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ON INTERPOLATION OF PAIRS OF GENERALIZED SPACES OF BESOV TYPE

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Abstract. We investigate generalized spaces of Besov type defined with the help of certain positive infinitely differentiable functions of polynomial growth and describe the K-functionals for pairs of $H-$spaces and for the pairs of B-spaces. We prove interpolation theorems for spaces with different anisotropy. Spaces of functions of mixed smoothness are characterized as "B-products" and interpolation theorems for these spaces are proved. Moreover, we establish embedding and trace theorems.

Introduction

The aim of this paper is to define and investigate the spaces $B^s_{p,q}(\mu; R_n)$ of Besov type defined with the help of *a* positive infinitely differentiable function μ with polynomial growth. If

$$\lim_{|\xi|\to\infty} \mu(\xi) = \infty \tag{5.1}$$

then we use the decomposition method (see [B-2, B-3]) to define the spaces $B^s_{p,q}(\mu; R_n)$ as in the classical case (see [T-2, T-3, B-L]). However, in order to investigate problems such as interpolation of the pairs of spaces with different anisotropy and the problem of traces, we have to define and study the spaces $B^s_{p,q}(\mu; R_n)$ with a function μ for which (5.1) does not hold. In this case we use an interpolation approach to define and investigate the appropriate B-spaces.

The crucial role here plays the property of quasi-linearizability of the pair $\left\{H^1_p(\mu_0), H^1_p(\mu_1)\right\}$ of spaces of Sobolev-Liouville type (see [B-1]). These spaces were investigated in [V-P] and [T-1]. Then we define B-spaces of Besov type as the real interpolation spaces between the appropriate spaces of H-type and prove interpolation formulas and embedding theorems for the spaces of B-type defined in this way. The introduced spaces have a number of important particular cases (see Section 2).

One of the goals of this paper is to prove interpolation theorems for the pairs of H and for the pairs of B-spaces with different anisotropy. This problem was formulated by H. Triebel in [T-3]. For the pairs of H-spaces the appropriate formula for the real method was proved in [B-4] using the property of quasi-linearizability. For the complex method an interpolation theorem for the pairs of H-spaces was proved in [T-1].

Considering the problem of interpolation for B-spaces with different anisotropy we note that the norm of the interpolation space is equivalent to a certain "mixture" of two norms of two spaces of B-type. Spaces of such type we call "B-products". These spaces ("B-products") have certain interesting properties which are established in Theorem 5.1 and its Corollaries.

In terms of "B-products" we prove interpolation formulas for the pairs of H and for the pairs of B-spaces with different anisotropy (see Theorems 5.2, 5.3, 6.4). In the considered case the pairs of H and the pairs of B-spaces have, in contrast to classical isotropic case, in general different real interpolation spaces.

Further we consider spaces of functions of mixed smoothness (see [L-N, Sc-T, Sc-S]) as "B-products". We give certain characterizations of these spaces and prove interpolation formulas. Applying the interpolation theorems we investigate the trace operator. We consider the H-space generated by a certain convex polyhedron $\Re$ and prove that the trace operator is a retract from this H-space onto the appropriate interpolation space, which is already characterized. It is remarkable that the space of traces depends only on a part of the polyhedron $\Re$. The rest of the polyhedron $\Re$ does not play any role.

We shall use the following notation:

N_0 – the set of nonnegative integers,

R_n – the Euclidean n-space,

$Z_n^+ = \underbrace{N_0 \times \ldots \times N_0}_{n}$ – the set of multi-indices,

S – the Schwartz space,

S' – its topological dual space,

M_p – the space of (p,p)-type Fourier multipliers,

C – the space of uniformly continuous and bounded functions with the standard norm,

F – the operator of the Fourier transform,

F^{-1} – its inverse,

the symbol $f \sim g$ means that $\exists c_1, c_2 > 0$ such that $c_1 g \leqslant f \leqslant c_2 g$ (f, g are positive functions),

the symbol $A_0 \subset A_1$ means that the Banach space A_0 is continuously embedded in the Banach space A_1,

the symbol $T : A_0 \to A_1$ means that T is a continuous operator from A_0 to A_1,

$L(A_0, A_1)$ – the space of linear and bounded operators from A_0 to A_1,

$A_0 + A_1$ – the sum of the Banach spaces A_0, A_1 in the sense of interpolation theory,

$A_0 \bigcap A_1$ – the intersection of the Banach spaces A_0, A_1 in the sense of interpolation theory,

$(A_0, A_1)_{\theta,q}$ – the real interpolation space,

$[A_0, A_1]_\theta$ – the complex interpolation space .

We shall assume that the same letter c may denote different constants.

1 Spaces with fixed anisotropy. The decomposition method

Let a convex polyhedron $\Re$ (with verticies $(0,\dots,0)$,$\alpha^j = \left(\alpha_1^j,\dots,\alpha_n^j\right) \in Z_n^+, j = 1,..,M$) be such that $\Re$ has verticies different from $(0,\dots,0)$ on all co-ordinate axes. We denote

$$\mu(\xi) = \left(\sum_{j=1}^{M}\prod_{i=1}^{n}|\xi_i|^{2\alpha_i^j}\right)^{1/2}. \tag{1.1}$$

Definition 1.1. Let $1 < p < \infty$. We define $\Phi(\mu, R^n)$ as the collection of all systems $\{\varphi_k\}_{k=0}^{\infty}$ with the following properties:
(i) $\varphi_k \in S(R^n)$, $(F\varphi_k)(\xi) \geqslant 0$, $k = 0, 1, \dots,$
(ii) $\mathrm{supp}F\varphi_k \subset \Omega_k \equiv \left\{\xi \in R^n; 2^{k-1} \leqslant \mu(\xi) \leqslant 2^{k+1}\right\}$, $k = 1, \dots,$
$\mathrm{supp}F\varphi_0 \subset \Omega_0 \equiv \{\xi \in R^n; \mu(\xi) \leqslant 2\}$,
(iii) $\exists c_1 > 0$ such that $\sum\limits_{k=0}^{\infty}(F\varphi_k)(\xi) \geqslant c_1$, $\xi \in R^n$,
(iv) $\exists c_2 > 0$ such that $|\xi^\gamma D^\gamma(F\varphi_k)(\xi)| \leqslant c_2$, $k = 1, \dots,$ $\prod\limits_{i=1}^{n}\xi_i \neq 0, \gamma_i = 0, 1$ $(i = 1, \dots, n)$.

We construct an example of such a system.

Example. Let ω be a non-negative, infinitely differentiable function on R with supp $\omega \subset [0;1]$ such that $\int\limits_{1/2}^{1}\omega(t)\,dt = 1$. We put

$$(F\varphi_k)(\xi) = \int\limits_{a_k(\xi)}^{b_k(\xi)}\omega(t)\,dt,$$

where $a_k(\xi) = 2 - 2^{-(k-1)}\mu(\xi)$, $b_k(\xi) = 4 - 2^{-(k-1)}\mu(\xi)$, $k = 1, \dots$.
We are going to prove that the system $\{\varphi_k\}_{k=0}^{\infty}$ (with an appropriate by chosen function φ_0) belongs the $\Phi(\mu, R^n)$.

Property (i) is clear. To prove property (ii) let us note that $b_k(\xi) < 0$ if $\mu(\xi) > 2^{k+1}$ and $a_k(\xi) > 1$ if $\mu(\xi) < 2^{k-1}$ (k=1,...,). In both cases $(F\varphi_k)(\xi) = 0$.

Let us check property (iii). For each ξ_0 ; $\mu(\xi_0) > \frac{3}{2}$, there exists an integer k_0 such that $3 \cdot 2^{k_0-2} \leqslant \mu(\xi_0) \leqslant 3 \cdot 2^{k_0-1}$ (this follows from the equality$\left[\frac{3}{2};\infty\right) = \bigcup\limits_{k=1}^{\infty}\left[3 \cdot 2^{k-2}; 3 \cdot 2^{k-1}\right]$). But

$$\left\{\begin{array}{l}\mu(\xi_0) \leqslant 3 \cdot 2^{k_0-1}\\ \mu(\xi_0) \geqslant 3 \cdot 2^{k_0-2}\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}a_{k_0}(\xi_0) \leqslant \frac{1}{2}\\ b_{k_0}(\xi_0) \geqslant 1\end{array}\right. .$$

So, for each ξ_0 ; $\mu(\xi_0) > \frac{3}{2}$, there exists an integer k_0 such that $[a_{k_0}(\xi_0); b_{k_0}(\xi_0)] \supset \left[\frac{1}{2}; 1\right]$.

Hence we have

$$F\varphi_{k_0}(\xi_0) = \int\limits_{a_{k_0}(\xi_0)}^{b_{k_0}(\xi_0)}\omega(t)\,dt \geqslant \int\limits_{1/2}^{1}\omega(t)\,dt = 1.$$

If the function φ_0 is chosen appropriately then for the system $\{\varphi_k\}_{k=0}^{\infty}$ property (iii) holds.

To check the last property (iv), we first of all note that for the function $\mu(\xi)$ from (1.1) there exists a positive number c' such that the following inequality holds

$$|\xi^{\gamma} D^{\gamma} \mu(\xi)| \leqslant c' \mu(\xi), \tag{1.2}$$

$$\prod_{i=1}^{n} \xi_i \neq 0, \gamma = (\gamma_1, \ldots, \gamma_n), \gamma_i = 0, 1 \, (i = 1, \ldots, n).$$

So for $\xi \in \Omega_k$ $(k = 1, ..,)$ and $i = 1, \ldots, n$

$$\left| \xi_i \frac{\partial F\varphi_k}{\partial \xi_i} \right| = 2^{-(k-1)} \left| \xi_i \frac{\partial \mu}{\partial \xi_i} \right| \cdot |\omega(b_k) - \omega(a_k)| \leqslant 8c' \max \omega(t).$$

For the mixed derivatives we have $(j = 1, \ldots, n)$

$$\left| \xi_i \xi_j \frac{\partial^2 F\varphi_k}{\partial \xi_i \partial \xi_j} \right| =$$

$$= 2^{-(k-1)} |\xi_i \xi_j| \cdot \left| \frac{\partial^2 \mu}{\partial \xi_i \partial \xi_j} [\omega(a_k) - \omega(b_k)] - 2^{-(k-1)} \frac{\partial \mu}{\partial \xi_i} \cdot \frac{\partial \mu}{\partial \xi_j} [\omega'(a_k) - \omega'(b_k)] \right| \leqslant$$

$$\leqslant 2^{-(k-1)} c' \mu(\xi) \cdot 2 \max \omega(t) + 2^{-2(k-1)} c'^2 \mu^2(\xi) \cdot 2 \max \omega'(t) \leqslant c_2.$$

In the same way we can also prove the appropriate inequalities for the remaining derivatives. So, we have proved that $\{\varphi_k\}_{k=0}^{\infty}$ belongs to $\Phi(\mu, R^n)$. □

If we put

$$(F\psi_k)(\xi) = (F\varphi_k)(\xi) \left[\sum_{k=0}^{\infty} (F\varphi_k)(\xi) \right]^{-1}, k = 0, 1, \ldots,$$

then the system $\{\psi_k\}_{k=0}^{\infty}$ satisfies

$$\sum_{k=0}^{\infty} (F\psi_k)(\xi) = 1, \ \ \xi \in R_n. \tag{1.3}$$

Using Lizorkin's multiplier theorem (see [L-1]) and property (iv) , we see that there exists a constant $c' > 0$ such that

$$\|F\varphi_k\|_{M_p} \leqslant c', k = 0, 1, \ldots. \tag{1.4}$$

Further we shall assume that the systems $\{\varphi_k\}_{k=0}^{\infty} \in \Phi(\mu, R_n)$ have properties (1.3), (1.4).

Definition 1.2. Let $1 < p < \infty$, $-\infty < s < \infty$. We put

$$H_p^s(\mu; R^n) \equiv H_p^s(\mu) = \left\{ f \in S'(R^n); \|f\|_{H_p^s(\mu)} = \left\| F^{-1} \left(1 + \mu^2\right)^{s/2} Ff \right\|_{L_p(R^n)} < \infty \right\}.$$

Definition 1.3. Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s < \infty$, $\{\varphi_k\}_{k=0}^{\infty} \in \Phi(\mu, R^n)$. We put

$$B^s_{p,q}(\mu) = \left\{ f \in S'(R^n);\|f\|_{B^s_{p,q}(\mu)} = \|\{f * \varphi_k\}\|_{l^s_q(L_p)} \equiv \|2^{sk}\|f * \varphi_k\|_{L_p(R^n)}\|_{l_q} < \infty \right\}.$$

If the vertices of $\Re$ instead of Z_n^+ (vectors, whose co-ordinates are non-negative integers or 0) belong to $\overset{+}{R}_n$(vectors with non-negative co-ordinates), then we define H and B-spaces as closurer of S in the norms introduces in definitions 1.2, 1.3.

If $\mu(\xi) = |\xi|$ then the defined spaces are classical spaces of Sobolev-Liouville and Besov.

Theorem 1.1. *Let* $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s_0 \neq s_1 < \infty$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$. *Then*

$$\left(H_p^{s_0}(\mu), H_p^{s_1}(\mu)\right)_{\theta,q} = B^s_{p,q}(\mu).$$

Proof. Let $f \in \left(H_p^{s_0}(\mu), H_p^{s_1}(\xi)\right)_{\theta,q}$. Then $f = f_0 + f_1$, where $f_i \in H_p^{s_i}(\mu)$, $i = 0,1$. We obtain

$$\|f * \varphi_k\|_{L_p} \leqslant \|f_0 * \varphi_k\|_{L_p} + \|f_1 * \varphi_k\|_{L_p} =$$

$$= \left\| F^{-1}\left\{ \frac{2^{ks_0} F\varphi_k}{(1+\mu^2)^{s_0/2}} F\left(2^{-ks_0} I_\mu^{s_0} f_0\right) \right\} \right\|_{L_p} + \left\| F^{-1}\left\{ \frac{2^{ks_1} F\varphi_k}{(1+\mu^2)^{s_1/2}} F\left(2^{-ks_1} I_\mu^{s_1} f_0\right) \right\} \right\|_{L_p} \leqslant$$

$$\leqslant c\left(2^{-ks_0}\left\|I_\mu^{s_0} f_0\right\|_{L_p} + 2^{-ks_1}\left\|I_\mu^{s_1} f_1\right\|_{L_p}\right),$$

where $I_\mu^{s_i} f_i = F^{-1}(1+\mu^2)^{s_i/2} F f_i$, $i = 0,1$. So, we obtain that

$$\|f * \varphi_k\|_{L_p} \leqslant c 2^{-ks_0} K\left(2^{k(s_0-s_1)}, f; H_p^{s_0}(\mu), H_p^{s_1}(\mu)\right).$$

Using the discrete K-method (Theorem 1.7a of [T-2]) we have

$$\|f\|_{B^s_{p,q}(\mu)} \leqslant c_1 \|f\|_{\left(H_p^{s_0}(\mu), H_p^{s_1}(\mu)\right)_{\theta,q}}.$$

Now we are going to use the discrete J-method (Theorem 1.7b of [T-2]). Let $f \in B^s_{p,q}(\mu)$. Then, using the properties of the system $\{\varphi_k\}_{k=0}^{\infty}$, we get

$$2^{k(s-s_0)} J\left(2^{k(s_0-s_1)}, f * \varphi_k; H_p^{s_0}(\mu), H_p^{s_1}(\mu)\right) =$$

$$= \max\left\{ 2^{k(s-s_0)} \|f * \varphi_k\|_{H_p^{s_0}(\mu)}, 2^{k(s-s_0)} \cdot 2^{k(s_0-s_1)} \|f * \varphi_k\|_{H_p^{s_1}(\mu)} \right\} \leqslant$$

$$\leqslant c 2^{k(s-s_0)} \max\left\{ 2^{ks_0} \|f * \varphi_k\|_{L_p}, 2^{ks_0} \|f * \varphi_k\|_{L_p} \right\} = c 2^{ks} \|f * \varphi_k\|_{L_p}.$$

This estimate shows that the inequality

$$\|f\|_{\left(H_p^{s_0}(\mu), H_p^{s_1}(\mu)\right)_{\theta,q}} \leqslant c_2 \|f\|_{B^s_{p,q}(\mu)}$$

will be proved if we check that $f=\sum\limits_{k=0}^{\infty} f*\varphi_k$ is in $H_p^{s_0}(\mu)+H_p^{s_1}(\mu)$. Let $s_0<s_1$ then $H_p^{s_0}(\mu)+H_p^{s_1}(\mu)= H_p^{s_0}(\mu)$. Using (1.3), (1.4) and Hölder's inequality, we have

$$\left\|\sum_{k=0}^{N} f*\varphi_k - f\right\|_{H_p^{s_0}(\mu)} \leqslant \sum_{k=N+1}^{\infty} \|f*\varphi_k\|_{H_p^{s_0}(\mu)} \leqslant$$

$$\leqslant c\sum_{k=N+1}^{\infty} 2^{ks_0}\|f*\varphi_k\|_{L_p} \leqslant c\,\|f\|_{B^{s_0}_{p,q}(\mu)}\left(\sum_{k=N+1}^{\infty} 2^{kq'(s_0-s_1)}\right)^{1/q'}.$$

Now it is clear that $f=\sum\limits_{k=0}^{\infty} f*\varphi_k$ in $H_p^{s_0}(\mu)+H_p^{s_1}(\mu)= H_p^{s_0}(\mu)$. □

Theorem 1.2. *Let* $1<p<\infty$, $1\leqslant q\leqslant\infty$, $-\infty<s<\infty$. *Then the space* $B^s_{p,q}(\mu)$ *is a retract (see [T-2], [B-L]) of the space* $l_q^s(L_p)$.

Proof. Let $\{\varphi_k\}_{k=0}^{\infty}\in\Phi(\mu,R^n)$. We put $\overline{\varphi}_k=\sum\limits_{j=-1}^{1}\varphi_{k+j}$ ($\varphi_k\equiv 0$ for $k<0$). Then

$$\overline{\varphi}_k*\varphi_k=\varphi_k,\quad k=0,1,\ldots.$$

For $f\in S'$and $g=\{g_j\}_{j=0}^{\infty}, g_j\in S', j=0,1,\ldots$ we define

$$Sf=\{f*\varphi_k\}_{k=0}^{\infty},\quad Rg=\sum_{j=0}^{\infty}\overline{\varphi}_j*g_j.$$

It is clear that $S\in L\left(B^s_{p,q}(\mu), l_q^s(L_p)\right)$ and $RS=E$. Using the properties of systems $\{\varphi_k\}_{k=0}^{\infty}$ and $\{\overline{\varphi}_k\}_{k=0}^{\infty}$, we have

$$\|Rg\|_{B^s_{p,q}}=\left(\sum_{k=0}^{\infty}2^{ksq}\left\|\sum_{j=k-1}^{k+1}\overline{\varphi}_j*\varphi_k*g_j\right\|^q_{L_p}\right)^{1/q}=$$

$$=\left(\sum_{k=0}^{\infty}2^{ksq}\left\|\sum_{r=-1}^{1}\overline{\varphi}_{k+r}*\varphi_k*g_{k+r}\right\|^q_{L_p}\right)^{1/q}\leqslant c\,\|\{g_j\}\|_{l_q^s(L_p)}.$$

So $R\in L\left(l_q^s(L_p), B^s_{p,q}(\mu)\right)$. □

Remark 1.1. *Let* μ *be a continuous positive function on* R^n, *infinitely differentiable outside of the co-ordinate axes, of polynomial growth and such that* $\lim\limits_{|\xi|\to\infty}\mu(\xi)=\infty$, $\lim\limits_{|\xi|\to 0}\mu(\xi)=0$ *and inequality* (1.2) *is satisfied. We denote the collection of all such functions by* G_0^+. *It is clear that Definitions 1.1-1.3 can be given and Theorems 1.1,1.2 are true also for functions* $\mu\in G_0^+$.

