we obtain a priori estimate in the space $C^1(\overline{D}_{P_0})$ for a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0}.

The following lemma is valid.

Lemma 2.2. *Under the conditions of Lemma 2.1 for a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0} the following estimate*

$$\|u\|_{C^1(\overline{D}_{P_0})} \le c_1 \tag{2.15}$$

holds with a positive constant $c_1 = c_1(P_0,\ c_0,\ \|f\|_{C(\overline{D}_{P_0})},\ \|\varphi\|_{C^1(\gamma_{P_0})},\ \|\psi\|_{C(\gamma_{P_0})})$.

Proof. Let u be a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0}. Then the limit equalities (2.4), (2.5) are valid, where u_n can be considered as a solution of problem (2.6), (2.7) with the right-hand side f_n and the initial Cauchy data $\varphi_n,\ \psi_n$ from (2.8).

Denote

$$l_i := \frac{\partial}{\partial t} + (-1)^i \frac{\partial}{\partial x}, \quad i = 1, 2.$$

For a fixed natural number n let us introduce the following functions

$$v_{in} := l_i u_n, \quad v_{3n} := u_n, \quad i = 1, 2, \tag{2.16}$$

which due to (1.3) satisfy the initial conditions

$$v_{1n}(x,0) = \psi_n(x) - \varphi_n'(x), v_{2n}(x,0) = \psi_n(x) + \varphi_n'(x), v_{3n}(x,0) = \varphi_n(x). \tag{2.17}$$

According to (1.1) and (2.16) the unknown functions $v_{in},\ i = 1,2,3$ satisfy the following system of first order partial differential equations

$$\begin{cases} l_2 v_{1n} = f_n(x,t) - \frac{1}{2} g(x,t,v_{3n})(v_{1n} + v_{2n}), \\ l_1 v_{2n} = f_n(x,t) - \frac{1}{2} g(x,t,v_{3n})(v_{1n} + v_{2n}), \\ l_1 v_{3n} = v_{1n}. \end{cases} \tag{2.18}$$

Let $P_\tau := (x - t + \tau, \tau),\ Q_\tau := (x + t - \tau, \tau)$. Integrating the equations of system (2.18) along the corresponding characteristic curves and taking into account initial conditions (2.17) we get

$$\begin{cases} v_{1n}(x,t) = -\frac{1}{2}\int_0^t g\big(P_\tau, v_{3n}(P_\tau)\big)\big(v_{1n}(P_\tau) + v_{2n}(P_\tau)\big)d\tau \\ +F_{1n}(x,t), \\ v_{2n}(x,t) = -\frac{1}{2}\int_0^t g\big(Q_\tau, v_{3n}(Q_\tau)\big)\big(v_{1n}(Q_\tau) + v_{2n}(Q_\tau)\big)d\tau \\ +F_{2n}(x,t), \\ v_{3n}(x,t) = \int_0^t v_{1n}(Q_\tau)d\tau + F_{3n}(x,t), \end{cases} \tag{2.19}$$

where

$$\begin{gathered} F_{1n}(x,t) := \psi_n(x-t) - \varphi_n'(x-t) + \int_0^t f_n(P_\tau)d\tau,\ F_{2n}(x,t) := \\ \psi_n(x+t) + \varphi_n'(x+t) + \int_0^t f_n(Q_\tau)d\tau,\ F_{3n}(x,t) := \varphi_n(x+t). \end{gathered} \tag{2.20}$$

Passing in equalities (2.19), (2.20) to the limit as $n \to \infty$ in the space $C(\overline{D}_{P_0})$ and taking into account (2.4), (2.5), (2.8) and (2.16) we obtain

$$\begin{cases} v_1(x,t) = -\frac{1}{2}\int_0^t g\big(P_\tau, v_3(P_\tau)\big)\big(v_1(P_\tau)+v_2(P_\tau)\big)d\tau + F_1(x,t), \\ v_2(x,t) = -\frac{1}{2}\int_0^t g\big(Q_\tau, v_3(Q_\tau)\big)\big(v_1(Q_\tau)+v_2(Q_\tau)\big)d\tau + F_2(x,t), \\ v_3(x,t) = \int_0^t v_1(Q_\tau)d\tau + F_3(x,t), \end{cases} \tag{2.21}$$

where $v_i := \lim\limits_{n\to\infty} v_{in}$ in the sense of the norm of the space $C(\overline{D}_{P_0}))$, $i = 1,2,3$, and

$$\begin{gathered} F_1(x,t) := \psi(x-t) - \varphi'(x-t) + \int_0^t f(P_\tau)d\tau, \\ F_2(x,t) := \psi(x+t) + \varphi'(x+t) + \int_0^t f(Q_\tau)d\tau, \quad F_3(x,t) := \varphi(x+t). \end{gathered} \tag{2.22}$$

Remark 2.1. Equalities (2.21) can be considered as a system of Volterra type nonlinear integral equations.

It is obvious that $v_3 = u$ and that it is a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0}. Besides, due to (2.16) the following equalities

$$v_1 = u_t - u_x, \ v_2 = u_t + u_x \tag{2.23}$$

are valid.

Let $G_{P_0} := \big\{(x,t,s) \in \mathbb{R}^3 : (x,t) \in \overline{D}_{P_0}, \ |s| \le c_0\big(\|f\|_{C(\overline{D}_{P_0})} + \|\varphi\|_{C^1(\gamma_{P_0})} + \|\psi\|_{C(\gamma_{P_0})}\big)\big\}$ and

$$K = K\big(P_0, c_0, f, \varphi, \psi\big) := \frac{1}{2} \sup_{(x,t,s)\in G_{P_0}} |g(x,t,s)| < +\infty. \tag{2.24}$$

Then due to a priori estimate (2.3) for a strong generalized solution $v_3 = u$ of problem (1.1), (1.3) of class C^1 in the domain D_{P_0} we get

$$|g\big(x,t,v_3(x,t)\big)| \le 2K, \quad (x,t) \in \overline{D}_{P_0}. \tag{2.25}$$

Let

$$\omega_i(t) := \sup_{(\xi,\tau)\in\overline{D}_{P_0,t}} |v_i(\xi,\tau)|, \ i = 1,2,3, \quad h(t) := \sup_{(\xi,\tau)\in\overline{D}_{P_0,t}} |f(\xi,\tau)|. \tag{2.26}$$