Remark 1.2. *With the help of Theorem 1.2 using interpolation theorems for the spaces* $l_q^s(L_p)$ *we find the interpolation formulas for the pairs of* B*-spaces. These formulas have the same form and the same proofs as the corresponding formulas in the classical case. We shall prove these formulas in a more general case.*

2 Interpolation approach. Estimation of the K-functional

Let μ be an infinitely differentiable on R^n positive function of polynomial growth satisfying inequality (1.2). We denote the collection of all such functions by G^+. As an example of a function from G^+we can consider the following function:

$$\left(1+\sum_{i=1}^{k}\xi_i^{2a_i}\right)^{s/2}\cdot\left(1+\sum_{j=1}^{m}\xi_j^{2b_j}\right)^{r/2},$$

where a_i,b_j $(j=1,\dots,m;\ i=1,\dots,k;1\leqslant k,m\leqslant n\ )$ non-negative are integers, $-\infty<s,r<\infty$ (See also Examples at the end of this section).

Definition 2.1. *Let μ $\in G^+$, $1<p<\infty$, $-\infty<s<\infty$. We put*

$$H_p^s\left(\mu;R^n\right)\equiv H_p^s\left(\mu\right)=\left\{f\in S'\left(R^n\right);\|f\|_{H_p^s(\mu)}=\left\|F^{-1}\mu^sFf\right\|_{L_p(R^n)}<\infty\right\}.$$

Remark 2.1. *Let us note that if the function $\mu \in G^+$is bounded then it belongs to M_p. This follows from inequality* (1.2) *and Lizorkin's multiplier theorem (see [L-1]) because*

$$\left|\xi^\gamma D^\gamma\mu\left(\xi\right)\right|\leqslant c\mu\left(\xi\right)\leqslant c',\ \xi\in R^n,$$

where c,c' are independent of ξ.

It is clear that $H_p^s\left(\mu\right)=H_p^1\left(\mu^s\right)$. For this reason we shall consider H- spaces with the upper index equal to one. The same fact will be valid also for B-spaces (see Section 3).

Theorem 2.1. *Let μ, $\nu\in G^+$, $1<p<\infty$. Then the interpolation pair $\left\{H_p^1\left(\mu\right),H_p^1\left(\nu\right)\right\}$is quasi-linearizable (see [T-2]). The appropriate operators are* $(t>0)$*:*

$$V_0\left(t\right)=F^{-1}\left\{\frac{t\nu}{\mu+t\nu}F\right\},V_1\left(t\right)=F^{-1}\left\{\frac{\mu}{\mu+t\nu}F\right\}.$$

Proof. Let us verify the quasi-linearizability (see Definition 1.8.4 of [T-2]). First of all, it is clear that

$$V_0\left(t\right)+V_1\left(t\right)=1,\ t>0.$$

Now we check that for all $t>0$

$$V_0\left(t\right)\in L\left(H_p^1\left(\mu\right)+H_p^1\left(\nu\right),H_p^1\left(\mu\right)\right),V_1\left(t\right)\in L\left(H_p^1\left(\mu\right)+H_p^1\left(\nu\right),H_p^1\left(\nu\right)\right).\qquad(2.1)$$

We have for $f\in H_p^1\left(\mu\right)+H_p^1\left(\nu\right)$, $f=f_0+f_1$, $f_0\in H_p^1\left(\mu\right)$, $f_1\in H_p^1\left(\nu\right)$

$$\left\|V_0\left(t\right)f\right\|_{H_p^1(\mu)}\leqslant\left\|V_0\left(t\right)f_0\right\|_{H_p^1(\mu)}+\left\|V_0\left(t\right)f_1\right\|_{H_p^1(\mu)}=$$

$$=\left\|F^{-1}\frac{t\nu}{\mu+t\nu}Ff_0\right\|_{H_p^1(\mu)}+t\left\|F^{-1}\frac{\mu}{\mu+t\nu}Ff_1\right\|_{H_p^1(\nu)}\leqslant$$

$$\leqslant c\left(\|f_0\|_{H_p^1(\mu)}+t\|f_1\|_{H_p^1(\nu)}\right)\leqslant c\max(1,t)\cdot\left(\|f_0\|_{H_p^1(\mu)}+\|f_1\|_{H_p^1(\nu)}\right). \tag{2.2}$$

Here we used that

$$\frac{t\nu}{\mu+t\nu}\in M_p, \frac{\mu}{\mu+t\nu}\in M_p, t>0, \tag{2.3}$$

with bounded norms with respect to t. Taking the infimum in (2.2) we see that $V_0(t)\in L\left(H_p^1(\mu)+H_p^1(\nu),H_p^1(\mu)\right)$. The same proof can be used also for $V_1(t)$. Hence (2.1) follows. All other conditions of quasi-linearizability can be checked in the same way with help of (2.3):

$$\begin{aligned}
\|V_0(t)f\|_{H_p^1(\mu)} &= \left\|F^{-1}\frac{t\mu\nu}{\mu+t\nu}Ff\right\|_{L_p}\leqslant ct\|f\|_{H_p^1(\nu)},\\
\|V_0(t)f\|_{H_p^1(\mu)} &= \left\|F^{-1}\frac{t\mu\nu}{\mu+t\nu}Ff\right\|_{L_p}\leqslant c\|f\|_{H_p^1(\mu)},\\
\|V_1(t)f\|_{H_p^1(\nu)} &= \left\|F^{-1}\frac{\mu\nu}{\mu+t\nu}Ff\right\|_{L_p}\leqslant \frac{c}{t}\|f\|_{H_p^1(\mu)},\\
\|V_1(t)f\|_{H_p^1(\nu)} &= \left\|F^{-1}\frac{\mu\nu}{\mu+t\nu}Ff\right\|_{L_p}\leqslant c\|f\|_{H_p^1(\nu)}.
\end{aligned}$$

The proof is complete. □

The main corollary of the property of quasi-linearizability for a given pair $\{A_0,A_1\}$ is the possibility to identify the K- functional of Peetre (see Lemma 1.8.4 of [T-2]):

$$K(t,f;A_0,A_1)\sim\|V_0(t)f\|_{A_0}+t\|V_1(t)f\|_{A_1}, f\in A_0+A_1, t>0.$$

Thus, we have

Corollary 2.1. *Let μ, $\nu\in G^+$, $1<p<\infty$. Then for $f\in H_p^1(\mu)+H_p^1(\nu)$, $t>0$*

$$K\left(t,f;H_p^1(\mu),H_p^1(\nu)\right)\sim\left\|F^{-1}\frac{t\mu\nu}{\mu+t\nu}Ff\right\|_{L_p}. \tag{2.4}$$

Corollary 2.2. *Let μ, $\nu\in G^+$, $1<p<\infty$. Then*

$$\begin{aligned}
&(i)\ \ H_p^1(\mu)\bigcap H_p^1(\nu)=H_p^1(\mu+\nu),\\
&(ii)\ \ H_p^1(\mu)+H_p^1(\nu)=H_p^1\left(\frac{\mu\nu}{\mu+\nu}\right).
\end{aligned}$$

Proof. Parts (i) and (ii) follow from (2.3), (2.4) respectively, with $t=1$. □

Now we are going to define the B-spaces of Besov type generated by functions in G^+. Let $I_\mu=F^{-1}\{\mu^{-1}F\}$, $\mu\in G^+$ be the lift operator.

Definition 2.2. *Let $1<p<\infty$, $1\leqslant q\leqslant\infty$, $-\infty<s<\infty$, $\mu\in G^+$. We put*

$$\begin{aligned}
&B_{p,q}^0(\mu;R_n)\equiv B_{p,q}^0(\mu)=\left(H_p^1(\mu),H_p^{-1}(\mu)\right)_{\frac{1}{2},q},\\
&B_{p,q}^s(\mu)=I_{\mu^s}B_{p,q}^0(\mu).
\end{aligned}$$

Remark 2.2. *If in Definition 2.2 we put $\mu = 1$, then for arbitrary $1 \leqslant q \leqslant \infty$ and $-\infty < s < -\infty$ we get that $B^s_{p,q}(1) = L_p$, $1 < p < \infty$.*

Corollary 2.3. *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s < \infty$, $\mu \in G^+$. Then*

$$(i)\ \ B^s_{p,q}(\mu) = \left\{ f \in S'(R^n)\,;\, \|f\|^{(i)}_{B^s_{p,q}(\mu)} = \left(\int\limits_0^\infty \left\| F^{-1} \frac{t^{1/2}\mu^{1+s}}{\mu^2 + t} Ff \right\|^q_{L_p(R^n)} \frac{dt}{t} \right)^{1/q} < \infty \right\},$$

$$(ii)\ \ B^s_{p,q}(\mu) = \left\{ f \in S'(R^n)\,;\, \|f\|^{(ii)}_{B^s_{p,q}(\mu)} = \left(\sum_{k=-\infty}^{\infty} \left\| F^{-1} \frac{2^{k/2}\mu^{1+s}}{\mu^2 + 2^k} Ff \right\|^q_{L_p(R^n)} \right)^{1/q} < \infty \right\}$$

(with usual modification if $q = \infty$).

Proof. Both of parts of the corollary can be obtained from Definition 2.2, using (2.4) and the K-method of real interpolation. In part (ii) we have to use the discrete K-method (see Theorem 1.7a of [T-2]). □

Remark 2.3. *If $\exists c_1 > 0$ such that $\mu(\xi) \geqslant c_1$ for all $\xi \in R^n$ then the sum in (ii) can be taken in $k = 0, 1, \ldots$; if $\exists c_2 > 0$ such that $\mu(\xi) \leqslant c_2$ for all $\xi \in R^n$ then the sum in (ii) can be taken in $k = 0, -1, -2, \ldots$.* □

Remark 2.4. *The following calculation*

$$\|f\|^2_{B^0_{2,2}(\mu)} \sim \int\limits_0^\infty \left\| \frac{t^{1/2}\mu}{\mu^2 + t} Ff \right\|^2_{L_2} \frac{dt}{t} \sim \int\limits_{R_n} \mu^2 \cdot (Ff)^2 \int\limits_0^\infty \frac{dt}{\mu^4 + t^2} d\xi \sim$$

$$\sim \int\limits_{R_n} (Ff)^2(\xi)\, d\xi \sim \|f\|^2_{L_2}$$

shows that $B^0_{2,2}(\mu) = L_2$ for arbitrary $\mu \in G^+$. □

Corollary 2.4. *Let $\mu \in G^+$, $1 < p < \infty$, $1 \leqslant q < \infty$, $1/p + 1/p' = 1/q + 1/q' = 1$. Then we have the following formula for the dual space*

$$\left(B^0_{p,q}(\mu)\right)' = B^0_{p',q'}(\mu).$$

Equality holds also in the case $q = \infty$ if instead of $B^0_{p,\infty}(\mu)$ we use the closure of S in $B^0_{p,\infty}(\mu)$.

Proof. From the duality theorem of the real method (see Theorem 1.11.2 in [T-2]) and the equality $\left(H^1_p(\mu)\right)' = H^{-1}_{p'}(\mu)$ (see Theorem 4.1/2 of [T-2]) we get

$$\left(B^0_{p,q}(\mu)\right)' = \left(H^1_p(\mu), H^{-1}_p(\mu)\right)'_{1/2,q} = \left(H^{-1}_{p'}(\mu), H^1_{p'}(\mu)\right)_{1/2,q'} = B^0_{p',q'}(\mu).$$

□

Proposition 2.1. *Let $\mu \in G^{+}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $a \neq 0$. Then*

$$B^{0}_{p,q}(\mu) = B^{0}_{p,q}(\mu^{a}).$$

Proof. *Step* 1. First we consider the case $a = -1$. Then we have

$$B^{0}_{p,q}\left(\mu^{-1}\right) = \left(H^{1}_{p}\left(\mu^{-1}\right), H^{-1}_{p}\left(\mu^{-1}\right)\right)_{\frac{1}{2},q} = \left(H^{-1}_{p}(\mu), H^{1}_{p}(\mu)\right)_{\frac{1}{2},q} =$$
$$= \left(H^{1}_{p}(\mu), H^{-1}_{p}(\mu)\right)_{\frac{1}{2},q} = B^{0}_{p,q}(\mu).$$

Step 2. Let $a > 1$. Because of $\frac{t^{1/2}\mu^{a}}{\mu^{2a}+t} \leqslant \frac{1}{2}$, $t > 0$, it holds $\left(\frac{t^{1/2}\mu^{a}}{\mu^{2a}+t}\right)^{1-\frac{1}{a}} \in M_{p}$, $1 - \frac{1}{a} > 0$ (see Remark 2.1). Therefore by Corollary 2.3 it follows that

$$\|f\|^{q}_{B^{0}_{p,q}(\mu^{a})} \sim \int_{0}^{\infty} \left\| F^{-1} \left(\frac{t^{1/2}\mu^{a}}{\mu^{2a}+t}\right)^{1-\frac{1}{a}} \left(\frac{t^{1/2}\mu^{a}}{\mu^{2a}+t}\right)^{\frac{1}{a}} Ff \right\|^{q}_{L_{p}} \frac{dt}{t} \leqslant$$

$$\leqslant c_{1} \int_{0}^{\infty} \left\| F^{-1} \left(\frac{t^{1/2}\mu^{a}}{\mu^{2a}+t}\right)^{\frac{1}{a}} Ff \right\|^{q}_{L_{p}} \frac{dt}{t} \leqslant c_{2} \int_{0}^{\infty} \left\| F^{-1} \frac{t^{1/2a}\mu}{\mu^{2}+t^{1/a}} Ff \right\|^{q}_{L_{p}} \frac{dt}{t}.$$

By the change of variables: $u = t^{1/a}$ in the last integral, we obtain for $f \in B^{0}_{p,q}(\mu)$

$$\|f\|_{B^{0}_{p,q}(\mu^{a})} \leqslant c\,\|f\|_{B^{0}_{p,q}(\mu)}.$$

We obtain the reverse inequality using duality arguments (see Corollary 2.4).
Step 3. Let $0 < a < 1$. It follows by *Step* 2 that

$$B^{0}_{p,q}(\mu^{a}) = B^{0}_{p,q}\left((\mu^{a})^{1/a}\right) = B^{0}_{p,q}(\mu).$$

Step 4. Let $a < 0$. Then the formula is a corollary of *Step* 1. □

Proposition 2.2. *Let $\mu \in G^{+}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s < \infty$, $0 < \theta < 1$. Then*

$$B^{s}_{p,q}(\mu) = \left\{ f \in S'(R^{n}) ; \|f\|^{(iii)}_{B^{s}_{p,q}(\mu)} = \left(\int_{0}^{\infty} \left\| F^{-1} \frac{t^{1-\theta}\mu^{\theta+s}}{\mu+t} Ff \right\|^{q}_{L_{p}(R^{n})} \frac{dt}{t}\right)^{1/q} < \infty \right\}$$

(with usual modification if $q = \infty$).

Proof. It is sufficient to prove the proposition in the case $s = 0$. The other cases can be obtained therefrom using the lift operator. First let us to prove that for $\theta = \frac{a}{a+b}$ $(a, b > 0)$

$$\left(H^{a}_{p}(\mu), H^{-b}_{p}(\mu)\right)_{\theta,q} = B^{0}_{p,q}(\mu). \tag{2.5}$$

Let us denote $\nu = \mu^{a+b}$. Theorem 2.1 implies

$$\|f\|^{q}_{\left(H^{a}_{p}(\mu), H^{-b}_{p}(\mu)\right)_{\theta,q}} \sim \int_{0}^{\infty} t^{-\theta q} \left\| F^{-1} \frac{t\mu^{a-b}}{\mu^{a} + t\mu^{-b}} Ff \right\|^{q}_{L_{p}} \frac{dt}{t} =$$

$$= \int_0^\infty \left\| F^{-1} \frac{t^{1-\theta}\mu^a}{\mu^{a+b}+t} Ff \right\|_{L_p}^q \frac{dt}{t} = \int_0^\infty \left\| F^{-1} \frac{t^{1-\theta}\nu^\theta}{\nu+t} Ff \right\|_{L_p}^q \frac{dt}{t}. \tag{2.6}$$

Let a number m be chosen in such a way that $\frac{1}{2m} < \theta$ and $\frac{1}{2m} < 1-\theta$. We shall check that

$$\frac{u^{m(1-\theta)}\nu^\theta}{\nu+u^m} \leqslant c \frac{u^{1/2}\nu^{1/(2m)}}{\nu^{1/m}+u}, u > 0, c > 0. \tag{2.7}$$

Let us denote $\alpha = 1-\theta-\frac{1}{2m}, \beta = 1-\theta+\frac{1}{2m}$. Because of $\frac{1}{2m} < \theta$ and $\frac{1}{2m} < 1-\theta$, we have $0 < \alpha, \beta < 1$. We rewrite inequality (2.7) as

$$u^{m(1-\theta)}\nu^{\theta+1/m} + u^{m(1-\theta)+1}\nu^\theta \leqslant c u^{1/2}\nu^{1/2m}\left(\nu+u^m\right). \tag{2.8}$$

We note that

$$u^{m(1-\theta)}\nu^{\theta+1/m} + u^{m(1-\theta)+1}\nu^\theta = u^{1/2}\nu^{1/2m}u^{m\alpha}\nu^{1-\alpha} + u^{1/2}\nu^{1/2m}u^{m\beta}\nu^{1-\beta} =$$

$$= u^{1/2}\nu^{1/2m}\left(u^{m\alpha}\nu^{1-\alpha} + u^{m\beta}\nu^{1-\beta}\right) \leqslant c u^{1/2}\nu^{1/2m}\left(\nu+u^m\right).$$

This implies (2.8) and hence (2.7). The change of variables: $t = u^m$ in the last integral of (2.6) leads to

$$\int_0^\infty \left\| F^{-1} \frac{t^{1-\theta}\nu^\theta}{\nu+t} Ff \right\|_{L_p}^q \frac{dt}{t} \sim \int_0^\infty \left\| F^{-1} \frac{u^{m(1-\theta)}\nu^\theta}{\nu+u^m} Ff \right\|_{L_p}^q \frac{du}{u} \leqslant$$

$$\leqslant c \int_0^\infty \left\| F^{-1} \frac{u^{1/2}\nu^{1/2m}}{\nu^{1/m}+u} Ff \right\|_{L_p}^q \frac{du}{u} \sim \|f\|^q_{B^0_{p,q}\left(\nu^{1/2m}\right)} \sim \|f\|^q_{B^0_{p,q}(\mu)},$$

using (2.7) and Proposition 2.1. From (2.6) we have

$$\left(H_p^a(\mu), H_p^{-b}(\mu)\right)_{\theta,q} \supset B^0_{p,q}(\mu).$$

Using duality arguments we obtain the reverse embedding. The proof of (2.5) is complete. From (2.5) and (2.6) we have

$$\|f\|^q_{B^0_{p,q}(\mu)} \sim \int_0^\infty \left\| F^{-1} \frac{t^{1-\theta}\nu^\theta}{\nu+t} Ff \right\|_{L_p}^q \frac{dt}{t}. \tag{2.9}$$

Finally, from Proposition 2.1, using (2.9) for the function $\mu^{1/(a+b)}$ we get

$$\|f\|^q_{B^0_{p,q}(\mu)} \sim \|f\|^q_{B^0_{p,q}\left(\mu^{1/(a+b)}\right)} \sim \int_0^\infty \left\| F^{-1} \frac{t^{1-\theta}\mu^\theta}{\mu+t} Ff \right\|_{L_p}^q \frac{dt}{t}.$$

The proof is complete. □

Theorem 2.2. *Let $\mu \in G^{+}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s_0 \neq s_1 < \infty$, $0 < \theta < 1$, $s = (1-\theta)\, s_0 + \theta s_1$. Then*

$$\left(H_p^{s_0}(\mu), H_p^{s_1}(\mu)\right)_{\theta,q} = B_{p,q}^{s}(\mu).$$

Proof. From Theorem 2.1 and Proposition 2.2 we have

$$\|f\|^q_{\left(H_p^{s_0}(\mu),H_p^{s_1}(\mu)\right)_{\theta,q}} \sim \int\limits_0^\infty t^{-\theta q} \left\| F^{-1} \frac{t\mu^{s_0+s_1}}{\mu^{s_0}+t\mu^{s_1}} Ff \right\|^q_{L_p} \frac{dt}{t} =$$

$$= \int\limits_0^\infty \left\| F^{-1} \frac{t^{1-\theta}\mu^{s_0-s}\mu^{s}}{\mu^{s_0-s_1}+t} Ff \right\|^q_{L_p} \frac{dt}{t} = \int\limits_0^\infty \left\| F^{-1} \frac{t^{1-\theta}\mu^{(s_0-s_1)\theta}}{\mu^{s_0-s_1}+t} \mu^s Ff \right\|^q_{L_p} \frac{dt}{t} \sim$$

$$\sim \int\limits_0^\infty \left\| F^{-1} \frac{t^{1/2}\mu}{\mu^2+t} \mu^s Ff \right\|^q_{L_p} \frac{dt}{t} \sim \|f\|^q_{B^s_{p,q}(\mu)}.$$

Remark 2.5. *Let $\mu \in G_0^{+}$, then $(1+\mu^2)^{1/2} \in G^{+}$. Theorem 2.2 and Theorem 1.1 show that for $\mu \in G_0^{+}$ both Definitions of B-spaces are equivalent (Definition 1.3 and Definition 2.2) : $B^s_{p,q}(\mu) = B^s_{p,q}\left((1+\mu^2)^{1/2}\right)$, $\mu \in G_0^{+}$,$1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s < \infty$. In particular if $\mu(\xi) = |\xi|$ then we obtain the classical spaces of Besov : $B^s_{p,q}(R_n) = B^s_{p,q}(|\xi|) = B^s_{p,q}\left(\left(1+|\xi|^2\right)^{1/2}\right)$.*

Corollary 2.5. *Let $\mu \in G^{+}$, $-\infty < s < \infty$, $1 < p < \infty$, $1 \leqslant q < \infty$, $1/p + 1/p' = 1/q + 1/q' = 1$. Then we have the following formula for the dual space*

$$\left(B^s_{p,q}(\mu)\right)' = B^{-s}_{p',q'}(\mu).$$

Equality holds also in the case $q = \infty$ if instead of $B^s_{p,\infty}(\mu)$ we use the closure S in $B^s_{p,\infty}(\mu)$.