By (2.21), due to (2.22), (2.25) and (2.26), it follows that

$$|v_1(x,t)| \le K\int_0^t (\omega_1(\tau)+\omega_2(\tau))d\tau + \|\varphi'\|_{C(\gamma_{P_0})} + \|\psi\|_{C(\gamma_{P_0})} + th(t),$$

$$|v_2(x,t)| \le K\int_0^t (\omega_1(\tau)+\omega_2(\tau))d\tau + \|\varphi'\|_{C(\gamma_{P_0})} + \|\psi\|_{C(\gamma_{P_0})} + th(t),$$

$$|v_3(x,t)| \le \int_0^t \omega_1(\tau)d\tau + \|\varphi\|_{C(\gamma_{P_0})}.$$

Whence for $(\xi,\tau)\in\overline{D}_{P_0,t}$ we have

$$|v_1(\xi,\tau)|\le K\int_0^\tau(\omega_1(\tau_1)+\omega_2(\tau_1))d\tau_1+\|\varphi'\|_{C(\gamma_{P_0})}+\|\psi\|_{C(\gamma_{P_0})}+\tau h(\tau),$$

$$|v_2(\xi,\tau)|\le K\int_0^\tau(\omega_1(\tau_1)+\omega_2(\tau_1))d\tau_1+\|\varphi'\|_{C(\gamma_{P_0})}+\|\psi\|_{C(\gamma_{P_0})}+\tau h(\tau),$$

$$|v_3(\xi,\tau)|\le\int_0^\tau\omega_1(\tau_1)d\tau_1+\|\varphi\|_{C(\gamma_{P_0})},$$

and, therefore, in view of (2.26) and the fact that $th(t)$ is a nondecreasing function, we have

$$\omega_1(t)\le K\int_0^t(\omega_1(\tau)+\omega_2(\tau))d\tau+\|\varphi'\|_{C(\gamma_{P_0})}+\|\psi\|_{C(\gamma_{P_0})}+th(t),$$

$$\omega_2(t)\le K\int_0^t(\omega_1(\tau)+\omega_2(\tau))d\tau+\|\varphi'\|_{C(\gamma_{P_0})}+\|\psi\|_{C(\gamma_{P_0})}+th(t),$$

$$\omega_3(t)\le\int_0^t\omega_1(\tau)d\tau+\|\varphi\|_{C(\gamma_{P_0})}.$$

Putting $\omega(t):=\max\limits_{1\le i\le 3}\omega_i(t)$, by obtained inequalities we have

$$\omega(t)\le(2K+1)\int_0^t\omega(\tau)d\tau+\|\varphi\|_{C^1(\gamma_{P_0})}+\|\psi\|_{C(\gamma_{P_0})}+th(t),$$

whence, using the Gronwall lemma, we obtain

$$\omega(t)\le\exp\big((2K+1)t\big)\big(\|\varphi\|_{C^1(\gamma_{P_0})}+\|\psi\|_{C(\gamma_{P_0})}+th(t)\big)$$

$$\le\exp\big((2K+1)t_0\big)\big(t_0\|f\|_{C(\overline{D}_{P_0})}+\|\varphi\|_{C^1(\gamma_{P_0})}+\|\psi\|_{C(\gamma_{P_0})}\big),\quad 0\le t\le t_0.$$

Now, by (2.23) it follows easily that

$$\|u\|_{C^1(\overline{D}_{P_0})}\le\|\omega\|_{C[0,t_0]}$$

$$\le\exp\big((2K+1)t_0\big)\big(t_0\|f\|_{C(\overline{D}_{P_0})}+\|\varphi\|_{C^1(\gamma_{P_0})}+\|\psi\|_{C(\gamma_{P_0})}\big).$$

The statement of Lemma 2.2 follows with

$$c_1:=\exp\big((2K+1)t_0\big)\big(t_0\|f\|_{C(\overline{D}_{P_0})}+\|\varphi\|_{C^1(\gamma_{P_0})}+\|\psi\|_{C(\gamma_{P_0})}\big),$$

where K is defined by (2.24). □

3 Equivalency of problem (1.1), (1.3) and Volterra type system of nonlinear integral equations (2.21)

In the second section we have reduced problem (1.1), (1.3) to a Volterra type system of nonlinear integral equations (2.21). Before consideration the solvability of problem (1.1), (1.3), let us prove the following lemma.

Lemma 3.1. *If a function $u \in C^1(\overline{D}_{P_0})$ is a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0}, then $v_1 := l_1 u,\ v_2 := l_2 u,\ v_3 := u$ is a continuous solution of Volterra type system of nonlinear integral equations (2.21) and vice versa, if v_1, v_2, v_3 is a continuous solution of system (2.21), then $u := v_3$ is a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0}, and the equalities $v_1 = l_1 u,\ v_2 = l_2 u$ are valid.*

Proof. Indeed, let $v_1, v_2, v_3 \in C(\overline{D}_{P_0})$ be a solution of system of equations (2.21). It is clear that functions v_1, v_2, v_3 satisfy the following system of partial differential equations of the first order

$$l_2 v_1 = H, \;\; l_1 v_2 = H, \quad l_1 v_3 = v_1, \tag{3.1}$$

where

$$2H := 2f(x,t) - g(x,t,v_3)(v_1 + v_2). \tag{3.2}$$

Moreover, let us show that

$$l_2 v_3 = v_2. \tag{3.3}$$

Indeed, due to equalities (3.1) for any test function $w \in C_0^\infty(D_{P_0})$ the following equalities are valid in the sense of the distribution theory of L. Schwartz [5]:

$$< l_1(v_2 - l_2 v_3), w >= - < v_2 - l_2 v_3, l_1 w >= - < v_2, l_1 w > + < l_2 v_3, l_1 w >$$

$$=< l_1 v_2, w > - < v_3, l_2 l_1 w >=< l_1 v_2, w > - < v_3, l_1 l_2 w >$$

$$=< l_1 v_2, w > + < l_1 v_3, l_2 w >=< l_1 v_2, w > + < v_1, l_2 w >$$

$$=< l_1 v_2, w > - < l_2 v_1, w >=< l_1 v_2 - l_2 v_1, w >= 0.$$

Whence, due to Theorem 1.4.2 in [5] we conclude that (3.3) takes place in the classical sense. Therefore, in view of (3.1) we have $v_3 \in C^1$ and

$$v_1 + v_2 = 2 v_{3t}. \tag{3.4}$$

Let us extend functions $f \in C(\overline{D}_{P_0})$ and $g \in C(\overline{D}_{P_0} \times \mathbb{R})$ continuously to $\mathbb{R}^2$, $\mathbb{R}^3$ respectively, preserving the same notation. Analogously we extend functions $v_i \in C(\overline{D}_{P_0}),\ i = 1, 2$ and $v_3 \in C^1(\overline{D}_{P_0})$ to $\mathbb{R}^2$ with preservation the same class of smoothness, not changing notation. Then the function H will be extended to the whole plane $\mathbb{R}^2$ [24, 4].