Proof. The same as of Corollary 2.4 using Theorem 2.2. □

Proposition 2.3. *Let μ, $\nu \in G^{+}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. Then*

$$(i)\;\; B^1_{p,q}(\mu) \bigcap B^1_{p,q}(\nu) = B^1_{p,q}(\mu+\nu),$$

$$(ii)\;\; B^1_{p,q}(\mu) + B^1_{p,q}(\nu) = B^1_{p,q}\left(\frac{\mu\nu}{\mu+\nu}\right).$$

Proof. Part (i) follows from the inequalities (see Remark 2.1 and Corollary 2.3 (i))

$$\frac{t^{1/2}\mu^2}{\mu^2+t} \leqslant \frac{t^{1/2}(\mu+\nu)^2}{(\mu+\nu)^2+t}, \frac{t^{1/2}\nu^2}{\nu^2+t} \leqslant \frac{t^{1/2}(\mu+\nu)^2}{(\mu+\nu)^2+t},$$

$$\frac{t^{1/2}(\mu+\nu)^2}{(\mu+\nu)^2+t} \leqslant 2\frac{t^{1/2}\mu^2}{\mu^2+t} + 2\frac{t^{1/2}\nu^2}{\nu^2+t}.$$

Part (ii) follows from part (i) using duality arguments (see Corollary 2.5). □

Theorem 2.3. *Let* $\mu \in G^+$, $0 < \theta < 1, s^* = (1-\theta)\, s_0 + \theta s_1, \frac{1}{q^*} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.
(i) If $1 < p < \infty$, $1 \leqslant r, q_0, q_1 \leqslant \infty$, $-\infty < s_0 \neq s_1 < \infty$, *then*

$$\left(B^{s_0}_{p,q_0}(\mu), B^{s_1}_{p,q_1}(\mu)\right)_{\theta,r} = B^{s^*}_{p,r}(\mu),$$

(ii) if $1 < p < \infty$, $1 \leqslant q_0 \neq q_1 \leqslant \infty$, $-\infty < s < \infty$, *then*

$$\left(B^{s}_{p,q_0}(\mu), B^{s}_{p,q_1}(\mu)\right)_{\theta,q^*} = B^{s}_{p,q^*}(\mu),$$

(iii) if $1 < p < \infty$, $1 \leqslant q, r \leqslant \infty$, $-\infty < s_0 \neq s_1 < \infty$, *then*

$$\left(B^{s_0}_{p,q}(\mu), H^{s_1}_{p}(\mu)\right)_{\theta,r} = B^{s^*}_{p,r}(\mu).$$

Proof. *Step* 1. To prove (i) we use Theorem 2.2 and the reiteration theorem (see Theorem 1.10.2 of [T-2]). We have

$$\left(B^{s_0}_{p,q_0}(\mu), B^{s_1}_{p,q_1}(\mu)\right)_{\theta,r} = \left(\left(H^{m_0}_p(\mu), H^{m_1}_p(\mu)\right)_{\theta_0,q_0}, \left(H^{m_0}_p(\mu), H^{m_1}_p(\mu)\right)_{\theta_1,q_1}\right)_{\theta,r} =$$

$$= \left(H^{m_0}_p(\mu), H^{m_1}_p(\mu)\right)_{\eta,r} = B^{s^*}_{p,r}(\mu),$$

where $s_i = (1-\theta_i)\, m_0 + \theta_i m_1 (i = 0, 1)$, $\eta = (1-\theta)\,\theta_0 + \theta\theta_1$.
Step 2. In the same way we prove part (iii):

$$\left(B^{s_0}_{p,q_0}(\mu), H^{s_1}_p(\mu)\right)_{\theta,r} = \left(\left(H^{s_1}_p(\mu), H^{s_2}_p(\mu)\right)_{\eta,q_0}, H^{s_1}_p(\mu)\right)_{\theta,r} =$$

$$= \left(H^{s_1}_p(\mu), H^{s_2}_p(\mu)\right)_{\eta(1-\theta),r} = B^{s^*}_{p,r}(\mu),$$

where $s_0 = (1-\eta)\, s_1 + \eta s_2$, $0 < \eta < 1$.
Step 3. To prove part (ii) we use Theorem 3.5.4 in [B-L]. Then we have using Theorem 2.2.

$$\left(B^{s}_{p,q_0}(\mu), B^{s}_{p,q_1}(\mu)\right)_{\theta,q^*} = \left(\left(H^{m_0}_p(\mu), H^{m_1}_p(\mu)\right)_{\eta,q_0}, \left(H^{m_0}_p(\mu), H^{m_1}_p(\mu)\right)_{\eta,q_1}\right)_{\theta,q^*} =$$

$$= \left(H^{m_0}_p(\mu), H^{m_1}_p(\mu)\right)_{\eta,q^*} = B^{s}_{p,q^*}(\mu),$$

where $s = (1-\eta)\, m_0 + \eta m_1$, $0 < \eta < 1$. □

To prove other interpolation formulas for B-spaces we start with the following proposition.

Proposition 2.4. *Let* $\mu \in G^+$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$,$-\infty < s < \infty$. *If a number* a *is chosen in such a way that* $a > 2\,|s|$, *then*

$$B^s_{p,q}(\mu) = \left\{ f \in S'(R^n)\,; \|f\|^{(iv)}_{B^s_{p,q}(\mu)} = \left(\sum_{k=-\infty}^{\infty} 2^{ksq} \left\| F^{-1}\left(\frac{2^{k/2}\mu^{1/2}}{\mu + 2^k}\right)^a Ff \right\|^q_{L_p(R^n)}\right)^{1/q} < \infty \right\}$$

(with usual modification if $q = \infty$*).*

Proof. We note that $0<\frac{1}{2}-\frac{s}{a}<1$ if $a>2|s|$. Let us put $\theta=\frac{1}{2}-\frac{s}{a}$. It follows from Theorem 2.2 that

$$B^s_{p,q}(\mu)=\left(H^{a/2}_p(\mu),H^{-a/2}_p(\mu)\right)_{\theta,q}. \tag{2.10}$$

We use the discrete K-method to describe the interpolation space in (2.10). As in Section 1.7 in [T-2] we have ($b>1$)

$$\|f\|^q_{B^s_{p,q}(\mu)}\sim\sum_{k=-\infty}^{\infty}b^{-kq\theta}K^q\left(b^k,f;H^{a/2}_p(\mu),H^{-a/2}_p(\mu)\right). \tag{2.11}$$

In contrast to Theorem 1.7 (a) of [T-2] we have used here the number $b>1$, instead of the number 2. It is clear that we are allowed to do this.

The K-functional in (2.11) can be calculated by Corollary 2.1. Then we have

$$\|f\|^q_{B^s_{p,q}(\mu)}\sim\sum_{k=-\infty}^{\infty}b^{-kq\left(\frac{1}{2}-\frac{s}{a}\right)}\left\|F^{-1}\frac{b^k\mu^{a/2}}{\mu^a+b^k}Ff\right\|^q_{L_p}. \tag{2.12}$$

Using $b=2^a$ in (2.12) we obtain

$$\|f\|^q_{B^s_{p,q}(\mu)}\sim\sum_{k=-\infty}^{\infty}2^{kqs}\left\|F^{-1}\frac{2^{ka/2}\mu^{a/2}}{\mu^a+2^{ka}}Ff\right\|^q_{L_p}\sim\sum_{k=-\infty}^{\infty}2^{kqs}\left\|F^{-1}\left(\frac{2^{k/2}\mu^{1/2}}{\mu+2^k}\right)^aFf\right\|^q_{L_p}.$$

The proof is complete.

Theorem 2.4. *Let $\mu\in G^+$, $1<p_0,p_1<\infty$, $1\leqslant q_0,q_1\leqslant\infty$, $-\infty<s_0,s_1<\infty$, $0<\theta<1$, $s^*=(1-\theta)s_0+\theta s_1$, $\frac{1}{p^*}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, $\frac{1}{q^*}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$. Then*

(i) $\left[B^{s_0}_{p_0,q_0}(\mu),B^{s_1}_{p_1,q_1}(\mu)\right]_\theta=B^{s^*}_{p^*,q^*}(\mu)$.

If in addition $p^*=q^*$, then

(ii) $\left(B^{s_0}_{p_0,q_0}(\mu),B^{s_1}_{p_1,q_1}(\mu)\right)_{\theta,q^*}=B^{s^*}_{p^*,q^*}(\mu)$.

Proof. It follows from Proposition 2.4 ($1<p<\infty,1\leqslant q\leqslant\infty,-\infty<m<\infty,a>2|m|$) that

$$\|f\|^q_{B^m_{p,q}(\mu)}\sim\sum_{k=-\infty}^{\infty}2^{kqm}\left\|F^{-1}\left(\frac{2^{k/2}\mu^{1/2}}{\mu+2^k}\right)^aFf\right\|^q_{L_p}\sim$$

$$\sim\sum_{k=0}^{\infty}2^{kqm}\left\|F^{-1}\left(\frac{2^{k/2}\mu^{1/2}}{\mu+2^k}+\frac{2^{k/2}\mu^{1/2}}{1+2^k\mu}\right)^aFf\right\|^q_{L_p}. \tag{2.13}$$

Let us denote $G\equiv G(f)=\{g_k\}$, where $g_k=F^{-1}\left\{\left(\frac{2^{k/2}\mu^{1/2}}{\mu+2^k}+\frac{2^{k/2}\mu^{1/2}}{1+2^k\mu}\right)^aFf\right\}$, $k=0,1,\ldots$. Then we can rewrite (2.13) as

$$\|f\|_{B^m_{p,q}(\mu)}\sim\left(\sum_{k=0}^{\infty}2^{kqm}\|g_k\|^q_{L_p}\right)^{1/q}=\|G\|_{l^m_q(L_p)}. \tag{2.14}$$

This shows that $G \in L\left(B_{p,q}^{m}(\mu); l_q^m(L_p)\right)$. Let us put $p = p_i, q = q_i, m = s_i\ (i = 0, 1)$ in (2.14). Here a is chosen in such a way that $a > 2|s_i|, i = 0, 1$. Using the interpolation property of the complex method, from Theorems 5.6.3 and 5.1.1 of [B-L] we obtain

$$\|f\|_{B^{s^*}_{p^*,q^*}(\mu)} \sim \|G\|_{l^{s^*}_{q^*}(L_{p^*})} \leqslant c\,\|f\|_{\left[B^{s_0}_{p_0,q_0}(\mu), B^{s_1}_{p_1,q_1}(\mu)\right]_\theta}.$$

We can get the reverse inequality using duality arguments (see Corollary 2.5). Formula (ii) can be proved in the same way using (2.14) and Theorems 5.6.2, 5.2.1 in [B-L]. The proof is complete.

Definition 2.1 contains a number of important spaces corresponding to various particular functions μ. All of these spaces are real interpolation spaces of the pairs of the appropriate H spaces.

Examples. (i) Let $\mu(\xi) = \left(1 + |\xi|^2\right)^{1/2}$. Then we get the classical Besov spaces (see [N, T-2, T-3, B-L]).

(ii) Let $\mu(\xi) = \left(1 + \sum\limits_{i=1}^{n} \xi_i^{2\lambda_i}\right)^{1/2}$, where $\lambda_i\ (i = 1, \ldots, n)$ are positive integers. Then we get the anisotropic Besov spaces (see [N, B-I-N] and Theorem 2.13.2 of [T-2]).

(iii) Let $\mu(\xi) = \left(1 + \sum\limits_{j=1}^{M} \xi^{2\alpha^j}\right)^{1/2}$, where $\alpha^j \in Z_n^+\ (j = 1, \ldots, M)$ are vertices of some polyhedron $\Re$ such that the point $(0, \ldots 0)$ is a vertex of $\Re$ and $\Re$ has vertices on each co-ordinate axis, different from $(0, \ldots 0)$. Then we have the B-spaces from [B-2, B-3] (see Theorem 1 of [B-2]).

(iv) Let $\mu(\xi_1, \xi_2) = (1 + \xi_1^2)^{1/2}(1 + \xi_2^2)^{1/2}$. Then we get the approximation spaces from [Sc-S] (see Proposition 5 of [Sc-S]).

(v) Let $\mu(\xi) = \left(1 + \prod\limits_{i=1}^{n} \xi_i^2\right)^{1/2}$. Then we get the approximation spaces from [D-K-T] (see Corollary 3.3 of [D-K-T]).

3 Embedding theorems

Since

$$B_{p,q}^{s}(\mu) = I_{\mu^s} B_{p,q}^{0}(\mu) = I_{\mu^s} B_{p,q}^{0}(\mu^s) = B_{p,q}^{1}(\mu^s),$$

$\mu \in G^+$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $s \neq 0$ (see Definition 2.2 and Proposition 2.1), we can restrict ourselves to B-spaces whose upper indices are 0 or 1.

Proposition 3.1. *Let μ, $\nu \in G^+$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. Then*
(i) $B_{p,q}^{0}(\mu) \bigcap B_{p,q}^{0}(\nu) = B_{p,q}^{0}(\mu + \nu) \bigcap B_{p,q}^{0}\left(\frac{\mu\nu}{\mu+\nu}\right)$,
(ii) $B_{p,q}^{0}(\mu) + B_{p,q}^{0}(\nu) = B_{p,q}^{0}(\mu + \nu) + B_{p,q}^{0}\left(\frac{\mu\nu}{\mu+\nu}\right)$.

Proof. From Proposition 2.1 we have

$$B_{p,q}^{0}\left(\frac{\mu\nu}{\mu + \nu}\right) = B_{p,q}^{0}\left(\frac{\mu + \nu}{\mu\nu}\right).$$

Then using Corollary 2.3(i) and Proposition 2.1 we get

$$\|f\|^q_{B^0_{p,q}(\mu+\nu)\bigcap B^0_{p,q}\left(\frac{\mu\nu}{\mu+\nu}\right)} \sim \int\limits_0^\infty \left\|F^{-1}\frac{t^{1/2}(\mu+\nu)}{(\mu+\nu)^2+t}Ff\right\|^q_{L_p}\frac{dt}{t}+$$

$$+\int\limits_0^\infty \left\|F^{-1}\frac{t^{1/2}\frac{\mu+\nu}{\mu\nu}}{(\frac{\mu+\nu}{\mu\nu})^2+t}Ff\right\|^q_{L_p}\frac{dt}{t}\leqslant$$

$$+\int\limits_0^\infty \left\|F^{-1}\frac{t^{1/2}\frac{1}{\nu}}{\left(\frac{1}{\mu}+\frac{1}{\nu}\right)^2+t}Ff\right\|^q_{L_p}\frac{dt}{t}+\int\limits_0^\infty \left\|F^{-1}\frac{t^{1/2}\frac{1}{\mu}}{\left(\frac{1}{\mu}+\frac{1}{\nu}\right)^2+t}Ff\right\|^q_{L_p}\frac{dt}{t}\leqslant$$

$$\leqslant c\left(\|f\|^q_{B^0_{p,q}(\mu)}+\|f\|^q_{B^0_{p,q}(\nu)}+\|f\|^q_{B^0_{p,q}\left(\frac{1}{\mu}\right)}+\|f\|^q_{B^0_{p,q}\left(\frac{1}{\nu}\right)}\right)\sim\|f\|^q_{B^0_{p,q}(\mu)\bigcap B^0_{p,q}(\nu)}.$$

In the same way we obtain

$$\|f\|^q_{B^0_{p,q}(\mu)\bigcap B^0_{p,q}(\nu)}\leqslant c\int\limits_0^\infty \left\|F^{-1}t^{1/2}\left[\frac{\mu}{\mu^2+t}+\frac{\nu}{\nu^2+t}\right]Ff\right\|^q_{L_p}\frac{dt}{t}=$$

$$=c\int\limits_0^\infty \left\|F^{-1}t^{1/2}\left[\frac{\mu\nu\,(\mu+\nu)+t\,(\mu+\nu)}{\mu^2\nu^2+t\,(\mu^2+\nu^2)+t^2}\right]Ff\right\|^q_{L_p}\frac{dt}{t}\leqslant$$

$$\leqslant c'\int\limits_0^\infty \left\|F^{-1}t^{1/2}\left[\frac{\mu\nu\,(\mu+\nu)}{\mu^2\nu^2+t\,(\mu^2+\nu^2)}\right]Ff\right\|^q_{L_p}\frac{dt}{t}+$$

$$+c'\int\limits_0^\infty \left\|F^{-1}t^{1/2}\left[\frac{t\,(\mu+\nu)}{t\,(\mu^2+\nu^2)+t^2}\right]Ff\right\|^q_{L_p}\frac{dt}{t}\sim$$

$$\sim\int\limits_0^\infty \left\|F^{-1}t^{1/2}\left[\frac{\frac{\mu\nu}{\mu+\nu}}{\left(\frac{\mu\nu}{\mu+\nu}\right)^2+t}\right]Ff\right\|^q_{L_p}\frac{dt}{t}+\int\limits_0^\infty \left\|F^{-1}t^{1/2}\left[\frac{\mu+\nu}{(\mu+\nu)^2+t}\right]Ff\right\|^q_{L_p}\frac{dt}{t}\sim$$

$$\sim\|f\|^q_{B^0_{p,q}\left(\frac{\mu\nu}{\mu+\nu}\right)}+\|f\|^q_{B^0_{p,q}(\mu+\nu)}\sim\|f\|^q_{B^0_{p,q}\left(\frac{\mu\nu}{\mu+\nu}\right)\bigcap B^0_{p,q}(\mu+\nu)}.$$

Thus, part (i) is proved. Part (ii) can be proved using duality arguments. □

Remark 3.1. *From Proposition 3.1 and Proposition 2.1 we have for* $\mu\in G^+$, $1<p<\infty$, $1\leqslant q\leqslant\infty$

$$B^0_{p,q}(\mu)=B^0_{p,q}\left(\frac{1}{\mu}\right)=B^0_{p,q}\left(\frac{1+\mu^2}{\mu}\right)=B^0_{p,q}\left(\frac{\mu}{1+\mu^2}\right).$$

Let us note that $\mu+\frac{1}{\mu}>1$*. Thus, considering the zero spaces we may assume that* $\mu(\xi)>1$ *or* $\mu(\xi)<1$ *(see Remark 2.3). If* $\mu=\frac{\lambda}{\rho}$ *then we have*

$$B^0_{p,q}\left(\frac{\lambda}{\rho}\right)=B^0_{p,q}\left(\frac{\rho}{\lambda}\right)=B^0_{p,q}\left(\frac{\lambda^2+\rho^2}{\lambda\rho}\right)=B^0_{p,q}\left(\frac{\lambda\rho}{\lambda^2+\rho^2}\right). \qquad (3.1)$$

Remark 3.2. *If we use in Proposition 3.1* $\mu=\frac{\lambda}{\rho}$, $\nu=1$*, then it follows (see Proposition 3.1, Remark 2.2 and Proposition 2.1) that*

$$\begin{aligned} B^0_{p,q}\left(\frac{\lambda}{\rho}\right)\bigcap L_p &= B^0_{p,q}\left(\frac{\lambda}{\lambda+\rho}\right)\bigcap B^0_{p,q}\left(\frac{\rho}{\lambda+\rho}\right), \\ B^0_{p,q}\left(\frac{\lambda}{\rho}\right)+L_p &= B^0_{p,q}\left(\frac{\lambda}{\lambda+\rho}\right)+B^0_{p,q}\left(\frac{\rho}{\lambda+\rho}\right) \end{aligned} \qquad (3.2)$$

for $\lambda,\ \rho\in G^+$, $1<p<\infty$, $1\leqslant q\leqslant\infty$.