Let

$$\theta_\varepsilon(y) = \begin{cases} C_\varepsilon \exp\left[-\frac{\varepsilon^2}{\varepsilon^2 - |y|^2}\right], & |y| < \varepsilon, \\ 0, & |y| \geq \varepsilon, \end{cases} \tag{3.5}$$

where $y := (x, t) \in \mathbb{R}^2$, and C_ε is the positive constant, which is defined by the equality

$$C_\varepsilon \varepsilon^2 \int_{|\xi|<1} \exp \Big[-\frac{1}{1-|\xi|^2} \Big] d\xi = 1, \quad \xi := (\xi_1, \xi_2);$$

ε is any positive number, and $|\cdot|$ is the norm in the Euclidian space $\mathbb{R}^2$.

Consider the sequence of functions

$$\overset{n}{v}_i := v_i * \theta_{\frac{1}{n}} \in C^\infty(\mathbb{R}^2), \quad i = 1, 2, 3, \quad \overset{n}{H} := H * \theta_{\frac{1}{n}}, \tag{3.6}$$

where $*$ denotes the convolution, and the function θ_ε is defined by (3.5).

In view of the properties of convolutions we have [5]

$$\begin{gathered} \lim_{n\to\infty} \| \overset{n}{v}_i - v_i \|_{C(\overline{D}_{P_0})} = 0, \quad i = 1, 2, \quad \lim_{n\to\infty} \| \overset{n}{v}_3 - v_3 \|_{C^1(\overline{D}_{P_0})} = 0, \\ \lim_{n\to\infty} \| \overset{n}{H} - H \|_{C(\overline{D}_{P_0})} = 0. \end{gathered} \tag{3.7}$$

Below we prove that the sequence of functions $u_n := \overset{n}{v}_3 \big|_{\overline{D}_{P_0}} \in C^2(\overline{D}_{P_0})$ satisfies conditions (2.4) and (2.5) and, therefore, $u := v_3$ is a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0} due to Definition 1.1.

According to the properties of convolution and equalities (3.1), (3.3), the functions $\overset{n}{v}_i$, $i = 1, 2, 3$, defined by (3.6), satisfy the following equalities

$$l_2 \overset{n}{v}_1 = \overset{n}{H}, \quad l_1 \overset{n}{v}_2 = \overset{n}{H}, \quad l_1 \overset{n}{v}_3 = \overset{n}{v}_1, \quad l_2 \overset{n}{v}_3 = \overset{n}{v}_2 . \tag{3.8}$$

Further, according to the definition of the operator L by (1.1) and equalities (3.8) we have

$$L \overset{n}{v}_3 = l_2 l_1 \overset{n}{v}_3 + g(x, t, \overset{n}{v}_3) \overset{n}{v}_{3t} = l_2 \overset{n}{v}_1 + g(x, t, \overset{n}{v}_3) \overset{n}{v}_{3t} = \overset{n}{H} + g(x, t, \overset{n}{v}_3) \overset{n}{v}_{3t} .$$

Whence due to equalities (3.7) it follows that $L \overset{n}{v}_3 \to H + g(x, t, v_3) v_{3t}$ as $n \to \infty$ in the space $C(\overline{D}_{P_0})$. At the same time due to (3.2), (3.4) we have $H + g(x, t, v_3)v_{3t} = f$. Therefore

$$\lim_{n\to\infty} \| L \overset{n}{v}_3 - f \|_{C(\overline{D}_{P_0})} = 0. \tag{3.9}$$

Now let us verify the validity of the initial conditions. Due to the properties of convolutions and equalities (2.21), (2.22) we have

$$\overset{n}{v}_3 (\cdot, 0) \to v_3(\cdot, 0) = \varphi \tag{3.10}$$

as $n \to \infty$ in the space $C^1(\gamma_{P_0})$.

Analogously, due to the properties of convolutions and equalities (3.4), (2.21) and (2.22), we have

$$\overset{n}{v}_{3t} (\cdot, 0) \to v_{3t}(\cdot, 0) = \frac{v_1(\cdot, 0) + v_2(\cdot, 0)}{2} = \psi \tag{3.11}$$

as $n \to \infty$ in the space $C(\gamma_{P_0})$.

The equalities $v_1 = l_1 u$, $v_2 = l_2 u$ immediately follow by the above considerations, which together with (3.9)-(3.11) prove the statement of Lemma 3.1, since the inverse proposition is obvious. □

4 Solvability of problem (1.1), (1.3)

First let us prove the local solvability of problem (1.1), (1.3), which is due to Lemma 3.1 equivalent to proving the local solvability of Volterra type system of nonlinear integral equations (2.21).

Let, together with (2.2), the following conditions be satisfied

$$\begin{gathered}\varphi_\infty := \max\Big\{\sup_{x\in\mathbb{R}}|\varphi(x)| < +\infty,\ \ \sup_{x\in\mathbb{R}}|\varphi'(x)| < +\infty\Big\},\\ \psi_\infty := \sup_{x\in\mathbb{R}}|\psi(x)| < +\infty,\end{gathered} \tag{4.1}$$

$$f_\infty := \sup_{(x,t)\in\overline{\Omega}}|f(x,t)| < +\infty \tag{4.2}$$

and

$$|g(x,t,s)| \le M(R), \quad |g(x,t,s_2) - g(x,t,s_1)| \le c(R)|s_2 - s_1| \tag{4.3}$$

for all $(x,t) \in \overline{\Omega}$, $|s|,|s_1|,|s_2| \le R$, where $M(R)$ and $c(R)$ are some nonnegative continuous functions of the argument $R \ge 0$.

Theorem 4.1. *Let the functions f, g, φ and ψ satisfy conditions (2.2), (4.1)-(4.3). Then there exists a positive number $t_* := t_*(f,g,\varphi,\psi)$, such that, for $t_0 \le t_*$ problem (1.1), (1.3) has at least one strong generalized solution u of class C^1 in the domain D_{P_0}.*

Proof. The unique solvability of system (2.21) we will proved below by the principle of contracted mappings [see e.g., 22, p. 390].