Theorem 3.1. *Let* $\mu\in G^+$, $1<p<\infty$*. Then*

$$B^0_{p,\min(p,2)}(\mu)\subset L_p\subset B^0_{p,\max(p,2)}(\mu). \qquad (3.3)$$

Proof. *Step* 1. For $1<p,q<\infty$ let us define

$$F^0_{p,q}(\mu)=\left\{f\in S';\|f\|_{F^0_{p,q}(\mu)}=\left\|\left(\sum_{k=-\infty}^{\infty}\left|F^{-1}\frac{2^{k/2}\mu}{\mu^2+2^k}Ff\right|^q\right)^{1/q}\right\|_{L_p}<\infty\right\}.$$

We shall prove that

$$L_p\subset F^0_{p,2}(\mu),1<p<\infty,\mu\in G^+. \qquad (3.4)$$

We apply Lizorkin's multiplier theorem for vector valued functions (see [L-2]). We show that there exists a positive number c such that

$$\left(\sum_{k=-\infty}^{\infty}\left|\xi^\alpha D^\alpha\frac{2^{k/2}\mu(\xi)}{\mu^2(\xi)+2^k}\right|^2\right)^{1/2}\leqslant c,\alpha\in Z_n^+,\alpha_i=0,1\,(i=1,\dots,n),\xi\in R_n.$$

From (1.2) we have

$$\left(\sum_{k=-\infty}^{\infty}\left|\xi^\alpha D^\alpha\frac{2^{k/2}\mu(\xi)}{\mu^2(\xi)+2^k}\right|^2\right)^{1/2}\leqslant c_0\left(\sum_{k=-\infty}^{\infty}\left(\frac{2^{k/2}\mu(\xi)}{\mu^2(\xi)+2^k}\right)^2\right)^{1/2}\leqslant$$

$$\leqslant c_1\left(\int_0^\infty\left(\frac{t^{1/2}\mu(\xi)}{\mu^2(\xi)+t}\right)^2\frac{dt}{t}\right)^{1/2}\sim\left(\mu^2(\xi)\int_0^\infty\frac{t}{\mu^4(\xi)+t^2}\frac{dt}{t}\right)^{1/2}=\sqrt{\frac{\pi}{2}}.$$

The second inequality we have got in the same way as in the proof of Lemma 3.1.3 of [B-L]. Now, Lizorkin's multiplier theorem implies embedding (3.4).

Step 2. It is clear (see Corollary 2.3 (ii)) that as in the classical case $(1 < p, q < \infty)$

$$B^0_{p,\min(p,q)}(\mu) \subset F^0_{p,q}(\mu) \subset B^0_{p,\max(p,q)}(\mu). \tag{3.5}$$

From (3.4), (3.5) we obtain

$$L_p \subset F^0_{p,2}(\mu) \subset B^0_{p,2}(\mu), 1 < p \leqslant 2,$$

$$L_p \subset F^0_{p,2}(\mu) \subset F^0_{p,p}(\mu) = B^0_{p,p}(\mu), 2 \leqslant p < \infty.$$

The right embedding in (3.3) is proved.
Step 3. The left embedding in (3.3) can be obtained from the right one using duality arguments (see Corollary 2.5). The proof is complete.

Theorem 3.2. *Let $\mu, \nu \in G^+$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. If there exists a positive number c such that*

$$\nu(\xi) \leqslant c\mu(\xi), \quad \xi \in R^n, \tag{3.6}$$

then embedding

$$B^1_{p,q}(\mu) \subset B^1_{p,q}(\nu) \tag{3.7}$$

holds.

Proof. Assume that inequality (3.6) holds. Then there exists $c' > 0$ such that

$$\frac{\nu^2}{\nu^2 + t} \leqslant c' \frac{\mu^2}{\mu^2 + t}, \quad t > 0.$$

Using Lizorkin's multiplier theorem (see [L-1]) and Corollary 2.3 (i) we see that

$$\|f\|^q_{B^1_{p,q}(\nu)} \sim \int_0^\infty \left\| F^{-1} \frac{t^{1/2}\nu^2}{\nu^2 + t} Ff \right\|^q_{L_p} \frac{dt}{t} \leqslant c'' \int_0^\infty \left\| F^{-1} \frac{t^{1/2}\mu^2}{\mu^2 + t} Ff \right\|^q_{L_p} \frac{dt}{t} \sim \|f\|^q_{B^1_{p,q}(\mu)}. \tag{3.8}$$

(3.8) means that embedding (3.7) holds. □

Remark 3.3. *If μ and ν are equivalent functions from G^+ $(\mu \sim \nu)$, then for arbitrary indices p, q, s; $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s < \infty$ it holds: $B^s_{p,q}(\mu) = B^s_{p,q}(\nu)$. If $s \neq 0$ then this follows from Theorem 3.2. If $s = 0$ then this follows from the Definition 2.2 and the equalities $H^{\pm 1}_p(\mu) = H^{\pm 1}_p(\nu)$ (see [V-P]).*

Theorem 3.3. *Let $\mu \in G^+$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $1/p + 1/p' = 1/q + 1/q' = 1$. The following embedding*

$$B^1_{p,q}(\mu) \subset C \tag{3.9}$$

holds if and only if

$$F^{-1}\left(\frac{1}{\mu}\right) \in B^0_{p',q'}(\mu). \tag{3.10}$$

If $q = \infty$, then instead of $B^1_{p,\infty}(\mu)$ in (3.9) we consider the closure of S in $B^1_{p,\infty}(\mu)$.

Proof. *Step* 1. Assume the embedding (3.9). Then

$$|f(0)| \leqslant c\,\|f\|_{B^1_{p,q}(\mu)}\,, f\in S(R_n)\,.$$

It means that the δ-function belongs to the dual space $\left(B^1_{p,q}(\mu)\right)'$ which is (see Corollary 2.5) $B^{-1}_{p',q'}(\mu)$. Then

$$\|\delta\|_{B^{-1}_{p',q'}(\mu)} \sim \|I_\mu\delta\|_{B^0_{p',q'}(\mu)} \sim \left\|F^{-1}\left(\frac{1}{\mu}\right)\right\|_{B^0_{p',q'}(\mu)}.$$

Step 2. Let (3.10) hold and $f\in B^1_{p,q}(\mu)$. From Theorem 2.2 we have $B^1_{p,q}(\mu) = \left(H^2_p(\mu), L_p\right)_{\frac12,q}$. We are going to use the J-method (see Theorem 1.6.1 of [T-2]). Let us represent f as $f=\int\limits_0^\infty u(t)\,\frac{dt}{t}$, where $u(t)$ is a function with values in $H^2_p(\mu)+L_p$. Then using Hölder's inequality we have

$$|f| \leqslant \int\limits_0^\infty \left| F^{-1}\left\{\frac{t^{1/2}\mu}{\mu^2+t}\cdot\frac{1}{\mu}\right\} * F^{-1}\left\{\frac{\mu^2+t}{t^{1/2}\mu}\mu Fu(t)\right\}\right|\frac{dt}{t} \leqslant$$

$$\leqslant \int\limits_0^\infty \left\| F^{-1}\left\{\frac{t^{1/2}\mu}{\mu^2+t}\cdot\frac{1}{\mu}\right\}\right\|_{L_{p'}} \left\|F^{-1}\left\{\frac{\mu^2+t}{t^{1/2}\mu}\mu Fu(t)\right\}\right\|_{L_p}\frac{dt}{t} \leqslant$$

$$\leqslant \left(\int\limits_0^\infty \left\| F^{-1}\left\{\frac{t^{1/2}\mu}{\mu^2+t}\cdot\frac{1}{\mu}\right\}\right\|^{q'}_{L_{p'}}\frac{dt}{t}\right)^{1/q'} \left(\int\limits_0^\infty t^{-q/2}\left\|F^{-1}\left\{\left(\mu^2+t\right)Fu(t)\right\}\right\|^q_{L_p}\frac{dt}{t}\right)^{1/q} \leqslant$$

$$\leqslant c\left\|F^{-1}\frac{1}{\mu}\right\|_{B^0_{p',q'}(\mu)} \left(\int\limits_0^\infty t^{-q/2}J^q\left(t,f;H^2_p(\mu),L_p\right)\frac{dt}{t}\right)^{1/q}.$$

Taking infimum over the all representations of the function f of the form $f=\int\limits_0^\infty u(t)\,\frac{dt}{t}$, from Theorem 2.2 and Theorem 1.6.1 of [T-2] we obtain

$$|f| \leqslant c\left\|F^{-1}\frac{1}{\mu}\right\|_{B^0_{p',q'}(\mu)} \|f\|_{\left(H^0_p(\mu),L_p\right)_{1/2,q}} \sim \left\|F^{-1}\frac{1}{\mu}\right\|_{B^0_{p',q'}(\mu)} \|f\|_{B^1_{p,q}(\mu)}\,.$$

This means that embedding (3.9) holds. □

4 Interpolation of pairs of spaces with different anisotropy

Theorem 4.1. *Let $\mu,\ \nu\in G^+$, $1<p<\infty$, $1\leqslant q\leqslant\infty$. Then the interpolation pair $\left\{B^1_{p,q}(\mu), B^1_{p,q}(\nu)\right\}$ is quasi-linearizable. The appropriate operators are ($t>0$):*

$$W_0(t) = F^{-1}\left\{\frac{t^2\nu^2}{\mu^2+t^2\nu^2}F\right\}, W_1(t) = F^{-1}\left\{\frac{\mu^2}{\mu^2+t^2\nu^2}F\right\}.$$

Proof. It is clear that $W_0(t) + W_1(t) = 1$, $t > 0$. As was shown in the proof of Theorem 2.2 (see (2.2)) we can obtain

$$W_0(t) : \left\{ \begin{array}{c} H_p^2(\nu) \overset{ct^2}{\to} H_p^2(\mu) \\ L_p \overset{c}{\to} L_p \end{array} \right. , W_1(t) : \left\{ \begin{array}{c} H_p^2(\mu) \overset{c/t^2}{\to} H_p^2(\nu) \\ L_p \overset{c}{\to} L_p \end{array} \right. .$$

Then, using the interpolation property of the real method, we obtain

$$W_0(t) : \left(H_p^2(\nu), L_p\right)_{1/2,q} \overset{ct}{\to} \left(H_p^2(\mu), L_p\right)_{1/2,q},$$
$$W_1(t) : \left(H_p^2(\mu), L_p\right)_{1/2,q} \overset{c/t}{\to} \left(H_p^2(\nu), L_p\right)_{1/2,q}.$$

Theorem 2.2 gives

$$W_0(t) : B_{p,q}^1(\nu) \overset{ct}{\to} B_{p,q}^1(\mu), W_1(t) : B_{p,q}^1(\mu) \overset{c/t}{\to} B_{p,q}^1(\nu).$$

In the same way we obtain

$$W_j(t) : B_{p,q}^1(\nu) \overset{c}{\to} B_{p,q}^1(\nu), W_j(t) : B_{p,q}^1(\mu) \overset{c}{\to} B_{p,q}^1(\mu), j = 0, 1.$$

The proof is complete. □

Corollary 4.1. *Let μ, $\nu \in G^+$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. Then for $f \in B_{p,q}^1(\mu) + B_{p,q}^1(\nu)$, $t > 0$*

$$K\left(t, f; B_{p,q}^1(\mu), B_{p,q}^1(\nu)\right) \sim \left\| F^{-1} \frac{t^2\nu^2}{\mu^2 + t^2\nu^2} Ff \right\|_{B_{p,q}^1(\mu)} + t \left\| F^{-1} \frac{\mu^2}{\mu^2 + t^2\nu^2} Ff \right\|_{B_{p,q}^1(\nu)}. \tag{4.1}$$

So in formulas (2.4) and (4.1) we have complete descriptions the K-functionals for the pairs of H and for the pairs B-spaces. Now we can formulate and prove the appropriate interpolation theorems. We start with following remark, which shows the difference between our case and the classical one.

Remark 4.1. *Let us prove that in general the interpolation spaces for the pairs of B-spaces depend on the second lower indexes (in contrast to the case of classical spaces). Otherwise, we have ($\mu, \nu \in G^+, 1 < p < \infty, 1 \leqslant q, q_0, q_1 \leqslant \infty, q_0 \neq q_1, 0 < \theta < 1$)*

$$\left(B_{p,q_0}^1(\nu), B_{p,q_0}^1(\mu+\nu)\right)_{\theta,q} = \left(B_{p,q_1}^1(\nu), B_{p,q_1}^1(\mu+\nu)\right)_{\theta,q}.$$

Then taking the intersections, from Proposition 2.3 and [M] we obtain

$$B_{p,q_0}^1(\mu+\nu) = \left(B_{p,q_0}^1(\mu) \bigcap B_{p,q_0}^1(\nu), B_{p,q_0}^1(\mu+\nu)\right)_{\theta,q} =$$

$$= \left(B_{p,q_0}^1(\mu), B_{p,q_0}^1(\mu+\nu)\right)_{\theta,q} \bigcap \left(B_{p,q_0}^1(\nu), B_{p,q_0}^1(\mu+\nu)\right)_{\theta,q} =$$

$$= \left(B_{p,q_1}^1(\mu), B_{p,q_1}^1(\mu+\nu)\right)_{\theta,q} \bigcap \left(B_{p,q_1}^1(\nu), B_{p,q_1}^1(\mu+\nu)\right)_{\theta,q} =$$

$$= \left(B^1_{p,q_1}(\mu)\bigcap B^1_{p,q_1}(\nu), B^1_{p,q_1}(\mu+\nu)\right)_{\theta,q} = B^1_{p,q_1}(\mu+\nu).$$

If we assume $\mu(\xi)+\nu(\xi) = \left(1+|\xi|^2\right)^{1/2}$*, then we get that* $B^1_{p,q_0} = B^1_{p,q_1}$ *for the classical Besov spaces (see Remark 2.5). This holds if and only if* $q_0 = q_1$ *(see Theorem 2.3.9 of [T-3]). In the same way assuming*

$$\left(H^1_p(\nu), H^1_p(\mu+\nu)\right)_{\theta,q} = \left(B^1_{p,q_1}(\nu), B^1_{p,q_1}(\mu+\nu)\right)_{\theta,q},$$

we get from [M] and Corollary 2.2 that $H^1_p = B^1_{p,q_1}$ *for the classical spaces. This holds if and only if* $p = q_1 = 2$ *(see Remark 2.3.9 of [T-3]). Thus, in general the interpolation spaces of the pairs of the spaces of Besov type depend on the second lower indices and the pairs of* H *and the pairs of* B*-spaces have different interpolation spaces.* □

Theorem 4.2. *Let* $\mu,\ \nu \in G^+$, $1<p<\infty$, $1 \leqslant q \leqslant \infty$, $0<\theta<1$. *Then*

$$\left(H^1_p(\mu), H^1_p(\nu)\right)_{\theta,q} = \left\{f \in S'(R_n); \|f\|^*_{\left(H^1_p(\mu),H^1_p(\nu)\right)_{\theta,q}} = \left\|F^{-1}\left\{\mu^{1-\theta}\nu^{\theta}Ff\right\}\right\|_{B^0_{p,q}\left(\frac{\mu}{\nu}\right)} < \infty\right\}.$$

Proof. From (2.4) and the definition of the K-method (see 1.3 of [T-2]) we have

$$\|f\|^q_{\left(H^1_p(\mu),H^1_p(\nu)\right)_{\theta,q}} \sim \int_0^\infty t^{-\theta q}\left\|F^{-1}\frac{t\mu\nu}{\mu+t\nu}Ff\right\|^q_{L_p}\frac{dt}{t} =$$

$$= \int_0^\infty \left\|F^{-1}\frac{t^{1-\theta}\left(\frac{\mu}{\nu}\right)^{\theta}}{\frac{\mu}{\nu}+t}\mu^{1-\theta}\nu^{\theta}Ff\right\|^q_{L_p}\frac{dt}{t} \sim \left\|F^{-1}\left\{\mu^{1-\theta}\nu^{\theta}Ff\right\}\right\|^q_{B^0_{p,q}\left(\frac{\mu}{\nu}\right)}.$$

At the last step we have used Proposition 2.2.

Theorem 4.3. *Let* $\mu,\ \nu \in G^+$, $1<p<\infty$, $1 \leqslant q \leqslant \infty$, $0<\theta<1$. *Then*

$$\left(B^1_{p,q}(\mu), B^1_{p,q}(\nu)\right)_{\theta,q} = \left\{f \in S'(R_n); \|f\|^*_{\left(B^1_{p,q}(\mu),B^s_{p,q}(\nu)\right)_{\theta,q}} = \right.$$

$$\left. = \left(\int_0^\infty \left\|F^{-1}\frac{t^{1/2}\mu\nu}{\mu^2+t\nu^2}\mu^{1-\theta}\nu^{\theta}Ff\right\|^q_{B^0_{p,q}(\mu)\bigcap B^0_{p,q}(\nu)}\frac{dt}{t}\right)^{1/q} < \infty\right\}.$$

Proof. From (4.1) and the definition of the $K-$method we have

$$\|f\|^q_{\left(B^1_{p,q}(\mu),B^1_{p,q}(\nu)\right)_{\theta,q}} \sim \int_0^\infty t^{-\theta q}\left\|F^{-1}\frac{t^2\nu^2}{\mu^2+t^2\nu^2}Ff\right\|^q_{B^1_{p,q}(\mu)}\frac{dt}{t}+$$

$$+t^q\int_0^\infty t^{-\theta q}\left\|F^{-1}\frac{\mu^2}{\mu^2+t^2\nu^2}Ff\right\|^q_{B^1_{p,q}(\nu)}\frac{dt}{t} \equiv A_0+A_1. \qquad (4.2)$$

Let us consider the first integral in (4.2) which we denote by A_0. The change of variables: $u=t^2$ and Corollary 2.3 lead to

$$A_0 \sim \int_0^\infty u^{-\frac{\theta q}{2}} \left\| F^{-1}\frac{u\nu^2}{\mu^2+u\nu^2}Ff\right\|^q_{B^1_{p,q}(\mu)} \frac{du}{u} \sim$$

$$\sim \int_0^\infty\int_0^\infty u^{-\frac{\theta q}{2}} \left\| F^{-1}\frac{u\nu^2\mu^2}{\mu^2+u\nu^2}\frac{t^{1/2}}{\mu^2+t}Ff\right\|^q_{L_p} \frac{du}{u}\frac{dt}{t}. \tag{4.3}$$

From (2.4) we can note that the expression $\left\| F^{-1}\frac{u\nu^2\mu^2}{\mu^2+u\nu^2}\frac{t^{1/2}}{\mu^2+t}Ff\right\|_{L_P}$ is equivalent to $K\left(u, F^{-1}\frac{t^{1/2}}{\mu^2+t}Ff; H^2_p(\mu), H^2_p(\nu)\right)$. Furthermore

$$A_0 \sim \int_0^\infty \left\| F^{-1}\frac{t^{1/2}}{\mu^2+t}Ff\right\|^q_{\left(H^2_p(\mu),H^2_p(\nu)\right)_{\theta/2,q}} \frac{dt}{t} \sim \int_0^\infty \left\| F^{-1}\frac{t^{1/2}\mu}{\mu^2+t}\mu^{1-\theta}\nu^\theta Ff\right\|^q_{B^0_{p,q}\left(\frac{\mu}{\nu}\right)} \frac{dt}{t}. \tag{4.4}$$

At last step we have used Theorem 4.2 and Proposition 2.1. In the same way we estimate the second integral in (4.2) which we denoted by A_1:

$$A_1 \sim \int_0^\infty u^{-\frac{\theta q}{2}} \left\| F^{-1}\frac{u^{1/2}\mu^2\nu}{\mu^2+u\nu^2}Ff\right\|^q_{B^0_{p,q}(\nu)} \frac{du}{u} \sim \int_0^\infty u^{-\frac{(1+\theta)q}{2}} \left\| F^{-1}\frac{u\mu^2\nu}{\mu^2+u\nu^2}Ff\right\|^q_{B^0_{p,q}(\nu)} \frac{du}{u} \sim$$

$$\sim \int_0^\infty\int_0^\infty u^{-\frac{(1+\theta)q}{2}} \left\| F^{-1}\frac{u\nu^2\mu^2}{\mu^2+u\nu^2}\frac{t^{1/2}}{\nu^2+t}Ff\right\|^q_{L_p} \frac{du}{u}\frac{dt}{t} \sim$$

$$\sim \int_0^\infty \left\| F^{-1}\frac{t^{1/2}}{\nu^2+t}Ff\right\|^q_{\left(H^2_p(\mu),H^2_p(\nu)\right)_{\frac{1+\theta}{2},q}} \frac{dt}{t} \sim \int_0^\infty \left\| F^{-1}\frac{t^{1/2}\nu}{\mu^2+t}\mu^{1-\theta}\nu^\theta Ff\right\|^q_{B^0_{p,q}\left(\frac{\mu}{\nu}\right)} \frac{dt}{t}. \tag{4.5}$$

From (4.2),(4.4),(4.5) we obtain

$$\|f\|_{\left(B^1_{p,q}(\mu),B^s_{p,q}(\nu)\right)_{\theta,q}} \sim \left(\int_0^\infty \left\| F^{-1}\left(\frac{t^{1/2}\mu}{\mu^2+t}+\frac{t^{1/2}\nu}{\nu^2+t}\right)\mu^{1-\theta}\nu^\theta Ff\right\|^q_{B^0_{p,q}\left(\frac{\mu}{\nu}\right)} \frac{dt}{t}\right)^{1/q} \sim$$

$$\sim \|f\|^*_{\left(B^1_{p,q}(\mu),B^s_{p,q}(\nu)\right)_{\theta,q}}. \tag{4.6}$$

The proof is complete.