Let $V := (v_1, v_2, v_3)$. Let us introduce the vector-operator $\Phi := (\Phi_1, \Phi_2, \Phi_3)$, by the formula

$$\begin{cases} (\Phi_1 V)(x,t) = -\frac{1}{2}\int_0^t g\big(P_\tau, v_3(P_\tau)\big)\big(v_1(P_\tau) + v_2(P_\tau)\big)d\tau + F_1(x,t),\\ (\Phi_2 V)(x,t) = -\frac{1}{2}\int_0^t g\big(Q_\tau, v_3(Q_\tau)\big)\big(v_1(Q_\tau) + v_2(Q_\tau)\big)d\tau + F_2(x,t),\\ (\Phi_3 V)(x,t) = \int_0^t v_1(Q_\tau)d\tau + F_3(x,t). \end{cases} \tag{4.4}$$

Then system (2.21) can be rewritten in the vector form

$$V = \Phi V. \tag{4.5}$$

Let

$$\|V\|_{X_{P_0}} := \max_{1\le i\le 3}\big\{\|v_i\|_{C(\overline{D}_{P_0})}\big\}, \quad V \in X_{P_0} := C(\overline{D}_{P_0};\mathbb{R}^3),$$

where $C(\overline{D}_{P_0};\mathbb{R}^3)$ is the set of all continuous vector-functions $V : \overline{D}_{P_0} \to \mathbb{R}^3$.

Denote by $B_R := \{V \in X_{P_0} : \|V\|_{X_{P_0}} \le R\}$ the closed ball with radius $R > 0$ in the Banach space X_{P_0} and the center at the null element.

Below we prove that for $R := 1 + \varphi_\infty + \psi_\infty$

(I) Φ maps the ball B_R into itself;

(II) Φ is a contracted map on B_R.

Indeed, in view of the first inequality (4.3), by (4.4) due to (2.22), for V such that $\|V\|_{X_{P_0}} \le R$, we have

$$|(\Phi_1 V)(x,t)| \le t_0\big(RM(R) + \|f\|_{C(\overline{D}_{P_0})}\big) + \|\varphi'\|_{C(\gamma_{P_0})} + \|\psi\|_{C(\gamma_{P_0})},$$

$$|(\Phi_2 V)(x,t)| \le t_0\big(RM(R) + \|f\|_{C(\overline{D}_{P_0})}\big) + \|\varphi'\|_{C(\gamma_{P_0})} + \|\psi\|_{C(\gamma_{P_0})},$$
$$|(\Phi_3 V)(x,t)| \le t_0 R + \|\varphi\|_{C(\gamma_{P_0})}.$$

By these estimates it follows that

$$\|\Phi V\|_{X_{P_0}} \le t_0\big(RM(R) + R + \|f\|_{C(\overline{D}_{P_0})}\big) + \|\varphi\|_{C(\gamma_{P_0})} + \|\varphi'\|_{C(\gamma_{P_0})}$$
$$+\|\psi\|_{C(\gamma_{P_0})} \le t_0\big(RM(R) + R + f_\infty\big) + \varphi_\infty + \psi_\infty,$$

where f_∞, φ_∞, ψ_∞ are defined by (4.1) and (4.2).

Now for $R := 1 + \varphi_\infty + \psi_\infty$ let us choose sufficiently small $t_0 > 0$, so that

$$t_0\big(RM(R) + R + f_\infty\big) \le 1, \tag{4.6}$$

hence $\Phi V \in B(R)$ and therefore condition (I) is satisfied.

Further, due to (4.3) by (4.4) for V^i such that $\|V^i\|_{X_{P_0}} \le R,\ i = 1, 2$, we have

$$|(\Phi_1 V^2 - \Phi_1 V^1)(x,t)| \le \frac{1}{2}\int_0^t \Big(|g(P_\tau, v_3^2(P_\tau)) - g(P_\tau, v_3^1(P_\tau))||v_1^2(P_\tau) + v_2^2(P_\tau)|$$
$$+|g(P_\tau, v_3^1(P_\tau))||v_1^2(P_\tau) - v_1^1(P_\tau) + v_2^2(P_\tau) - v_2^1(P_\tau)|\Big)d\tau$$
$$\le t_0\big(Rc(R) + M(R)\big)\|V^2 - V^1\|_{X_{P_0}}.$$

Analogously,

$$|(\Phi_2 V^2 - \Phi_2 V^1)(x,t)| \le t_0\big(Rc(R) + M(R)\big)\|V^2 - V^1\|_{X_{P_0}}$$

and

$$|(\Phi_3 V^2 - \Phi_3 V^1)(x,t)| \le \int_0^t |v_1^2(Q_\tau) - v_1^1(Q_\tau)|d\tau \le t_0\|V^2 - V^1\|_{X_{P_0}}.$$

By decreasing the number $t_0 > 0$ we can assume that together with (4.6) the following inequality

$$\max\ \big\{t_0, t_0\big(Rc(R) + M(R)\big)\big\} \le \frac{1}{2} < 1, \tag{4.7}$$

is valid and hence

$$\|\Phi V^2 - \Phi V^1\|_{X_{P_0}} \le \frac{1}{2}\|V^2 - V^1\|_{X_{P_0}}.$$

Thus the operator Φ is a contracted mapping on the set $B(R)$, hence condition (II) is satisfied.

By (4.6) and (4.7) it follows that if $0 < t_0 \le t_*$, where

$$t_* := \min\left\{\frac{1}{RM(R) + R + f_\infty}, \frac{1}{2}, \frac{1}{2\big(Rc(R) + M(R)\big)}\right\}, \tag{4.8}$$

then

$$\|\Phi V\|_{X_{P_0}} \le R$$

and

$$\|\Phi V^2 - \Phi V^1\|_{X_{P_0}} \le \frac{1}{2}\|V^2 - V^1\|_{X_{P_0}}$$

for $V, V^1, V^2 \in B(R)$. Therefore due to the principle of contracted mappings there exists a solution V of equation (4.5) in the space X_{P_0} for $t_0 \in (0, t_*]$. $\square$

In view of the local solvability of problem (1.1), (1.3) and the global a priori estimate (2.15), using standard considerations [see e.g., 20], we get the validity of the following theorem on the global solvability of this problem.