Later we shall see that in the norm $\|f\|^*_{\left(B^1_{p,q}(\mu),B^s_{p,q}(\nu)\right)_{\theta,q}}$ we may use each of the spaces $B^0_{p,q}(\mu), B^0_{p,q}(\nu)$ instead of the intersection $B^0_{p,q}(\mu)\bigcap B^0_{p,q}(\nu)$ (see Corollary 5.3). In Section 5, with the help of so-called “B-products”, we shall obtain more symmetric representations for the interpolation spaces $\left(B^1_{p,q}(\mu), B^1_{p,q}(\nu)\right)_{\theta,q}$ and prove a more general interpolation theorem for the pairs of spaces of Besov type.

5 "B-products"

Definition 5.1. *Let $\mu,\ \nu \in G^{+}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s, m < \infty$. The following space*

$$B_{p,q}^{s}(\mu) \cdot B_{p,q}^{m}(\nu) = I_{\mu^{s}\nu^{m}}\left(B_{p,q}^{0}(\mu) \cdot B_{p,q}^{0}(\nu)\right),$$

where

$$B_{p,q}^{0}(\mu) \cdot B_{p,q}^{0}(\nu) = \Bigg\{ f \in S'(R^{n});\, \|f\|_{B_{p,q}^{0}(\mu)\cdot B_{p,q}^{0}(\nu)} =$$

$$= \left(\int_{0}^{\infty}\int_{0}^{\infty} \left\| F^{-1} \frac{t^{1/2}\mu}{\mu^{2}+t} \cdot \frac{u^{1/2}\nu}{\nu^{2}+u} Ff \right\|_{L_p(R^n)}^{q} \frac{dt}{t}\frac{du}{u} \right)^{1/q} < \infty \Bigg\}$$

(with usual modification if $q = \infty$) we call "B-product" of the spaces $B_{p,q}^{s}(\mu)$ and $B_{p,q}^{m}(\nu)$.

We start the considerations with the case $s = m = 0$. This "operation" has in fact certain properties of multiplication of the real numbers, which follow from the definition (see also Remark 5.1):

(i) $B_{p,q}^{0}(\mu) \cdot B_{p,q}^{0}(\nu) = B_{p,q}^{0}(\nu) \cdot B_{p,q}^{0}(\mu)$ (commutativity),

(ii) $\left[B_{p,q}^{0}(\mu) \cdot B_{p,q}^{0}(\nu)\right] \cdot B_{p,q}^{0}(\lambda) = B_{p,q}^{0}(\mu) \cdot \left[B_{p,q}^{0}(\nu) \cdot B_{p,q}^{0}(\lambda)\right]$ (associativity),

(iii) $B_{p,q}^{0}(\lambda) \cdot \left[B_{p,q}^{0}(\mu) \bigcap B_{p,q}^{0}(\nu)\right] = B_{p,q}^{0}(\lambda) \cdot B_{p,q}^{0}(\mu) \bigcap B_{p,q}^{0}(\lambda) \cdot B_{p,q}^{0}(\nu)$ (distributivity),

(iv) $B_{p,q}^{0}(\mu) \cdot L_p = B_{p,q}^{0}(\mu)$ (existence of the unity).

In addition we have also the following property of the "B-product" (it will be formulated as Corollary 5.1):

(v) $B_{p,q}^{0}(\mu) \cdot B_{p,q}^{0}(\mu) = B_{p,q}^{0}(\mu)$.

Remark 5.1. *It is clear that the norm of the "B-product" $B_{p,q}^{0}(\mu) \cdot B_{p,q}^{0}(\nu)$ is equivalent to the each of the expressions (see Corollary 2.3)*

$$\left(\int_{0}^{\infty} \left\| F^{-1} \frac{t^{1/2}\mu}{\mu^{2}+t} Ff \right\|_{B_{p,q}^{0}(\nu)}^{q} \frac{dt}{t} \right)^{1/q}, \left(\int_{0}^{\infty} \left\| F^{-1} \frac{t^{1/2}\nu}{\nu^{2}+t} Ff \right\|_{B_{p,q}^{0}(\mu)}^{q} \frac{dt}{t} \right)^{1/q}.$$

Theorem 5.1. *Let $\mu,\ \nu \in G^{+}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. Then*

$$B_{p,q}^{0}(\mu) \cdot B_{p,q}^{0}(\nu) = B_{p,q}^{0}(\mu\nu) \cdot B_{p,q}^{0}\left(\frac{\mu}{\nu}\right).$$

Proof. From Definition 5.1 we have

$$\|f\|_{B_{p,q}^{0}(\mu)\cdot B_{p,q}^{0}(\nu)}^{q} = \int_{0}^{\infty}\int_{0}^{\infty} \left\| F^{-1} \frac{(ut)^{1/2}\mu\nu}{(\mu\nu)^{2} + ut + t\nu^{2} + u\mu^{2}} Ff \right\|_{L_p}^{q} \frac{dt}{t}\frac{du}{u}. \tag{5.1}$$

Let us put $k = (ut)^{1/2}\mu\nu$, $l = (\mu\nu)^{2} + ut$, $m = t\nu^{2} + u\mu^{2}$.

Then using the following simple inequality:

$$\frac{k}{l+m} = \frac{\frac{k}{l}\cdot\frac{k}{m}}{\frac{k}{l}+\frac{k}{m}} \leqslant \frac{1}{2}\left(\frac{k}{l}\right)^{\frac{1}{2}}\left(\frac{k}{m}\right)^{\frac{1}{2}}, k, l, m > 0,$$

from (5.1) we obtain

$$\|f\|^q_{B^0_{p,q}(\mu)\cdot B^0_{p,q}(\nu)} \leqslant c\int_0^\infty\int_0^\infty \left\|F^{-1}\left(\frac{(ut)^{1/2}\mu\nu}{(\mu\nu)^2+ut}\right)^{1/2}\cdot\left(\frac{(ut)^{1/2}\mu\nu}{t\nu^2+u\mu^2}\right)^{1/2}Ff\right\|^q_{L_p}\frac{dt}{t}\frac{du}{u} =$$

$$= c\int_0^\infty\int_0^\infty \left\|F^{-1}\left(\frac{(ut)^{1/2}\mu\nu}{(\mu\nu)^2+ut}\right)^{1/2}\cdot\left(\frac{\left(\frac{t}{u}\right)^{1/2}\frac{\mu}{\nu}}{\left(\frac{\mu}{\nu}\right)^2+\frac{t}{u}}\right)^{1/2}Ff\right\|^q_{L_p}\frac{dt}{t}\frac{du}{u}. \tag{5.2}$$

The change of variables: $x=(ut)^{1/2}$, $y=\left(\frac{t}{u}\right)^{1/2}$ and Proposition 2.1 lead to.

$$\|f\|^q_{B^0_{p,q}(\mu)\cdot B^0_{p,q}(\nu)} \leqslant c\int_0^\infty\int_0^\infty \left\|F^{-1}\left(\frac{x\mu\nu}{(\mu\nu)^2+x^2}\right)^{1/2}\cdot\left(\frac{y\frac{\mu}{\nu}}{\left(\frac{\mu}{\nu}\right)^2+y^2}\right)^{1/2}Ff\right\|^q_{L_p}\frac{dx}{x}\frac{dy}{y} \sim$$

$$\sim \int_0^\infty\int_0^\infty \left\|F^{-1}\frac{x^{1/2}(\mu\nu)^{1/2}}{\mu\nu+x}\cdot\frac{y^{1/2}\left(\frac{\mu}{\nu}\right)^{1/2}}{\frac{\mu}{\nu}+y}Ff\right\|^q_{L_p}\frac{dx}{x}\frac{dy}{y} \sim$$

$$\sim \int_0^\infty\int_0^\infty \left\|F^{-1}\frac{x^{1/2}\mu\nu}{(\mu\nu)^2+x}\cdot\frac{y^{1/2}\frac{\mu}{\nu}}{\left(\frac{\mu}{\nu}\right)^2+y}Ff\right\|^q_{L_p}\frac{dx}{x}\frac{dy}{y}. \tag{5.3}$$

Inequality (5.3) means that

$$B^0_{p,q}(\mu\nu)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right) \subset B^0_{p,q}(\mu)\cdot B^0_{p,q}(\nu). \tag{5.4}$$

The reverse embedding we can obtain from (5.4), Proposition 2.1 and Remark 5.1:

$$B^0_{p,q}(\mu\nu)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right) \supset B^0_{p,q}\left(\mu^2\right)\cdot B^0_{p,q}\left(\nu^2\right) = B^0_{p,q}(\mu)\cdot B^0_{p,q}(\nu).$$

The proof is complete.

Corollary 5.1. *Let* $\mu \in G^+$, $1<p<\infty$, $1\leqslant q\leqslant\infty$. *Then*

$$B^0_{p,q}(\mu)\cdot B^0_{p,q}(\mu) = B^0_{p,q}(\mu).$$

Proof. From Theorem 5.1, Proposition 2.1, Remark 2.2 and the property (iv) of the "B-product" we have

$$B^0_{p,q}(\mu)\cdot B^0_{p,q}(\mu) = B^0_{p,q}\left(\mu^2\right)\cdot B^0_{p,q}(1) = B^0_{p,q}(\mu)\cdot L_p = B^0_{p,q}(\mu).$$

Corollary 5.2. *Let μ, $\nu \in G^{+}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. Then*

$$B^{0}_{p,q}(\mu\nu) \cdot B^{0}_{p,q}\left(\frac{\mu}{\nu}\right) = B^{0}_{p,q}(\mu+\nu) \cdot B^{0}_{p,q}\left(\frac{\mu\nu}{\mu+\nu}\right). \tag{5.5}$$

Proof. From Remarks 3.1 (see (3.1)) and 3.2 we have (see also Remark 5.1)

$$B^{0}_{p,q}(\mu\nu) \cdot B^{0}_{p,q}\left(\frac{\mu}{\nu}\right) = B^{0}_{p,q}(\mu\nu) \cdot B^{0}_{p,q}\left(\frac{\mu\nu}{\mu^2+\nu^2}\right) = B^{0}_{p,q}(\mu\nu) \cdot B^{0}_{p,q}\left(\frac{\mu\nu}{(\mu+\nu)^2}\right). \tag{5.6}$$

From Theorem 5.1 using Proposition 2.1 we obtain

$$B^{0}_{p,q}(\mu\nu) \cdot B^{0}_{p,q}\left(\frac{\mu\nu}{(\mu+\nu)^2}\right) = B^{0}_{p,q}\left(\left(\frac{\mu\nu}{\mu+\nu}\right)^2\right) \cdot B^{0}_{p,q}\left(\frac{1}{(\mu+\nu)^2}\right) =$$

$$= B^{0}_{p,q}\left(\frac{\mu\nu}{\mu+\nu}\right) \cdot B^{0}_{p,q}(\mu+\nu). \tag{5.7}$$

Equalities (5.6) and (5.7) give (5.5). □

Corollary 5.3. *Let μ, $\nu \in G^{+}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. Then*

$$B^{0}_{p,q}(\mu) \cdot B^{0}_{p,q}\left(\frac{\mu}{\nu}\right) = B^{0}_{p,q}(\nu) \cdot B^{0}_{p,q}\left(\frac{\mu}{\nu}\right) = \tag{5.8a}$$

$$= B^{0}_{p,q}(\mu) \cdot B^{0}_{p,q}(\mu\nu) = B^{0}_{p,q}(\nu) \cdot B^{0}_{p,q}(\mu\nu) = \tag{5.8b}$$

$$= B^{0}_{p,q}(\mu\nu) \cdot B^{0}_{p,q}\left(\frac{\mu}{\nu}\right) = B^{0}_{p,q}(\mu) \cdot B^{0}_{p,q}(\nu). \tag{5.8c}$$

The analogous equalities also hold if instead of the spaces $B^{0}_{p,q}(\mu), B^{0}_{p,q}(\nu)$ we use the spaces

$$B^{0}_{p,q}\left(\frac{\mu\nu}{\mu+\nu}\right), B^{0}_{p,q}(\mu+\nu).$$

Proof. *Step* 1.We have (see explanations below)

$$B^{0}_{p,q}(\mu) \cdot B^{0}_{p,q}\left(\frac{\mu}{\nu}\right) = B^{0}_{p,q}\left(\frac{\mu}{\nu}\right) \cdot B^{0}_{p,q}(\mu) \cdot B^{0}_{p,q}\left(\frac{\mu}{\nu}\right) = B^{0}_{p,q}\left(\frac{\mu}{\nu}\right) \cdot B^{0}_{p,q}(\mu^2) \cdot B^{0}_{p,q}\left(\frac{\mu}{\nu}\right) =$$

$$= B^{0}_{p,q}\left(\frac{\mu}{\nu}\right) \cdot B^{0}_{p,q}(\mu\nu) \cdot B^{0}_{p,q}\left(\frac{\mu^3}{\nu}\right) = B^{0}_{p,q}(\mu) \cdot B^{0}_{p,q}(\nu) \cdot B^{0}_{p,q}\left(\frac{\mu^3}{\nu}\right) =$$

$$= B^{0}_{p,q}\left(\frac{\mu^3}{\nu}\right) \cdot B^{0}_{p,q}(\mu) \cdot B^{0}_{p,q}(\nu) \cdot B^{0}_{p,q}\left(\frac{\mu}{\nu}\right). \tag{5.8}$$

The first equality of (5.8) follows by Corollary 5.1 and properties (i),(ii) of "B-products". The second one follows by Proposition 2.1 and Remark 5.1. The third and forth equalities follow by Theorem 5.1. The last one follows by property (i), Theorem 5.1 and Corollary 5.1. In the same way we obtain (using the analogues of the three steps in (5.8))

$$B^{0}_{p,q}(\nu) \cdot B^{0}_{p,q}\left(\frac{\mu}{\nu}\right) = B^{0}_{p,q}\left(\frac{\mu}{\nu}\right) \cdot B^{0}_{p,q}(\mu\nu) \cdot B^{0}_{p,q}\left(\frac{\nu^3}{\mu}\right). \tag{5.9}$$

Let us insert (5.9) in (5.8). Then we have

$$B^0_{p,q}(\mu)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right)=B^0_{p,q}\left(\frac{\mu^3}{\nu}\right)\cdot B^0_{p,q}(\mu)\cdot B^0_{p,q}\left(\frac{\nu^3}{\mu}\right)\cdot B^0_{p,q}(\mu\nu)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right)=$$

$$=B^0_{p,q}\left(\frac{\mu^3}{\nu}\right)\cdot B^0_{p,q}\left(\frac{\nu^3}{\mu}\right)\cdot B^0_{p,q}(\mu)\cdot B^0_{p,q}(\nu)\,.$$

The obtained expression is symmetric with respect to the functions μ and ν. This same expression we can get for $B^0_{p,q}(\nu)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right)$. This means that (5.8a) holds.
Step 2. Let us use (5.8a) for the functions μ and $\frac{1}{\nu}$. Then we obtain from Proposition 2.1

$$B^0_{p,q}(\mu)\cdot B^0_{p,q}(\mu\nu)=B^0_{p,q}\left(\frac{1}{\nu}\right)\cdot B^0_{p,q}(\mu\nu)=B^0_{p,q}(\nu)\cdot B^0_{p,q}(\mu\nu)\,.$$

This proves (5.8b).
Step 3. To prove that the expressions in (5.8a) and (5.8c) are equal we use Theorem 5.1, Corollary 5.1 and equalities (5.8a), (5.8b). We obtain

$$B^0_{p,q}(\mu)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right)=B^0_{p,q}(\mu)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}(\mu)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right)=$$

$$=B^0_{p,q}(\mu\nu)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right)=B^0_{p,q}(\mu\nu)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right).$$

In the same way we prove that the expressions in (5.8b) and (5.8c) are equal.
Step 4. For the spaces $B^0_{p,q}\left(\frac{\mu\nu}{\mu+\nu}\right)$, $B^0_{p,q}(\mu+\nu)$ the proofs are the same (with the help of Corollary 5.2). The proof is complete.

Corollary 5.4. *Let* $\mu,\ \nu\in G^+$, $1<p<\infty$, $1\leqslant q\leqslant\infty$,$-\infty<s,r<\infty$, $s+r\neq 0$. *Then*

$$B^0_{p,q}(\mu^s\nu^r)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right)=B^0_{p,q}(\mu)\cdot B^0_{p,q}(\nu)\,.$$

Proof. If either $s=0$ or $r=0$, then the statement follows from Corollary 5.3 and Proposition 2.1. Let $r\neq 0$. We have (explanations below)

$$B^0_{p,q}(\mu^s\nu^r)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right)=B^0_{p,q}(\mu^s\nu^r)\cdot B^0_{p,q}\left(\left(\frac{\mu}{\nu}\right)^r\right)=$$

$$=B^0_{p,q}\left(\mu^{s+r}\right)\cdot B^0_{p,q}\left(\left(\frac{\mu}{\nu}\right)^r\right)=B^0_{p,q}(\mu)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right)=B^0_{p,q}(\mu)\cdot B^0_{p,q}(\nu)\,. \tag{5.10}$$

The first and third equalities in (5.10) follow by Proposition 2.1. The second and forth equalities follow by Corollary 5.3. □

Corollary 5.5. *Let* $\mu,\ \nu\in G^+$, $1<p<\infty$.
(i) If $1\leqslant q\leqslant\min(p,2)$, *then* $B^0_{p,q}(\mu)\cdot B^0_{p,q}(\nu)\subset B^0_{p,q}(\mu)$.
(ii) If $\max(p,2)\leqslant q\leqslant\infty$, *then* $B^0_{p,q}(\mu)\subset B^0_{p,q}(\mu)\cdot B^0_{p,q}(\nu)$.