Theorem 4.2. *If the conditions (2.1), (2.2), (4.1)-(4.3) are satisfied, then problem (1.1), (1.3) is globally solvable in the class C^1 in the sense of Definition 1.2, i.e. for any $P_0 \in \Omega$ this problem has a strong generalized solution of class C^1 in the domain D_{P_0}.*

Remark 4.1. We give the examples of functions $g = g(x,t,s)$, satisfying the conditions of Theorem 4.2, or, which is the same, conditions (2.1), $g \in C(\overline{\Omega} \times \mathbb{R})$ and (4.3). The function

$$g(x,t,s) = \sum_{k=1}^{n} \alpha_k(x,t)|s|^{\beta_k},$$

where $\alpha_k \in C(\overline{\Omega}),\ k = 1,...,n;\ \alpha_1(x,t) \geq const > 0,\ |\alpha_i(x,t)| \leq const$ for $(x,t) \in \overline{\Omega}$ and $\beta_1 > \beta_i \geq 1,\ i = 2,...,n$; and also the function $g(x,t,s) = \alpha(x,t)g_0(s)$, where $\alpha \in C(\overline{\Omega}),\ |\alpha(x,t)| \leq const$ for $(x,t) \in \overline{\Omega},\ g_0 \in Lip_{loc}(\mathbb{R})$ and $\liminf_{|s|\to+\infty} g_0(s) > -\infty$, satisfy the conditions of Theorem 4.2.

5 The uniqueness and the existence theorems

Theorem 5.1. *Let conditions (2.2) and (4.3) be satisfied. Then for any fixed point $P_0 \in \Omega$ problem (1.1), (1.3) cannot have more than one strong generalized solution of the class C^1 in the domain D_{P_0}.*

Proof. Indeed, suppose that problem (1.1), (1.3) has two different strong generalized solutions u^1 and u^2 of class C^1 in the domain D_{P_0}. Then according to Definition 1.1 there exists a sequence of functions $u_n^i \in C^2(\overline{D}_{P_0})$, such that

$$\begin{gathered}\lim_{n\to\infty} \|u_n^i - u^i\|_{C^1(\overline{D}_{P_0})} = 0, \quad \lim_{n\to\infty} \|Lu_n^i - f\|_{C(\overline{D}_{P_0})} = 0,\\ \lim_{n\to\infty} \|u_n^i(\cdot,0) - \varphi\|_{C^1(\gamma_{P_0})} = 0, \quad \lim_{n\to\infty} \|u_{nt}^i(\cdot,0) - \psi\|_{C(\gamma_{P_0})} = 0,\\ \lim_{n\to\infty} \|g(x,t,u_n^i)u_{nt}^i - g(x,t,u^i)u_t^i\|_{C(\overline{D}_{P_0})} = 0, \quad i = 1,2.\end{gathered} \tag{5.1}$$

Let us use the well-known notation $\Box := \partial^2/\partial t^2 - \partial^2/\partial x^2$ and let $\omega_n := u_n^2 - u_n^1$. It is easy to see that the function $\omega_n \in C^2(\overline{D}_{P_0})$ satisfies the following equalities

$$\Box\, \omega_n + g_n = f_n, \tag{5.2}$$

$$\omega_n\big|_{\gamma_{P_0}} = \tau_n, \quad \omega_{nt}\big|_{\gamma_{P_0}} = \nu_n, \tag{5.3}$$

where

$$\begin{gathered}g_n := g(x,t,u_n^2)u_{nt}^2 - g(x,t,u_n^1)u_{nt}^1, \quad f_n := Lu_n^2 - Lu_n^1,\\ \tau_n := (u_n^2 - u_n^1)\big|_{\gamma_{P_0}}, \quad \nu_n := (u_n^2 - u_n^1)_t\big|_{\gamma_{P_0}}.\end{gathered} \tag{5.4}$$

Due to the first equality of (5.1) there exists the number $A := const > 0$, independent of indices i and n, such that

$$\|u_n^i\|_{C^1(\overline{D}_{P_0})} \leq A. \tag{5.5}$$

By virtue of equalities (5.1) and (5.4) we have

$$\lim_{n\to\infty}\|\tau_n\|_{C^1(\gamma_{P_0})}=0,\quad \lim_{n\to\infty}\|\nu_n\|_{C(\gamma_{P_0})}=0,\quad \lim_{n\to\infty}\|f_n\|_{C(\overline{D}_{P_0})}=0. \tag{5.6}$$

By (4.3), (5.5) and the first equality of (5.4) it is easy to see that

$$\begin{aligned} g_n^2=\Big(g(x,t,u_n^2)\omega_{nt}+\big(g(x,t,u_n^2)-g(x,t,u_n^1)\big)u_{nt}^1\Big)^2\\ \le 2M^2(A)\omega_{nt}^2+2A^2c^2(A)\omega_n^2. \end{aligned} \tag{5.7}$$

Multiplying both sides of the equality (5.2) by ω_{nt} and integrating the obtained equality over the domain $D_{P_0',\tau}$, $0<\tau<t_0'$, where $P_0':=P_0'(x_0',t_0')\in D_{P_0}$, in view of (5.3), in the same way as getting equality (2.9) from (2.6), (2.7), we have

$$\begin{aligned} v_n(\tau):=\int\limits_{\Omega_{P_0',\tau}}(\omega_{nx}^2+\omega_{nt}^2)dx+\sum_{i=1}^2\int\limits_{\gamma_{i,P_0',\tau}}\nu_t^{-1}\big(\nu_t\omega_{nx}-\nu_x\omega_{nt}\big)^2ds\\ =2\int\limits_{D_{P_0',\tau}}(f_n-g_n)\omega_{nt}dxdt+\|\tau_n'\|^2_{L_2(\gamma_{P_0'})}+\|\nu_n\|^2_{L_2(\gamma_{P_0'})}. \end{aligned} \tag{5.8}$$

Due to estimate (5.7) and the Cauchy inequality we get

$$\begin{aligned} 2\int\limits_{D_{P_0',\tau}}(f_n-g_n)\omega_{nt}dxdt\le 2\int\limits_{D_{P_0',\tau}}\omega_{nt}^2dxdt+\int\limits_{D_{P_0,\tau}}f_n^2dxdt+\int\limits_{D_{P_0',\tau}}g_n^2dxdt\\ \le 2\big(1+M^2(A)\big)\int\limits_{D_{P_0',\tau}}\omega_{nt}^2dxdt+\|f_n\|^2_{L_2(D_{P_0,\tau})}+2A^2c^2(A)\int\limits_{D_{P_0',\tau}}\omega_n^2dxdt. \end{aligned} \tag{5.9}$$

Further, according to the first equality (5.3) it is easy to see that

$$\omega_n(x,t)=\tau_n(x)+\int\limits_0^t\omega_{nt}(x,\sigma)d\sigma,\quad (x,t)\in\overline{D}_{P_0',\tau}.$$

Squaring both sides of this equality and using the Cauchy-Schwartz inequality, we get

$$|\omega_n(x,t)|^2\le 2\tau_n^2(x)+2t\int_0^t\omega_{nt}^2(x,\sigma)d\sigma,\quad (x,t)\in\overline{D}_{P_0',\tau}.$$