Proof. This follows from Theorem 3.1, Remark 5.1 and property of "B-products" (iv):

$$B^0_{p,q}(\mu)\cdot B^0_{p,q}(\nu)\subset B^0_{p,q}(\mu)\cdot L_p=B^0_{p,q}(\mu)\,,1\leqslant q\leqslant\min(p,2)\,.$$

In the same way we prove (ii). □

Coming back to the interpolation spaces, we can reformulate Theorem 4.2 in terms of "B-products".

Theorem 5.2. *Let $\mu,\ \nu\in G^+$, $1<p<\infty$, $1\leqslant q\leqslant\infty$, $0<\theta<1$. Then*

$$\left(H^1_p(\mu)\,,H^1_p(\nu)\right)_{\theta,q}=B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot H^1_p\left(\mu^{1-\theta}\nu^{\theta}\right).$$

Theorem 5.3. *Let $\mu,\ \nu\in G^+$, $1<p<\infty$, $1\leqslant q\leqslant\infty$, $0<\theta<1$. Then*

$$\left(B^1_{p,q}(\mu)\,,B^1_{p,q}(\nu)\right)_{\theta,q}=B^{1-\theta}_{p,q}(\mu)\cdot B^{\theta}_{p,q}(\nu)\,. \tag{5.11}$$

Proof. From (4.6), Definition 5.1, property (iii) of "B-products" and Corollary (5.3) we have

$$\left(B^1_{p,q}(\mu)\,,B^1_{p,q}(\nu)\right)_{\theta,q}=I_{\mu^{1-\theta}\nu^{\theta}}\left(B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}(\mu)\bigcap B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}(\nu)\right)=$$

$$=I_{\mu^{1-\theta}\nu^{\theta}}\left(B^0_{p,q}(\mu)\cdot B^0_{p,q}(\nu)\right)=B^{1-\theta}_{p,q}(\mu)\cdot B^{\theta}_{p,q}(\nu)\,.$$

The proof is complete.

Remark 5.2. *In the same way using Corollary 5.4 we have*

$$\left(B^1_{p,q}(\mu)\,,B^1_{p,q}(\nu)\right)_{\theta,q}=I_{\mu^{1-\theta}\nu^{\theta}}\left(B^0_{p,q}(\mu)\cdot B^0_{p,q}(\nu)\right)=$$

$$=I_{\mu^{1-\theta}\nu^{\theta}}\left(B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}\left(\mu^{1-\theta}\nu^{\theta}\right)\right)=B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B^1_{p,q}\left(\mu^{1-\theta}\nu^{\theta}\right).$$

Coming back to the spaces generated by the functions in G^+_0 (see Remark 1.1) we can rewrite Theorem 5.3 for this type of spaces.

Theorem 5.4. *Let μ_0,$\nu_0\in G^+_0$ (see Remark 1.1), $\{\varphi_k\}_{k=0}^{\infty}\in\Phi(\mu_0;R^n)$, $\{\psi_j\}_{j=0}^{\infty}\in\Phi(\nu_0;R^n)$ (see Definition 1.1), $\mu=(1+\mu_0^2)^{1/2}$, $\nu=(1+\nu_0^2)^{1/2}$,$1<p<\infty$, $1\leqslant q\leqslant\infty$, $0<\theta<1$. Then*

$$\left(B^1_{p,q}(\mu)\,,B^1_{p,q}(\nu)\right)_{\theta,q}=\left\{f\in S'(R_n)\,;\|f\|^{**}_{\left(B^1_{p,q}(\mu),B^1_{p,q}(\nu)\right)_{\theta,q}}=\right.$$

$$\left.=\left(\sum_{k=0}^{\infty}2^{kq(1-\theta)}\sum_{j=0}^{\infty}2^{jq\theta}\left\|\varphi_k*\psi_j*f\right\|^q_{L_p(R_n)}\right)^{1/q}<\infty\right\}$$

(with usual modification if $q=\infty$).

Proof. From Definition 1.3 and Theorem 1.1 we have

$$\|f\|^{**}_{\left(B^1_{p,q}(\mu),B^1_{p,q}(\nu)\right)_{\theta,q}} = \left(\sum_{k=0}^{\infty} 2^{kq(1-\theta)} \|\varphi_k * f\|^q_{B^0_{p,q}(\nu_0)}\right)^{1/q} \sim$$

$$\sim \left(\sum_{k=0}^{\infty} 2^{kq(1-\theta)} \|\varphi_k * f\|^q_{\left(L_p,H^1_p(\nu_0)\right)_{\theta,q}}\right)^{1/q}. \tag{5.12}$$

Here spaces $H^1_p(\nu_0)$ and $B^0_{p,q}(\nu_0)$ are understood in sense of Definitions 1.2, 1.3 (see Remark 2.5). On the other hand, the interpolation space in the (5.12) can be described by Theorem 2.2. Then from (5.12), Definition 5.1 and Corollary 2.3 we have

$$\|f\|^{**}_{\left(B^1_{p,q}(\mu),B^1_{p,q}(\nu)\right)_{\theta,q}} \sim \left(\sum_{k=0}^{\infty} 2^{kq(1-\theta)} \int_0^\infty \left\|F^{-1}\frac{t^{1/2}\nu^{1+\theta}}{\nu^2+t}F\varphi_k Ff\right\|^q_{L_p(R_n)} \frac{dt}{t}\right)^{1/q} \sim$$

$$\sim \left(\int_0^\infty \left\|F^{-1}\frac{t^{1/2}\nu^{1+\theta}}{\nu^2+t}Ff\right\|^q_{B^{1-\theta}_{p,q}(\mu_0)} \frac{dt}{t}\right)^{1/q} \sim \left(\int_0^\infty \left\|F^{-1}\frac{t^{1/2}\nu^{1+\theta}}{\nu^2+t}Ff\right\|^q_{\left(H^1_p(\mu_0),L_p\right)_{\theta,q}} \frac{dt}{t}\right)^{1/q} \sim$$

$$\sim \left(\int_0^\infty\int_0^\infty \left\|F^{-1}\frac{u^{1/2}\mu^{2-\theta}}{\mu^2+u}\cdot\frac{t^{1/2}\nu^{1+\theta}}{\nu^2+t}Ff\right\|^q_{L_p(R_n)} \frac{dt}{t}\frac{du}{u}\right)^{1/q} = \|f\|_{B^{1-\theta}_{p,q}(\mu)\cdot B^{\theta}_{p,q}(\nu)}.$$

The proof is complete due to Theorem 5.3. □

Remark 5.3. *In the same way we can also rewrite other interpolation formulas and formulas with "B-products" for the spaces generated by functions in G^+_0. This can be done also for the appropriate formulas in Section 6.*

6 Interpolation of pairs of B-spaces with different anisotropy. Interpolation of "B-products"

Now we would like to describe the interpolation space $\left(B^1_{p,q}(\mu), B^1_{p,q}(\nu)\right)_{\theta,r}$ in terms of "B-products". We need to define "B-products" of the spaces with different second lower indices and prove some interpolation formulas for "B-products".

Definition 6.1. *Let $\mu,\nu \in G^+$, $1 \leqslant q,r \leqslant \infty$, $-\infty < s,m < \infty$. The following space*

$$B^s_{p,q}(\mu)\cdot B^m_{p,r}(\nu) = I_{\mu^s\nu^m}\left(B^0_{p,q}(\mu)\cdot B^0_{p,r}(\nu)\right),$$

where

$$B^0_{p,q}(\mu)\cdot B^0_{p,r}(\nu) =$$

$$= \left\{f \in S'(R_n);\, \|f\|_{B^0_{p,q}(\mu)\cdot B^0_{p,r}(\nu)} = \left(\int_0^\infty \left\|F^{-1}\frac{t^{1/2}\mu}{\mu^2+t}Ff\right\|^q_{B^0_{p,r}(\nu)} \frac{dt}{t}\right)^{1/q} < \infty\right\},$$

(with usual modification if $q = \infty$) we call "B-product" of the spaces $B^s_{p,q}(\mu)$ and $B^m_{p,r}(\nu)$.

Let us fix $B \equiv B^{s_*}_{p,q_*}(\rho)$, where $\rho \in G^+$, $1 \leqslant q_* \leqslant \infty$, $-\infty < s_* < \infty$.

Definition 6.2. *Let $\mu \in G^+$. We put*

$$H^1_p(\mu) \cdot B = \left\{ f \in S'(R_n) ; \|f\|_{H^1_p(\mu)\cdot B} = \left\|F^{-1}\mu F f\right\|_B < \infty \right\}.$$

Remark 6.1. *If in Section 2 we start with the space $H^1_p(\mu) \cdot B$ instead of $H^1_p(\mu) = H^1_p(\mu) \cdot L_p$ then almost all of the propositions in Section 2 have their analogues for "B-products" with the fixed second space B. Below we formulate those of them that we shall use.*

Proposition 6.1. *Let $\mu, \nu \in G^+$. Then for $f \in H^1_p(\mu) \cdot B + H^1_p(\nu) \cdot B$, $t > 0$*

$$K\left(t, f; H^1_p(\mu) \cdot B, H^1_p(\nu) \cdot B\right) \sim \left\| F^{-1} \frac{t\mu\nu}{\mu + t\nu} F f \right\|_B. \tag{6.1}$$

Proof. It is clear that as in Remark 2.1, if a function $\lambda \in G^+$ is bounded, then for some $c > 0$

$$\left\|F^{-1}\lambda F f\right\|_B \leqslant c \|f\|_B, \quad f \in B.$$

Then using this fact we have

$$K\left(t, f; H^1_p(\mu) \cdot B, H^1_p(\nu) \cdot B\right) \leqslant \left\| F^{-1} \frac{t\nu}{\mu + t\nu} F f \right\|_{H^1_p(\mu)\cdot B} +$$

$$+ t \left\| F^{-1} \frac{\mu}{\mu + t\nu} F f \right\|_{H^1_p(\nu)\cdot B} = 2 \left\| F^{-1} \frac{t\mu\nu}{\mu + t\nu} F f \right\|_B. \tag{6.2}$$

On the other hand for $f = f_0 + f_1$, $f_0 \in H^1_p \cdot B$, $f_1 \in H^1_p(\nu) \cdot B$ we have

$$\left\| F^{-1} \frac{t\mu\nu}{\mu + t\nu} F f \right\|_B \leqslant \left\| F^{-1} \frac{t\mu\nu}{\mu + t\nu} F f_0 \right\|_B + \left\| F^{-1} \frac{t\mu\nu}{\mu + t\nu} F f_1 \right\|_B =$$

$$= \left\| F^{-1} \frac{t\nu}{\mu + t\nu} F f_0 \right\|_{H^1_p(\mu)\cdot B} + t \left\| F^{-1} \frac{\mu}{\mu + t\nu} F f_1 \right\|_{H^1_p(\nu) B} \leqslant$$

$$\leqslant c \left(\|f_0\|_{H^1_p(\mu)\cdot B} + t \|f_1\|_{H^1_p(\nu)\cdot B} \right).$$

Taking infimum over all representations $f = f_0 + f_1$, $f_0 \in H^1_p \cdot B$, $f_1 \in H^1_p(\nu) \cdot B$ we get the reverse of inequality (6.2). This proves (6.1).

Theorem 6.1. *Let $\mu \in G^+$.*

(i) If $1 \leqslant q \leqslant \infty$, $-\infty < s_0 \neq s_1 < \infty$, $0 < \theta < 1$, $s = (1-\theta) s_0 + \theta s_1$, then

$$\left(H^{s_0}_p(\mu) \cdot B, H^{s_1}_p(\mu) \cdot B\right)_{\theta,q} = B^s_{p,q}(\mu) \cdot B.$$

(ii) If, $1 \leqslant r, q_0, q_1 \leqslant \infty$, $-\infty < s_0 \neq s_1 < \infty$, $0 < \theta < 1$, $s = (1-\theta) s_0 + \theta s_1$, then

$$\left(B^{s_0}_{p,q_0}(\mu) \cdot B, B^{s_1}_{p,q_1}(\mu) \cdot B\right)_{\theta,r} = B^s_{p,r}(\mu) \cdot B.$$

Proof. The same as for the appropriate formulas for H and B-spaces in Section 2 (see Theorem 2.2 and Theorem 2.3 (i)). In the proof of part (i) we use (6.1). Part (ii) is proved with the help of part (i) and the reiteration theorem (see Theorem 1.10.2 of [T-2]).

Theorem 6.2. *Let $\mu,\ \nu \in G^{+}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $0 < \theta < 1$. Then*

$$\left(H_p^1(\mu)\cdot B, H_p^1(\nu)\cdot B\right)_{\theta,q} = I_{\mu^{1-\theta}\nu^{\theta}}\left[B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B\right]. \tag{6.3}$$

Proof. From (6.1) and the definition of the K-method (see 1.3 of [T-2]) we have

$$\|f\|^q_{\left(H_p^1(\mu)\cdot B, H_p^1(\nu)\cdot B\right)_{\theta,q}} \sim \int_0^\infty t^{-\theta q}\left\|F^{-1}\frac{t\mu\nu}{\mu+t\nu}Ff\right\|_B^q\frac{dt}{t} =$$

$$= \int_0^\infty \left\|F^{-1}\frac{t^{1-\theta}\left(\frac{\mu}{\nu}\right)^{\theta}}{\frac{\mu}{\nu}+t}\mu^{1-\theta}\nu^{\theta}Ff\right\|_B^q\frac{dt}{t}. \tag{6.4}$$

In the same way as in Proposition 2.2 we get that (6.4) is equivalent to

$$\left\|F^{-1}\left\{\mu^{1-\theta}\nu^{\theta}Ff\right\}\right\|^q_{B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B}\cdot\left\|F^{-1}\left\{\mu^{1-\theta}\nu^{\theta}Ff\right\}\right\|^q_{B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B}.$$

This proves (6.3). □

Proposition 6.2. *Let $\mu\in G^{+}$,$1 < p < \infty$, $1 \leqslant q, r \leqslant \infty$. Then*

$$B^0_{p,q}(\mu)\cdot B^0_{p,r}(\mu) = B^0_{p,q}(\mu). \tag{6.5}$$

Proof. We use formula (i) of Theorem 2.3. Then we have

$$\|f\|^q_{B^0_{p,q}(\mu)} \sim \|f\|^q_{\left(B^1_{p,r}(\mu),B^{-1}_{p,r}(\mu)\right)_{1/2,q}} \sim \int_0^\infty t^{-\frac{q}{2}}K^q\left(t,f;B^1_{p,r}(\mu),B^{-1}_{p,r}(\mu)\right)\frac{dt}{t}. \tag{6.6}$$

The proof of (6.1) gives

$$K\left(t,f;B^1_{p,r}(\mu),B^{-1}_{p,r}(\mu)\right) \sim \left\|F^{-1}\frac{t\mu}{\mu^2+t}Ff\right\|_{B^0_{p,r}(\mu)}. \tag{6.7}$$

If we insert (6.7) in (6.6), then we obtain (6.5). □

Proposition 6.3. *Let $\mu,\nu \in G^{+}$, $1 \leqslant q, r \leqslant \infty$. Then*
(i) $B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,r}(\mu) = B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,r}(\nu) = B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,r}(\mu\nu)$,
(ii) $B^0_{p,q}(\mu\nu)\cdot B^0_{p,r}(\mu) = B^0_{p,q}(\mu\nu)\cdot B^0_{p,r}(\nu) = B^0_{p,q}(\mu\nu)\cdot B^0_{p,r}\left(\frac{\mu}{\nu}\right)$.

Proof. *Step* 1. Let us verify that

$$B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,r}(\mu)=B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot\left[B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,r}(\mu)\right]. \tag{6.8}$$

Let us use Theorem 6.1 (ii) with $B=B^0_{p,r}(\mu)$. We have

$$\|f\|^q_{B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B}\sim\|f\|^q_{\left(B^1_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B,B^{-1}_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B\right)_{1/2,q}}=$$

$$=\int\limits_0^\infty t^{-\frac{q}{2}}K^q\left(t,f;B^1_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B,B^{-1}_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B\right)\frac{dt}{t}. \tag{6.9}$$

As in (6.1), (6.7) we have

$$K\left(t,f;B^1_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B,B^{-1}_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B\right)\sim\left\|F^{-1}\frac{t\frac{\mu}{\nu}}{\left(\frac{\mu}{\nu}\right)^2+t}Ff\right\|_{B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B}. \tag{6.10}$$

If we insert (6.10) in (6.9), then we obtain (6.8).
Step 2. From Corollary 5.3 and (6.8) we have

$$B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,r}(\mu)=B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot\left[B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,r}(\mu)\right]=$$

$$=B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot\left[B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,r}(\nu)\right]=B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,r}(\nu)=$$

$$=B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot\left[B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,r}(\mu\nu)\right]=B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,r}(\mu\nu).$$

Step 3. In the same way we prove part (ii). □

Instead of the property of commutativity (see Section 5) in the case of different lower indices for "B-products" we only have the following proposition (see Theorem 5.1).

Proposition 6.4. *Let μ, $\nu\in G^+$, $1<p<\infty$, $1\leqslant r<q\leqslant\infty$. Then*
(i) $B^0_{p,r}(\mu)\cdot B^0_{p,q}(\nu)\subset B^0_{p,q}(\nu)\cdot B^0_{p,r}(\mu)$,
(ii) $B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}(\mu\nu)\subset B^0_{p,q}(\mu)\cdot B^0_{p,r}(\nu)$.