Whence putting

$$v(x,t)=\begin{cases}\omega_{nt}(x,t), & (x,t)\in\overline{D}_{P_0',\tau},\\ 0, & (x,t)\notin\overline{D}_{P_0',\tau},\end{cases}$$

and taking into account that $t \leq \tau$ for $(x,t) \in \overline{D}_{P_0',\tau}$, we obtain

$$\int\limits_{D_{P_0',\tau}} \omega_n^2 dxdt \leq 2\tau \|\tau_n\|^2_{L_2(\gamma_{P_0})} + 2\tau \int_{x_0'-t_0'}^{x_0'+t_0'} dx \int_0^\tau \Big(\int_0^\tau v^2(x,\sigma)d\sigma \Big) dt$$

$$= 2\tau \|\tau_n\|^2_{L_2(\gamma_{P_0})} + 2\tau^2 \int_{x_0'-t_0'}^{x_0'+t_0'} dx \int_0^\tau v^2(x,t)dt \tag{5.10}$$

$$= 2\tau \|\tau_n\|^2_{L_2(\gamma_{P_0})} + 2\tau^2 \int\limits_{D_{P_0',\tau}} \omega_{nt}^2 dxdt.$$

By (5.8)-(5.10) it follows that

$$v_n(\tau) \leq 2\big(2\tau^2 A^2 c^2(A) + M^2(A) + 1\big) \int\limits_{D_{P_0',\tau}} \omega_{nt}^2 dxdt + \|f_n\|^2_{L_2(D_{P_0,\tau})} + \|\tau_n'\|^2_{L_2(\gamma_{P_0})}$$

$$+\|\nu_n\|^2_{L_2(\gamma_{P_0})} + 4\tau A^2 c^2(A) \|\tau_n\|^2_{L_2(\gamma_{P_0})} \leq 2\big(2t_0^2 A^2 c^2(A) + M^2(A) + 1\big) \int\limits_0^\tau v_n(\sigma)d\sigma$$

$$+\|f_n\|^2_{L_2(D_{P_0})} + \|\tau_n'\|^2_{L_2(\gamma_{P_0})} + \|\nu_n\|^2_{L_2(\gamma_{P_0})} + 4t_0 A^2 c^2(A) \|\tau_n\|^2_{L_2(\gamma_{P_0})}.$$

Therefore, due to the Gronwall lemma we get

$$v_n(\tau) \leq c_2\big(\|f_n\|^2_{L_2(D_{P_0})} + \|\tau_n'\|^2_{L_2(\gamma_{P_0})} + \|\nu_n\|^2_{L_2(\gamma_{P_0})} + 4t_0 A^2 c^2(A) \|\tau_n\|^2_{L_2(\gamma_{P_0})}\big),$$

where $c_2 := \exp\big\{2t_0(2t_0^2 A^2 c^2(A) + M^2(A) + 1)\big\}$. Whence, taking into account (5.8), we have

$$\int\limits_{\gamma_{1,P_0'}} (\nu_t \omega_{nx} - \nu_x \omega_{nt})^2 ds \leq \sqrt{2} c_2\big(\|f_n\|^2_{L_2(D_{P_0})} + \|\tau_n'\|^2_{L_2(\gamma_{P_0})} + \|\nu_n\|^2_{L_2(\gamma_{P_0})}$$

$$+4t_0 A^2 c^2(A) \|\tau_n\|^2_{L_2(\gamma_{P_0})}\big).$$

Further, by the same considerations as those used for getting estimate (2.14), we find

$$|\omega_n(x_0', t_0')|^2 \leq 2\tau_n^2(x_0' - t_0') + 4t_0^2 c_2 \Big(t_0 \|f_n\|^2_{C(\overline{D}_{P_0})} + 2\|\tau_n'\|^2_{C(\gamma_{P_0})}$$

$$+2\|\nu_n\|^2_{C(\gamma_{P_0})} + 8t_0 A^2 c^2(A) \|\tau_n\|^2_{C(\gamma_{P_0}}\Big).$$

Therefore, in view of (5.6) we have $\lim\limits_{n\to\infty} |\omega_n(x_0', t_0')| = 0$, i.e. $u^2(x_0', t_0') = u^1(x_0', t_0')$ for any $(x_0', t_0') \in \overline{D}_{P_0}$. □

By Theorems 4.2 and 5.1 the following theorem immediately follows.

Theorem 5.2. *If conditions (2.1), (2.2), (4.1)-(4.3) are satisfied, then problem (1.1), (1.2) has a unique global strong generalized solution of class C^1 in the sense of Definition 1.3.*

Proof. By Theorems 4.2 and 5.1, in the domain D_{P_0} for $t_0 = k \in \mathbb{N}$ problem (1.1), (1.3) has a unique strong generalized solution u^k of class C^1 in the sense of Definition 1.1. Since u^{k+1} is also a strong generalized solution to problem (1.1), (1.3) of class C^1 in the domain $D_{x_0,k}$, due to Theorem 5.1 we have $u^{k+1}\big|_{D_{x_0,k}} = u^k$. Therefore the function u, built in the domain Ω according to the rule $u(x,t) = u^k(x,t)$ for $k = [t]+1$, where $[t]$ is the integer part of the number t, and the point $(x,t) \in \Omega$, will be a unique global strong generalized solution of problem (1.1), (1.2) of class C^1 in the sense of Definition 1.3. □

Remark 5.1. Under the conditions of Theorem 4.1 there exists the positive number $T_* := T_*(f,g,\varphi,\psi) > 0$, such that problem (1.1), (1.2) in the strip $\Omega_1 := \mathbb{R} \times (0,T_*)$ has a unique strong generalized solution u of class C^1 in the domain Ω_1, in the sense that for any point $P_0 \in \Omega_1$ the function $u\big|_{D_{P_0}}$ represents a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0} in the sense of Definition 1.1. The proof of this statement immediately follows by the uniqueness Theorem 5.1, Theorem 4.1 and considerations analogous to those given in the proof of Theorem 5.2.
Remark 5.2. From the proofs of Theorems 4.1, 4.2 and 5.2 it easily follows that they remain valid without conditions (4.1), (4.2), under only the conditions of smoothness: $\varphi \in C^1(\mathbb{R}),\ \psi \in C(\mathbb{R}),\ f \in C(\overline{\Omega})$.

6 The case of absence of a global solution of the problem (1.1), (1.3)

Remark 6.1. Violation of condition (2.1), generally speaking, may cause an absence of global solvability of problem (1.1), (1.3) in the sense of Definition 1.2. Indeed, let $g(x,t,s) = -|s|^\alpha s,\ (x,t) \in \overline{\Omega},\ s \in \mathbb{R}$ with the exponent of nonlinearity $\alpha > -1$. Below we show that under certain conditions on the functions $f \in C(\overline{\Omega}),\ \varphi \in C^1(\mathbb{R}),\ \psi \in C(\mathbb{R})$ for any fixed $x_0 \in \mathbb{R}$ there exists a number $t^* := t^*(x_0; f,\varphi,\psi) > 0$, such that for $t_0 \in (0,t^*)$ problem (1.1), (1.3) has a strong generalized solution of the class C^1 in the domain D_{P_0}, while for $t_0 > t^*$ it does not have such a solution in this domain.

Lemma 6.1. *Let u be a strong generalized solution of problem (1.1), (1.3) of class C^1 in the domain D_{P_0} in the sense of Definition 1.1. Then the following integral equality*

$$\begin{aligned}\int\limits_{D_{P_0}} u\Box\chi dxdt = \int_{x_0-t_0}^{x_0+t_0} [\psi(x)\chi(x,0) - \varphi(x)\chi_t(x,0)]dx \\ + \int\limits_{D_{P_0}} |u|^\alpha u u_t \chi dxdt + \int\limits_{D_{P_0}} f\chi dxdt\end{aligned} \tag{6.1}$$

is valid for any function χ, such that

$$\chi \in C^2(\overline{D}_{P_0}), \quad \chi\big|_{\gamma_{i,P_0}} = 0, \quad i = 1,2. \tag{6.2}$$

Proof. According to the definition of a strong generalized solution u of problem (1.1), (1.3) of class C^1 in the domain D_{P_0}, the function $u \in C^1(\overline{D}_{P_0})$ and there exists a sequence of functions $u_n \in C^2(\overline{D}_{P_0})$, such that it is valid equalities (2.4) and (2.5) for $g(x,t,s) = -|s|^\alpha s,\ (x,t) \in \overline{\Omega},\ s \in \mathbb{R}$.

Let $f_n := Lu_n$. Multiply both sides of the equality $Lu_n = f_n$ by function χ and integrate the obtained equality over the domain D_{P_0}. By integrating by parts of the left-hand side of this equality, due to (6.2) and conditions (2.7), we have

$$\int\limits_{D_{P_0}} u_n \Box \chi dxdt = \int_{x_0-t_0}^{x_0+t_0} [\psi_n(x)\chi(x,0) - \varphi_n(x)\chi_t(x,0)]dx$$

$$+ \int\limits_{D_{P_0}} |u_n|^\alpha u_n u_{nt} \chi dxdt + \int\limits_{D_{P_0}} f_n \chi dxdt.$$

Passing in the last equality to the limit as $n \to \infty$, due to (2.4), we get (6.1). $\square$

Let us use the method of test functions (see, e.g., [16], pp. 10-12). Consider the function $\chi^0 := \chi^0(x,t)$, such that

$$\chi^0 \in C^2(\overline{D}_{(0,1)}), \quad \chi^0 + \chi^0_t \le 0, \quad \chi^0\big|_{D_{(0,1)}} > 0, \quad \chi^0\big|_{\gamma_{i,(0,1)}} = 0, \quad i = 1,2, \tag{6.3}$$

и

$$\kappa_0 := \int\limits_{D_{(0,1)}} \frac{|\Box \chi^0|^{p'}}{|\chi^0|^{p'-1}} dxdt < +\infty, \quad p' = \frac{\alpha+2}{\alpha+1}. \tag{6.4}$$

It is easy to verify that for the function χ^0, satisfying conditions (6.3) and (6.4), one may consider the function

$$\chi^0 = \chi^*(x,t) := [(1-t)^2 - x^2]^n, \quad (x,t) \in \overline{D}_{(0,1)}, \tag{6.5}$$

for sufficiently large natural number n.

Now, putting $\chi_{P_0}(x,t) = \chi^0\big(\frac{x-x_0}{t_0}, \frac{t}{t_0}\big)$, in view of (6.3), it is easy to see that

$$\begin{gathered} \chi_{P_0} \in C^2(\overline{D}_{P_0}), \ \ \chi_{P_0} + t_0 \frac{\partial \chi_{P_0}}{\partial t} \le 0, \ \ \chi_{P_0}\big|_{D_{P_0}} > 0, \\ \chi_{P_0}\big|_{\gamma_{i,P_0}} = 0, \ \ i = 1,2. \end{gathered} \tag{6.6}$$

For fixed functions f, φ, ψ and a number x_0, consider the following function of one variable t_0

$$\begin{aligned} \zeta(t_0) :=& \int_{x_0-t_0}^{x_0+t_0} \left[\psi(x)\chi_{P_0}(x,0) - \varphi(x)\frac{\partial \chi_{P_0}(x,0)}{\partial t}\right]dx \\ &+ \int\limits_{D_{P_0}} f\chi_{P_0} dxdt - \frac{1}{\alpha+2}\int_{x_0-t_0}^{x_0+t_0} |\varphi(x)|^{\alpha+2}\chi_{P_0}(x,0)dx. \end{aligned} \tag{6.7}$$

The following theorem on the absence of global solvability of problem (1.1), (1.3) is valid.

Theorem 6.1. *Let* $g(x,t,s) = -|s|^\alpha s,\ (x,t) \in \overline{\Omega},\ s \in \mathbb{R},\ \alpha > -1,\ f \in C(\overline{\Omega})$, *and the function* $u \in C^1(\overline{D}_{P_0})$ *be a strong generalized solution of problem (1.1), (1.3) of class* C^1 *in the domain* D_{P_0}. *If*

$$\liminf_{t_0 \to +\infty} \zeta(t_0) > 0, \tag{6.8}$$

then there exists a positive number $t^0 := t^0(x_0; f, \varphi, \psi) > 0$, *such that for* $t_0 > t^0$ *problem (1.1), (1.3) cannot have a strong generalized solution of class* C^1 *in the domain* D_{P_0}.