Proof. Let us prove part (ii). As in the proof of Theorem 5.1 we have

$$\|f\|^q_{B^0_{p,q}(\mu)\cdot B^0_{p,r}(\nu)}\leqslant c\int\limits_0^\infty\left[\int\limits_0^\infty\left\|F^{-1}\left(\frac{(ut)^{1/2}\mu\nu}{(\mu\nu)^2+ut}\right)^{1/2}\cdot\left(\frac{\left(\frac{t}{u}\right)^{1/2}\frac{\mu}{\nu}}{\left(\frac{\mu}{\nu}\right)^2+\frac{t}{u}}\right)^{1/2}Ff\right\|^r_{L_p}\frac{dt}{t}\right]^{\frac{q}{r}}\frac{du}{u}.$$

The change of variables: $\left(\frac{t}{u}\right)^{1/2}=x$ leads to

$$\|f\|^q_{B^0_{p,q}(\mu)\cdot B^0_{p,r}(\nu)}\leqslant c_1\int\limits_0^\infty\left[\int\limits_0^\infty\left\|F^{-1}\left(\frac{ux\mu\nu}{(\mu\nu)^2+(ux)^2}\right)^{1/2}\cdot\left(\frac{x\frac{\mu}{\nu}}{\left(\frac{\mu}{\nu}\right)^2+x^2}\right)^{1/2}Ff\right\|^r_{L_p}\frac{dx}{x}\right]^{\frac{q}{r}}\frac{du}{u}.$$

Using Minkowski's inequality we may change the order of integration. With the help of the change of variables: $ux = y$ and Proposition 2.1 we obtain

$$\|f\|^r_{B^0_{p,q}(\mu)\cdot B^0_{p,r}(\nu)}$$

$$\leqslant c_2 \int_0^\infty \left[\int_0^\infty \left\| F^{-1} \left(\frac{y\mu\nu}{(\mu\nu)^2 + y^2} \right)^{1/2} \cdot \left(\frac{x^{1/2}\frac{\mu}{\nu}}{\left(\frac{\mu}{\nu}\right)^2 + x} \right)^{1/2} Ff \right\|^q_{L_p} \frac{dy}{y} \right]^{\frac{r}{q}} \frac{dx}{x} \sim \|f\|^r_{B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}(\mu\nu)}.$$

Part (ii) is proved. To prove part (i) we use only Minkowski's inequality. The proof is complete. □

Remark 6.2. *It is clear that using Propositions 6.2, 6.3 we can get other embeddings similar to the proved ones.*

Theorem 6.3. *Let μ, $\nu \in G^+$, $1 < p < \infty$, $1 \leqslant q, r \leqslant \infty$, $0 < \theta < 1$. Then*

$$\left(B^1_{p,q}(\mu), B^1_{p,q}(\nu)\right)_{\theta,r} = I_{\mu^{1-\theta}\nu^\theta}\left(B^0_{p,r}\left(\frac{\mu}{\nu}\right) \cdot \left[B^0_{p,q}(\mu) \bigcap B^0_{p,q}(\mu)\right]\right). \tag{6.11}$$

Proof. From (4.1) and the definition of the K-method we obtain

$$\|f\|^r_{\left(B^1_{p,q}(\mu),B^1_{p,q}(\nu)\right)_{\theta,r}} \sim \int_0^\infty t^{-\theta r} \left\| F^{-1} \frac{t^2\nu^2}{\mu^2 + t^2\nu^2} Ff \right\|^r_{B^1_{p,q}(\mu)} \frac{dt}{t} +$$

$$+ t^r \int_0^\infty t^{-\theta q} \left\| F^{-1} \frac{\mu^2}{\mu^2 + t^2\nu^2} Ff \right\|^r_{B^1_{p,q}(\nu)} \frac{dt}{t} \equiv A_0 + A_1. \tag{6.12}$$

Let us estimate A_0. As in the proof of Theorem 4.3 by change of variables we get

$$A_0 \sim \int_0^\infty t^{-\theta r} \left\| F^{-1} \frac{t^2\mu^2\nu^2}{\mu^2 + t^2\nu^2} Ff \right\|^r_{B^{-1}_{p,q}(\mu)} \frac{dt}{t} = c \int_0^\infty u^{-\frac{\theta r}{2}} \left\| F^{-1} \frac{u\mu^2\nu^2}{\mu^2 + u\nu^2} Ff \right\|^r_{B^{-1}_{p,q}(\mu)} \frac{du}{u}.$$

It is clear that (see (6.1))

$$\left\| F^{-1} \frac{u\mu^2\nu^2}{\mu^2 + u\nu^2} Ff \right\|_{B^{-1}_{p,q}(\mu)} \sim K\left(u, f; H^2_p(\mu) \cdot B^{-1}_{p,q}(\mu), H^2_p(\nu) \cdot B^{-1}_{p,q}(\mu)\right).$$

Then from Theorem 6.2 we get

$$A_0 \sim \left\|F^{-1}\mu^{2-\theta}\nu^\theta\right\|^r_{B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^{-1}_{p,q}(\mu)} = \left\|F^{-1}\mu^{1-\theta}\nu^\theta\right\|^r_{B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}(\mu)}. \tag{6.13}$$

In the same way for A_1 we have

$$A_1 \sim \int_0^\infty u^{-\frac{(1+\theta)r}{2}} \left\| F^{-1} \frac{u\mu^2\nu^2}{\mu^2 + u\nu^2} Ff \right\|^r_{B^{-1}_{p,q}(\nu)} \frac{du}{u} \sim \left\|F^{-1}\mu^{1-\theta}\nu^\theta\right\|^r_{B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}(\nu)}. \tag{6.14}$$

If we insert the obtained estimates (6.13), (6.14) for A_0 and A_1 in (6.12), then we obtain (6.11). The proof is complete. □

Based on the proved properties of the "B-products" we can give other representations for the interpolation space $\left(B^1_{p,q}(\mu), B^1_{p,q}(\nu)\right)_{\theta,r}$.

Theorem 6.4. *Let $\mu,\ \nu \in G^+$, $1 < p < \infty$, $1 \leqslant q, r \leqslant \infty$, $0 < \theta < 1$. Then*

(i) $\left(B^1_{p,q}(\mu), B^1_{p,q}(\nu)\right)_{\theta,r} = I_{\mu^{1-\theta}\nu^\theta}\left[B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}(\mu\nu)\right]$,

(ii) $\left(B^1_{p,q}(\mu), B^1_{p,q}(\nu)\right)_{\theta,r} = B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^1_{p,q}\left(\mu^{1-\theta}\nu^\theta\right)$,

(iii) $\left(B^1_{p,q}(\mu), B^1_{p,q}(\nu)\right)_{\theta,r} = B^{1-\theta}_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^1_{p,q}(\nu)$.

Proof. *Step* 1. Part (i) follows from Theorem 6.3 and Proposition 6.3.
Step 2. To prove part (ii) we use (6.8) and Corollary 5.4. We have

$$B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}(\mu\nu) = B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}(\mu\nu) =$$

$$= B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}\left(\mu^{1-\theta}\nu^\theta\right).$$

Now part (ii) follows from part (i).
Step 3. Part (iii) follows from part (i), Proposition 6.3 and Proposition 2.1. We have

$$I_{\mu^{1-\theta}\nu^\theta}\left[B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}(\mu\nu)\right] = I_{\mu^{1-\theta}\nu^\theta}\left[B^0_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^0_{p,q}(\nu)\right] =$$

$$= \left[I_{\left(\frac{\mu}{\nu}\right)^{1-\theta}} B^0_{p,r}\left(\frac{\mu}{\nu}\right)\right]\cdot\left[I_\nu B^0_{p,q}(\nu)\right] = B^{1-\theta}_{p,r}\left(\frac{\mu}{\nu}\right)\cdot B^1_{p,q}(\nu).$$

The proof is complete. □

The following proposition shows that "B-products" are the interpolation spaces for certain B-spaces.

Proposition 6.5. *Let $\mu,\ \nu \in G^+$, $1 < p < \infty$, $1 \leqslant q, r \leqslant \infty$, $s, m \neq 0$, $0 < \theta < 1$. Then*

$$B^s_{p,r}(\mu)\cdot B^m_{p,q}(\nu) = \left(B^1_{p,q}\left(\mu^{s/(1-\theta)}\nu^m\right), B^m_{p,q}(\nu)\right)_{\theta,r}.$$

Proof. This follows from Theorem 6.4 (iii). □

Theorem 6.5. *Let $\mu,\ \nu \in G^+$, $1 < p < \infty$, $1 \leqslant q, r < \infty$, $\frac{1}{p}+\frac{1}{p'} = \frac{1}{q}+\frac{1}{q'} = \frac{1}{r}+\frac{1}{r'} = 1$, $-\infty < s, m < \infty$. Then*

$$\left(B^s_{p,r}(\mu)\cdot B^m_{p,q}(\nu)\right)' = B^{-s}_{p',r'}(\mu)\cdot B^{-m}_{p',q'}(\nu). \tag{6.15}$$

Equality also holds in case $r = \infty$ (or $q = \infty$) if instead of $B^s_{p,\infty}(\mu)$ (or $B^m_{p,\infty}(\nu)$) we use the closure S in $B^s_{p,\infty}(\mu)$ (or $B^m_{p,\infty}(\nu)$).

Proof. *Step* 1. Let $s, m \neq 0$. Then from Proposition 6.5, Corollary 2.5 and Theorem 3.7.1 of [B-L] we have

$$\left(B^s_{p,r}(\mu)\cdot B^m_{p,q}(\nu)\right)' = \left(\left(B^1_{p,q}\left(\mu^{s/(1-\theta)}\nu^m\right), B^m_{p,q}(\nu)\right)_{\theta,r}\right)' =$$

$$= \left(B^{-1}_{p',q'}\left(\mu^{s/(1-\theta)}\nu^m\right), B^{-m}_{p',q'}(\nu)\right)_{\theta,r'} = B^{-s}_{p',r'}(\mu)\cdot B^{-m}_{p',q'}(\nu).$$

If $r = \infty$ (or $q = \infty$) then we use Remark 3.7.1.

Step 2. Let $m \neq 0, s = 0$. We put $B = B^{m}_{p,q}(\nu)$. Then from Theorem 6.1 (ii) and *Step* 1 we have

$$\left(B^{0}_{p,r}(\mu) \cdot B^{m}_{p,q}(\nu)\right)' = \left(B^{0}_{p,r}(\mu) \cdot B\right)' = \left(\left(B^{1}_{p,q}(\mu) \cdot B, B^{-1}_{p,q}(\mu) \cdot B\right)_{1/2,r}\right)' =$$

$$= \left(\left(B^{1}_{p,q}(\mu) \cdot B\right)', \left(B^{-1}_{p,q}(\mu) \cdot B\right)'\right)_{1/2,r'} = B^{0}_{p',r'}(\mu) \cdot B^{-m}_{p',q'}(\nu).$$

Step 3. Let $m = 0, s = 0$. As in *Step* 2 we have

$$\left(B^{0}_{p,r}(\mu) \cdot B^{0}_{p,q}(\nu)\right)' = \left(B^{0}_{p,r}(\mu) \cdot B\right)' = \left(\left(B^{1}_{p,q}(\mu) \cdot B, B^{-1}_{p,q}(\mu) \cdot B\right)_{1/2,r}\right)' =$$

$$= \left(\left(B^{1}_{p,q}(\mu) \cdot B\right)', \left(B^{-1}_{p,q}(\mu) \cdot B\right)'\right)_{1/2,r'} =$$

$$= \left(\left(B^{1}_{p,q}(\mu) \cdot B^{0}_{p,q}(\nu)\right)', \left(B^{-1}_{p,q}(\mu) \cdot B^{0}_{p,q}(\nu)\right)'\right)_{1/2,r'} =$$

$$= \left(\left(B^{0}_{p,q}(\nu) \cdot B^{1}_{p,q}(\mu)\right)', \left(B^{0}_{p,q}(\nu) \cdot B^{-1}_{p,q}(\mu)\right)'\right)_{1/2,r'}, \tag{6.16}$$

where $B = B^{0}_{p,q}(\nu)$. Using *Step* 2 and Theorem 6.1(ii) from (6.16) we obtain (6.15). □

7 Spaces of functions of mixed smoothness as "B-products". Interpolation formulas

In this Section we show that spaces of functions of mixed smoothness (see [L-N, Sc-T, Sc-S]) can be considered as "B-products" of approximation spaces (see [L-N, Sc-S]) and classical Besov spaces. In this way we obtain the new characterizations of spaces of functions of mixed smoothness. We start with certain particular spaces of $H$and B-type.

Definition 7.1. *Let $1 < p < \infty$, $-\infty < s < \infty$, $i = 1, 2$. We put*

$$H^{s,i}_{p}(R_2) = \left\{ f \in S'(R_2); \|f\|_{H^{s,i}_{p}(R_2)} = \left\| F^{-1}\left(1+\xi_i^2\right)^{s/2} Ff(\xi_1,\xi_2)\right\|_{L_p(R_2)} < \infty \right\}.$$

In terms of Definition 2.1 we have $H^{s,i}_{p}(R_2) = H^{s}_{p}\left(\left(1+\xi_i^2\right)^{1/2}; R_2\right)$.

Definition 7.2. *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s < \infty$, $i = 1, 2$. We put*

$$B^{s,i}_{p,q}(R_2) =$$

$$= \left\{ f \in S'(R_2); \|f\|_{B^{s,i}_{p,q}(R_2)} = \left(\sum_{k=0}^{\infty} 2^{ksq} \left\|F^{-1}F_1\varphi_k(\xi_i) Ff(\xi_1,\xi_2)\right\|^{q}_{L_p(R_2)}\right)^{1/q} < \infty \right\}$$

(with usual modification if $q = \infty$), where $\{\varphi_k\} \in \Phi(R)$ (see Definition 2.3.1/2 of [T-2]). F_1 denoted the Fourier transform in one variable.

Proposition 7.1. *Let* $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s_0 \neq s_1 < \infty$, $0 < \theta < 1$, $s = (1-\theta)\, s_0 + \theta s_1$, $i = 1, 2$. *Then*

$$\left(H_p^{s_0,i}(R_2), H_p^{s_1,i}(R_2)\right)_{\theta,q} = B_{p,q}^{s,i}(R_2).$$

Proof The same as the proof of Theorem 1.1. ☐

Corollary 7.1. *Let* $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s < \infty$, $\mu_i = \left(1+\xi_i^2\right)^{1/2}$, $i = 1, 2$. *Then*

$$B_{p,q}^{s,i}(R_2) = B_{p,q}^{s}(\mu_i),$$

where $B_{p,q}^{s}(\mu_i)$ *are the spaces defined in Definition 2.2.*

Proof. This follows from Proposition 7.1 and Theorem 2.2. ☐

Definition 7.3. *Let* $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s < \infty$. *We put*

$$S_{p,q}^{s}B(R_2) = \Bigg\{ f \in S'(R_2);\ \|f\|_{S_{p,q}^{s}B(R_2)} =$$

$$= \left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} 2^{sq(j+k)} \left\|F^{-1}\left\{F_1\varphi_k(\xi_1)\, F_1\varphi_j(\xi_2)\, Ff(\xi_1,\xi_2)\right\}\right\|_{L_p(R_2)}^{q}\right)^{1/q} < \infty \Bigg\}$$

(with usual modification if $q = \infty$*).*

Remark 7.1. *It is clear that we can rewrite the norm in Definition 7.3 as follows:*

$$\|f\|_{S_{p,q}^{s}B(R_2)} = \left(\sum_{j=0}^{\infty} 2^{sqj} \left\|F^{-1}\left\{F_1\varphi_j(\xi_2)\, Ff(\xi_1,\xi_2)\right\}\right\|_{B_{p,q}^{s,1}(R_2)}^{q}\right)^{1/q}.$$

Theorem 7.1. *Let* $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s < \infty$, $\mu_i = \left(1+\xi_i^2\right)^{1/2}$, $i = 1, 2$. *Then*

$$S_{p,q}^{s}B(R_2) = B_{p,q}^{s}(\mu_1) \cdot B_{p,q}^{s}(\mu_2).$$

This means that

$$S_{p,q}^{s}B(R_2) = \Bigg\{ f \in S'(R_2);\ \|f\|_{S_{p,q}^{s}(R_2)}^{*} =$$

$$= \left(\int_0^{\infty}\int_0^{\infty} \left\|F^{-1}\frac{t^{1/2}\mu_1^{1+s}}{\mu_1^2+t}\cdot\frac{u^{1/2}\mu_2^{1+s}}{\mu_2^2+u}Ff\right\|_{L_p(R_2)}^{q} \frac{dt}{t}\frac{du}{u}\right)^{1/q} < \infty \Bigg\}$$

(with usual modification if $q = \infty$*).*

Proof. From Corollary 7.1, Remark 7.1 and Corollary 2.3 we have

$$\|f\|_{S_{p,q}^{0}B(R_2)} \sim \left(\sum_{j=0}^{\infty} \left\|F^{-1}\left\{F_1\varphi_j(\xi_2)\, Ff\right\}\right\|_{B_{p,q}^{0}(\mu_1)}^{q}\right)^{1/q} \sim$$

$$\sim \left(\sum_{j=0}^{\infty} \int_{0}^{\infty} \left\| F^{-1} \frac{t^{1/2}\mu_1}{\mu_1^2+t} \varphi_j(\xi_2) Ff \right\|_{L_p(R_2)}^q \frac{dt}{t} \right)^{1/q} \sim$$

$$\sim \left(\int_{0}^{\infty} \left\| F^{-1} \frac{t^{1/2}\mu_1}{\mu_1^2+t} Ff \right\|_{B^0_{p,q}(\mu_2)}^q \frac{dt}{t} \right)^{1/q} \sim$$

$$\sim \left(\int_{0}^{\infty} \int_{0}^{\infty} \left\| F^{-1} \frac{t^{1/2}\mu_1}{\mu_1^2+t} \cdot \frac{u^{1/2}\mu_2}{\mu_2^2+u} Ff \right\|_{L_p(R_2)}^q \frac{dt}{t}\frac{du}{u} \right)^{1/q} \sim \|f\|_{B^0_{p,q}(\mu_1)\cdot B^0_{p,q}(\mu_2)}\,.$$

This proves the theorem in the case $s=0$. Using an operator of lifting type $I_{(1+\rho^2)^{s/2}}$ where $\rho = (1+\rho_0^2)^{1/2}$, $\rho_0(\xi_1,\xi_2) = (\xi_1^2+\xi_2^2+\xi_1^2\xi_2^2)^{1/2}$ we obtain

$$S^s_{p,q}B(R_2) = I_{\rho^s} S^0_{p,q}B(R_2) = I_{\rho^s}\left[B^0_{p,q}(\mu_1)\cdot B^0_{p,q}(\mu_2)\right] =$$

$$= \left[I_{(1+\xi_1^2)^{s/2}} B^0_{p,q}(\mu_1)\right]\cdot\left[I_{(1+\xi_2^2)^{s/2}} B^0_{p,q}(\mu_2)\right] = B^s_{p,q}(\mu_1)\cdot B^s_{p,q}(\mu_2)\,.$$

Corollary 7.2. *Let $1<p<\infty$, $1\leqslant q\leqslant\infty$, $\mu_i = (1+\xi_i^2)^{1/2}$, $i=1,2$. Then*

$$S^0_{p,q}B(R_2) = B^0_{p,q}(\mu_1\mu_2)\cdot B^0_{p,q}(\mu_1+\mu_2) = B^0_{p,q}(\mu_1\mu_2)\cdot B^0_{p,q}(R_2)\,,$$

where $B^0_{p,q}(R_2)$ is the classical Besov space (see [T-2, T-3, B-L]) and the space $B^0_{p,q}(\mu_1\mu_2)$ is defined in Definition 2.2.

Proof. From Theorem 7.1 and Corollary 5.3 we have

$$S^0_{p,q}B(R_2) = B^0_{p,q}(\mu_1)\cdot B^0_{p,q}(\mu_2) =$$

$$= B^0_{p,q}(\mu_1\mu_2)\cdot B^0_{p,q}(\mu_1+\mu_2) = B^0_{p,q}(\mu_1\mu_2)\cdot B^0_{p,q}(R_2)\,.$$

The last equality follows from Remark 3.3 using that $\mu_1+\mu_2 \sim \left(1+|\xi|^2\right)^{1/2}$. Corollary 7.2 and Remark 2.5 show that we can give another characterization of the spaces $S^0_{p,q}B(R_2)$.

Theorem 7.2. *Let $1<p<\infty$, $1\leqslant q\leqslant\infty$, $-\infty<s<\infty$. Then*

$$S^s_{p,q}B(R_2) = \Big\{ f\in S'(R_2)\,;\, \|f\|^{**}_{S^s_{p,q}B(R_2)} =$$

$$= \left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} 2^{jsq} \left\|F^{-1}\{F\varphi_k F\psi_j Ff\}\right\|^q_{L_p(R_2)} \right)^{1/q} < \infty \Big\} \tag{7.1}$$

(with usual modification if $q=\infty$), where $\{\varphi_k\}\in\Phi(R_2)$ (see Definition 2.3.1/2 of [T-2]), $\{\psi_j\}\in\Phi(\rho_0;R_2)$, $\rho_0(\xi_1,\xi_2) = (\xi_1^2+\xi_2^2+\xi_1^2\xi_2^2)^{1/2}$ (see Definition 1.1).

Proof. We put $\rho = (1+\rho_0^2)^{1/2}$, $\mu_i = (1+\xi_i^2)^{1/2}$, $i = 1, 2$. Then

$$\mu_1 + \mu_2 \sim \left(1+|\xi|^2\right)^{1/2}, \mu_1\mu_2 \sim \rho. \tag{7.2}$$

From Theorem 6.2.4 of [B-L] and Theorem 2.2 we have

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left\|F^{-1}\left\{F\varphi_k F\psi_j Ff\right\}\right\|_{L_p(R_2)}^q = \sum_{j=0}^{\infty}\left\|F^{-1}\left\{F\psi_j Ff\right\}\right\|_{B_{p,q}^0(R_2)}^q \sim$$

$$\sim \sum_{j=0}^{\infty}\left\|F^{-1}\left\{F\psi_j Ff\right\}\right\|_{\left(H_p^1(R_2), H_p^{-1}(R_2)\right)_{1/2,q}}^q \sim$$

$$\sim \sum_{j=0}^{\infty}\int_0^{\infty}\left\|F^{-1}\frac{t^{1/2}(\mu_1+\mu_2)}{(\mu_1+\mu_2)^2+t}F\psi_j Ff\right\|_{L_p(R_2)}^q \frac{dt}{t} \sim$$

$$\sim \int_0^{\infty}\int_0^{\infty}\left\|F^{-1}\frac{t^{1/2}(\mu_1+\mu_2)}{(\mu_1+\mu_2)^2+t}\cdot\frac{u^{1/2}\rho}{\rho^2+u}Ff\right\|_{L_p(R_2)}^q \frac{dt}{t}\frac{du}{u} \sim$$

$$\sim \|f\|_{B_{p,q}^0(\mu_1+\mu_2)\cdot B_{p,q}^0(\rho)}^q.$$

Applying Corollary 7.2 we obtain (7.1) in the case $s = 0$. Using the lift property of the spaces $S_{p,q}^0 B(R_2)$ and $B_{p,q}^s(\rho)$ (see property (ii) of Definition 1.1) we obtain (7.1). □

Let us define the approximation spaces $A_{p,q}^s(R_2)$ (see [Sc-S]).