Proof. Suppose that under the conditions of this theorem there exists a strong generalized solution u of problem (1.1), (1.3) of class C^1 in the domain D_{P_0}. Then due to Lemma 6.1 we have equality (6.1), in which, in view of (6.6), for χ one may take the function $\chi = \chi_{P_0}$, i.e.

$$\int\limits_{D_{P_0}} u\Box\chi_{P_0}dxdt = \int\limits_{D_{P_0}} |u|^\alpha u u_t \chi_{P_0}dxdt + \int_{x_0-t_0}^{x_0+t_0} \Big[\psi(x)\chi_{P_0}(x,0) - \varphi(x)\frac{\partial \chi_{P_0}(x,0)}{\partial t}\Big]dx + \int\limits_{D_{P_0}} f\chi_{P_0}dxdt. \tag{6.9}$$

Taking into account (1.3) and (6.6), we have

$$\int\limits_{D_{P_0}} |u|^\alpha u u_t \chi_{P_0}dxdt = \frac{1}{\alpha+2}\int\limits_{D_{P_0}} \chi_{P_0}\frac{\partial}{\partial t}|u|^{\alpha+2}dxdt$$

$$= -\frac{1}{\alpha+2}\Big(\int_{x_0-t_0}^{x_0+t_0} |\varphi(x)|^{\alpha+2}\chi_{P_0}(x,0)dx + \int\limits_{D_{P_0}} |u|^{\alpha+2}\frac{\partial\chi_{P_0}}{\partial t}dxdt\Big)$$

$$\geq \frac{1}{\alpha+2}\Big(\frac{1}{t_0}\int\limits_{D_{P_0}} |u|^{\alpha+2}\chi_{P_0}dxdt - \int_{x_0-t_0}^{x_0+t_0} |\varphi(x)|^{\alpha+2}\chi_{P_0}(x,0)dx\Big).$$

Whence, due to (6.7), from (6.9) it follows that

$$\frac{1}{pt_0}\int\limits_{D_{P_0}} |u|^p\chi_{P_0}dxdt \leq \int\limits_{D_{P_0}} u\Box\chi_{P_0}dxdt - \zeta(t_0), \quad p := \alpha + 2 > 1. \tag{6.10}$$

If in the Young inequality with the parameter $\varepsilon > 0$

$$ab \leq \frac{\varepsilon}{p}a^p + \frac{1}{p'\varepsilon^{p'-1}}b^{p'}; \quad a, b \geq 0, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad p > 1$$

we take $a = |u|\chi_{P_0}^{\frac{1}{p}}, \quad b = \frac{|\Box\chi_{P_0}|}{\chi_{P_0}^{\frac{1}{p}}}$ and $\varepsilon = \frac{1}{t_0}$, then we get

$$|u\Box\chi_{P_0}| = |u|\chi_{P_0}^{\frac{1}{p}}\frac{|\Box\chi_{P_0}|}{\chi_{P_0}^{\frac{1}{p}}} \leq \frac{1}{pt_0}|u|^p\chi_{P_0} + \frac{t_0^{p'-1}}{p'}\frac{|\Box\chi_{P_0}|^{p'}}{\chi_{P_0}^{p'-1}}.$$

By virtue of (6.10) and the last inequality we have

$$0 \leq \frac{t_0^{p'-1}}{p'} \int\limits_{D_{P_0}} \frac{|\Box \chi_{P_0}|^{p'}}{\chi_{P_0}^{p'-1}} dxdt - \zeta(t_0). \tag{6.11}$$

By (6.3), (6.4), after the following change of variables $x = x_0 + t_0 x_1$, $t = t_0 t_1$, it is easy to verify that

$$\int\limits_{D_{P_0}} \frac{|\Box \chi_{P_0}|^{p'}}{\chi_{P_0}^{p'-1}} dxdt = \frac{1}{t_0^{2(p'-1)}} \int\limits_{D_{(0,1)}} \frac{|\Box \chi^0|^{p'}}{|\chi^0|^{p'-1}} dx_1 dt_1 = \frac{\kappa_0}{t_0^{2(p'-1)}}.$$

Whence due to (6.11) we get

$$0 \leq \frac{\kappa_0}{p' t_0^{p'-1}} - \zeta(t_0). \tag{6.12}$$

Since $p' = \frac{p}{p-1} > 1$ and due to (6.4) we have

$$\lim_{t_0 \to +\infty} \frac{\kappa_0}{p' t_0^{p'-1}} = 0.$$

Therefore in view of (6.8) there exists a positive number $t^0 := t^0(x_0; f, \varphi, \psi) > 0$, such that for $t_0 > t^0$ the right-hand side of inequality (6.12) is negative, whereas the left-hand side of this inequality is zero. This implies that if there exists a strong generalized solution u of problem (1.1), (1.3) of class C^1 in the domain D_{P_0}, then $t_0 \leq t^0$ necessarily, and this proves Theorem 6.1. $\square$

Remark 6.2. In Remark 6.1 let us denote by $t^* := t^*(x_0; f, \varphi, \psi)$ the supremum of those $t_0 > 0$, for which problem (1.1), (1.3) is solvable in the domain D_{P_0}. By Theorems 4.1 and 6.1 it follows that $0 < t^* \leq t^0$, and that problem (1.1), (1.3) is solvable in the domain D_{P_0} for $t_0 < t^*$ and does not have a solution for $t_0 > t^*$.

Remark 6.3. It is easy to verify that if $\varphi \equiv 0$, $f \geq 0$, $\psi \geq 0$ and one of the following conditions:

$$1)\ f(x,t) \geq c,\ (x,t) \in \overline{\Omega}; \quad 2)\ \psi(x) \geq c,\ x \in \mathbb{R}, \tag{6.13}$$

is satisfied, where $c := const > 0$, and for function χ_{P_0} we take $\chi_{P_0}(x,t) = \chi^*\big(\frac{x-x_0}{t_0}, \frac{t}{t_0}\big)$, where χ^* is defined by equality (6.5), then condition (6.8) will be satisfied, and therefore in this case problem (1.1), (1.3) for sufficiently large t_0 will not have a strong generalized solution u of class C^1 in the domain D_{P_0}.

Indeed, by considering in the first integral in (6.7) the transformation of the independent variable $x = x_0 + t_0 \tau$, in the case in which, for example, the second of conditions (6.13) is satisfied, after some transformations we have

$$\begin{aligned} \zeta(t_0) &\geq \int_{x_0-t_0}^{x_0+t_0} \psi(x) \chi_{P_0}(x,0) dx = t_0 \int_{-1}^{1} \psi(x_0 + t_0 \tau) \chi^*(\tau, 0) d\tau \\ &\geq c t_0 \int_{-1}^{1} (1-\tau^2)^n d\tau = 2 c t_0 \int_0^1 (1-\tau^2)^n d\tau = c t_0 B(2^{-1}, n+1) > 0, \end{aligned} \tag{6.14}$$

where $B(a,b)$ is the well-known Euler integral of the first kind (see e.g., [2], p. 750). By (6.14) immediately follows the validity of inequality (6.8). Analogously is considered the case in which the first condition in (6.13) is satisfied.

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