We consider the set

$$H_m = \left\{(\xi_1,\xi_2)\in R_2; \exists r\in\{0,\dots,m\}; |\xi_1|\leqslant 2^r\pi, |\xi_2|\leqslant 2^{m-r}\pi\right\}.$$

Using these sets we define the hyperbolic best approximation of order m in $L_p(R_2)$ as

$$E_m(f, L_p) = \inf\|f-g\|_{L_p(R_2)}$$

where the infimum is taken with respect to all functions $g \in L_p(R_2)$ such that $\mathrm{supp}Fg \subset H_m$.

Definition 7.4. *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $s > 0$. We put*

$$A_{p,q}^s(R_2) = \left\{f\in L_p(R_2); \|f\|_{A_{p,q}^s(R_2)} = \|f\|_{L_p} + \left(\sum_{m=0}^{\infty}2^{msq}\left[E_m(f,L_p)\right]^q\right)^{1/q} < \infty\right\}$$

(with usual modification if $q = \infty$).

Theorem 7.3. *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $s > 0$. Then*

$$S_{p,q}^s B(R_2) = A_{p,q}^s(R_2)\cdot B_{p,q}^0(R_2) \tag{7.3}$$

where $B_{p,q}^0(R_2)$ is the classical Besov space.

Proof. Let us consider the functions ρ and μ_i (see (7.2)):

$$\rho_0(\xi_1,\xi_2)=\left(\xi_1^2+\xi_2^2+\xi_1^2\xi_2^2\right)^{1/2},\rho=\left(1+\rho_0^2\right)^{1/2},\mu_i=\left(1+\xi_i^2\right)^{1/2},i=1,2.$$

From Corollary 7.2 and the lift property of the spaces $S^s_{p,q}B(R_2)$ and $B^s_{p,q}(\mu_1\mu_2)$ (see (7.2)) we have

$$S^s_{p,q}B(R_2)=I_{\rho^s}S^0_{p,q}B(R_2)=I_{\rho^s}\left[B^0_{p,q}(\mu_1\mu_2)\cdot B^0_{p,q}(R_2)\right]=$$
$$=\left[I_{\rho^s}B^0_{p,q}(\mu_1\mu_2)\right]\cdot B^0_{p,q}(R_2)=B^s_{p,q}(\mu_1\mu_2)\cdot B^0_{p,q}(R_2). \tag{7.4}$$

On the other hand from Proposition 5 of [Sc-S] and Theorem 2.2 we have $A^s_{p,q}(R_2)=B^s_{p,q}(\mu_1\mu_2)$. If we insert this equality in (7.4) we obtain (7.3). ☐

As in (13) of [Sc-S] we put

$$S^H_j f(x)=F^{-1}\left[\chi_{H_j}(\xi)Ff(\xi)\right](x),$$
$$S^H_{j,k}f(x)=F^{-1}\left[\chi_{H_j}(\xi)\chi_kFf(\xi)\right](x),j,k\in N_0,$$

where χ_{H_j} is the characteristic function of H_j and $\chi_k\equiv\chi_{K_k}$ is the characteristic function of the following set

$$K_k=\left\{(\xi_1,\xi_2)\in R_2;|\xi_i|\leqslant 2^k,i=1,2\right\}-\left\{(\xi_1,\xi_2)\in R_2;|\xi_i|\leqslant 2^{k-1},i=1,2\right\}.$$

Applying Lizorkin's representation of B-spaces (see [L-3] and Theorem 2.5.4 of [T-3]) and Proposition 3 of [Sc-S] we obtain the following characterization of spaces $S^s_{p,q}B(R_2)$.

Proposition 7.2. *Let* $1<p<\infty$, $1\leqslant q\leqslant\infty$, $s>0$. *Then*

$$S^s_{p,q}B(R_2)=\Bigg\{f\in B^0_{p,q}(R_2);\|f\|^{***}_{S^s_{p,q}B(R_2)}=$$
$$=\|f\|_{B^0_{p,q}(R_2)}+\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}2^{jsq}\left\|S^H_{j+1,k}f-S^H_{j,k}f\right\|^q_{L_p(R_2)}\right)^{1/q}<\infty\Bigg\}.$$

(with usual modification if $q=\infty$*).*

Proof. From Theorem 7.3 and Proposition 3 of [Sc-S] we have

$$\|f\|_{S^s_{p,q}B(R_2)}\sim\left(\sum_{k=0}^{\infty}\left\|F^{-1}\{\chi_kFf\}\right\|^q_{A^s_{p,q}(R_2)}\right)^{1/q}\sim$$
$$\sim\left(\sum_{k=0}^{\infty}\left\|F^{-1}\{\chi_kFf\}\right\|^q_{L_p(R_2)}\right)^{1/q}+$$
$$+\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}2^{jsq}\left\|S^H_{j+1}\left[F^{-1}(\chi_kFf)\right]-S^H_j\left[F^{-1}(\chi_kFf)\right]\right\|^q_{L_p(R_2)}\right)^{1/q}\sim$$

$$\sim \|f\|_{B^0_{p,q}(R_2)} + \left(\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{jsq} \left\| S^H_{j+1,k} f - S^H_{j,k} f \right\|^q_{L_p(R_2)} \right)^{1/q}.$$

□

Applying the results of Section 5 to the spaces of functions of mixed smoothness characterised as "B-products" we obtain following two propositions.

Proposition 7.3. *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $s \neq 0$, $0 < \theta < 1$, $\mu_i = (1 + \xi_i^2)^{1/2}$, $i = 1, 2$. Then*

$$S^s_{p,q} B (R_2) = \left(B^{\frac{s}{1-\theta}}_{p,q} (\mu_1), B^{\frac{s}{\theta}}_{p,q} (\mu_2) \right)_{\theta,q},$$

where the spaces $B^{\frac{s}{1-\theta}}_{p,q} (\mu_1)$ and $B^{\frac{s}{\theta}}_{p,q} (\mu_2)$ are defined in Definition 2.2 (see also Corollary 7.1).

Proof. This follows from Theorem 7.1 and Theorem 5.3. □

Proposition 7.4. *Let $1 < p < \infty$.*

(i) If $1 \leqslant q \leqslant \min(p, 2)$ then $S^0_{p,q} B (R_2) \subset B^0_{p,q} (R_2)$,

(ii) if $\max(p, 2) \leqslant q \leqslant \infty$ then $B^0_{p,q} (R_2) \subset S^0_{p,q} B (R_2)$, where $B^0_{p,q} (R_2)$ is the classical Besov space. The embeddings of the same type also hold if instead of $B^0_{p,q} (R_2)$ we use the space $B^0_{p,q} (\mu_1 \mu_2)$ from Corollary 7.2.

Proof. This follows from Corollary 7.2 and Corollary 5.5. Now we are going to formulate and prove the interpolation formulas for spaces of functions of mixed smoothness. □

Theorem 7.4. *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s_1 \neq s_2 < \infty$, $\mu_i = (1 + \xi_i^2)^{1/2}$, $i = 1, 2$. Then the pair $\left\{ S^{s_1}_{p,q} B, S^{s_2}_{p,q} B \right\}$ is quasi-linearizable. The appropriate operators are ($t > 0$):*

$$U_0 (t) = F^{-1} \left\{ \frac{t (\mu_1 \mu_2)^{s_2}}{(\mu_1 \mu_2)^{s_1} + t (\mu_1 \mu_2)^{s_2}} F \right\}, \quad U_1 (t) = F^{-1} \left\{ \frac{(\mu_1 \mu_2)^{s_1}}{(\mu_1 \mu_2)^{s_1} + t (\mu_1 \mu_2)^{s_2}} F \right\}.$$

Proof. We verify all required properties of quasi-linearizability in the same way as in the proof of Theorem 2.1 (see also the proof of Proposition 6.1). For the K-functional we have (see Corollary 2.1):

$$K \left(t, f; S^{s_1}_{p,q} B, S^{s_2}_{p,q} B \right) \sim \left\| F^{-1} \frac{t (\mu_1 \mu_2)^{s_1 + s_2}}{(\mu_1 \mu_2)^{s_1} + t (\mu_1 \mu_2)^{s_2}} F f \right\|_{S^0_{p,q} B}, \tag{7.5}$$

$$f \in S^{s_1}_{p,q} B + S^{s_2}_{p,q} B, \quad t > 0.$$

Theorem 7.5. *Let $1 < p < \infty$, $1 \leqslant q, r \leqslant \infty$, $-\infty < s_1 \neq s_2 < \infty$, $0 < \theta < 1$, $s = (1 - \theta) s_1 + \theta s_2$. Then*

$$\left(S^{s_1}_{p,q} B, S^{s_2}_{p,q} B \right)_{\theta,r} = B^s_{p,r} (\mu_1 \mu_2) \cdot S^0_{p,q} B. \tag{7.6}$$

Proof. From the definition of the K-method (see [T-2] or [B-L]) and (7.5) we have

$$\|f\|^r_{\left(S^{s_1}_{p,q}B,S^{s_2}_{p,q}B\right)_{\theta,r}} \sim \int\limits_0^\infty t^{-\theta r}\left\|F^{-1}\frac{t\left(\mu_1\mu_2\right)^{s_1+s_2}}{\left(\mu_1\mu_2\right)^{s_1}+t\left(\mu_1\mu_2\right)^{s_2}}Ff\right\|^r_{S^0_{p,q}B}\frac{dt}{t}=$$

$$=\int\limits_0^\infty\left\|F^{-1}\frac{t^{1-\theta}\left(\mu_1\mu_2\right)^{(s_1-s_2)\theta}}{\left(\mu_1\mu_2\right)^{s_1-s_2}+t}\left(\mu_1\mu_2\right)^s Ff\right\|^r_{S^0_{p,q}B}\frac{dt}{t}.$$

Applying Theorem 7.1 and Propositions 2.1, 2.2 we obtain (7.6). □

Remark 7.2. *If we define appropriate "product" $A^s_{p,r}\cdot S^0_{p,q}B$, then using Proposition 5 of [Sc-S] we can rewrite* (7.6) *as*

$$\left(S^{s_1}_{p,q}B,S^{s_2}_{p,q}B\right)_{\theta,r}=A^s_{p,r}\cdot S^0_{p,q}B.$$

To formulate other interpolation formulas we generalize Definition 7.3.

Definition 7.5. *Let $1<p<\infty$, $1\leqslant q_1,q_2\leqslant\infty$, $-\infty<s_1,s_2<\infty$, $\overline{q}=(q_1,q_2)$, $\overline{s}=(s_1,s_2)$, $\{\varphi_k\}\in\Phi(R)$ (see Definition 2.3.1/2 of [T-2]). We put*

$$S^{\overline{s}}_{p,\overline{q}}B\left(R_2\right)=\Big\{f\in S'\left(R_2\right);\|f\|_{S^{\overline{s}}_{p,\overline{q}}B(R_2)}=$$

$$=\left(\sum_{k=0}^\infty 2^{ks_1q_1}\left(\sum_{j=0}^\infty 2^{js_2q_2}\left\|F^{-1}\left\{F_1\varphi_k\left(\xi_1\right)F_1\varphi_j\left(\xi_2\right)Ff\left(\xi_1,\xi_2\right)\right\}\right\|^{q_2}_{L_p(R_2)}\right)^{\frac{q_1}{q_2}}\right)^{1/q_1}<\infty\Big\}$$

(with usual modification if $q=\infty$).

Remark 7.3. *Using the terms of Definition 7.2 we can rewrite the norm in Definition 7.5 as*

$$\|f\|_{S^{\overline{s}}_{p,\overline{q}}B(R_2)}=\left(\sum_{k=0}^\infty 2^{ks_1q_1}\left\|F^{-1}F_1\varphi_k\left(\xi_1\right)Ff\left(\xi_1,\xi_2\right)\right\|^{q_1}_{B^{s_2,2}_{p,q_2}(R_2)}\right)^{1/q_1}=$$

$$=\left\|F^{-1}F_1\varphi_k\left(\xi_1\right)Ff\left(\xi_1,\xi_2\right)\right\|_{l^{s_1}_{q_1}\left(B^{s_2,2}_{p,q_2}(R_2)\right)}. \tag{7.7}$$

Let $\{\varphi_k\}^\infty_{k=0}\in\Phi(R)$. As in Section 1 we put $\overline{\varphi}_k=\sum\limits_{j=-1}^{1}\varphi_{k+j}$ ($\varphi_k\equiv0$ for $k<0$). Then

$$\overline{\varphi}_k*\varphi_k=\varphi_k,\quad k=0,1,\ldots.$$

For $f\in S'\left(R_2\right)$ and $g=\{g_j\}^\infty_{j=0}, g_j\in S'\left(R_2\right), j=0,1,\ldots$ we put

$$Sf=\left\{F^{-1}F_1\varphi_k\left(\xi_1\right)Ff\left(\xi_1,\xi_2\right)\right\}^\infty_{k=0},\quad Rg=\sum_{j=0}^\infty F^{-1}F_1\overline{\varphi}_j\left(\xi_1\right)Fg_j\left(\xi_1,\xi_2\right). \tag{7.8}$$

Theorem 7.6. *Let $1 < p < \infty$, $\overline{q} = (q_1, q_2)$, $\overline{s} = (s_1, s_2)$, $1 \leqslant q_i \leqslant \infty$, $-\infty < s_i < \infty$, $i = 1, 2$. Then space $S^{\overline{s}}_{p,\overline{q}}B(R_2)$ is a retract of the space $l^{s_1}_{q_1}\left(B^{s_2,2}_{p,q_2}(R_2)\right)$. The appropriate operators are defined in* (7.8).

Proof. From (7.7), (7.8) we see that $S \in L\left(S^{\overline{s}}_{p,\overline{q}}B(R_2), l^{s_1}_{q_1}\left(B^{s_2,2}_{p,q_2}(R_2)\right)\right)$ and $RS = E$. Using the properties of the systems $\{\varphi_k\}_{k=0}^{\infty}$ and $\{\overline{\varphi}_k\}_{k=0}^{\infty}$ we have (see Remark 7.3)

$$\|Rg\|_{S^{\overline{s}}_{p,\overline{q}}B(R_2)} = \left(\sum_{k=0}^{\infty} 2^{ks_1q_1}\left\|\sum_{j=k-1}^{k+1} F^{-1}F_1\varphi_k(\xi_1)\,F_1\overline{\varphi}_j(\xi_1)\,Fg_j(\xi_1,\xi_2)\right\|^{q_1}_{B^{s_2,2}_{p,q_2}(R_2)}\right)^{1/q_1} =$$

$$= \left(\sum_{k=0}^{\infty} 2^{ks_1q_1}\left\|\sum_{r=-1}^{1} F^{-1}F_1\varphi_k(\xi_1)\,F_1\overline{\varphi}_{k+r}(\xi_1)\,Fg_{k+r}(\xi_1,\xi_2)\right\|^{q_1}_{B^{s_2,2}_{p,q_2}(R_2)}\right)^{1/q_1} \leqslant$$

$$\leqslant c\left(\sum_{k=0}^{\infty} 2^{ks_1q_1}\|g_k\|^{q_1}_{B^{s_2,2}_{p,q_2}(R_2)}\right)^{1/q_1} = c\,\|\{g_j\}\|_{l^{s_1}_{q_1}\left(B^{s_2,2}_{p,q_2}(R_2)\right)}.$$

So $R \in L\left(l^{s_1}_{q_1}\left(B^{s_2,2}_{p,q_2}(R_2)\right), S^{\overline{s}}_{p,\overline{q}}B(R_2)\right)$. □

Theorem 7.7. *Let $\overline{s} = (s_1, s_2)$, $\overline{m} = (m_1, m_2)$, $\overline{q} = (q_1, q_2)$, $\overline{r} = (r_1, r_2)$, $1 \leqslant q_i, r_i \leqslant \infty$, $-\infty < s_i, m_i < \infty$, $i = 1, 2$, $1 < p, p_0, p_1 < \infty$, $0 < \theta < 1$. If $s^* = (1-\theta)s_1 + \theta m_1$, $m^* = (1-\theta)s_2 + \theta m_2$, $\frac{1}{q^*} = \frac{1-\theta}{q_1} + \frac{\theta}{r_1}$, $\frac{1}{r^*} = \frac{1-\theta}{q_2} + \frac{\theta}{r_2}$, $\frac{1}{p^*} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ then*

(i) $\left[S^{\overline{s}}_{p_0,\overline{q}}B, S^{\overline{m}}_{p_1,\overline{r}}B\right]_{\theta} = S^{(s^*,m^*)}_{p^*,(q^*,r^*)}B$,

(ii) $\left(S^{\overline{s}}_{p,\overline{q}}B, S^{\overline{m}}_{p,\overline{r}}B\right)_{\theta,q^*} = S^{(s^*,m^*)}_{p,(q^*,q^*)}B$.

If in addition $p^* = q^*$, then

(iii) $\left(S^{\overline{s}}_{p_0,\overline{q}}B, S^{\overline{m}}_{p_1,\overline{r}}B\right)_{\theta,q^*} = S^{(s^*,m^*)}_{p^*,(q^*,q^*)}B$.

Proof. *Step* 1. From Theorem 7.6 we have for a certain interpolation functor Ψ

$$\|f\|_{\Psi\left(S^{\overline{s}}_{p_0,\overline{q}}B, S^{\overline{m}}_{p_1,\overline{r}}B\right)} \sim \left\|\left\{F^{-1}F_1\varphi_k(\xi_1)\,Ff(\xi_1,\xi_2)\right\}\right\|_{\Psi\left(l^{s_1}_{q_1}\left(B^{s_2,2}_{p_0,q_2}\right) l^{m_1}_{r_1}\left(B^{m_2,2}_{p_1,r_2}\right)\right)}. \tag{7.9}$$

Let us denote $A_0 = B^{s_2,2}_{p_0,q_2}(R_2)$, $A_1 = B^{m_2,2}_{p_1,r_2}(R_2)$. Then using, as the functor Ψ, the functor of the complex method, from (7.9) and Theorem 5.6.3 of [B-L] we get

$$\|f\|_{\left[S^{\overline{s}}_{p_0,\overline{q}}B, S^{\overline{m}}_{p_1,\overline{r}}B\right]_{\theta}} \sim \left\|\left\{F^{-1}F_1\varphi_k(\xi_1)\,Ff(\xi_1,\xi_2)\right\}\right\|_{\left[l^{s_1}_{q_1}\left(B^{s_2,2}_{p_0,q_2}\right) l^{m_1}_{r_1}\left(B^{m_2,2}_{p_1,r_2}\right)\right]_{\theta}} \sim$$

$$\sim \left\|\left\{F^{-1}F_1\varphi_k(\xi_1)\,Ff(\xi_1,\xi_2)\right\}\right\|_{l^{s^*}_{q^*}\left([A_0,A]_{\theta}\right)}. \tag{7.10}$$

We use the description of the interpolation space $[A_0, A_1]_{\theta}$ given in Theorem 2.4 (i) (see also Corollary 7.1). Then from (7.10) we have

$$\|f\|_{\left[S^{\overline{s}}_{p_0,\overline{q}}B, S^{\overline{m}}_{p_1,\overline{r}}B\right]_{\theta}} \sim \left\|\left\{F^{-1}F_1\varphi_k(\xi_1)\,Ff(\xi_1,\xi_2)\right\}\right\|_{l^{s^*}_{q^*}\left(B^{m^*,2}_{p^*,r^*}\right)}. \tag{7.11}$$

Now (i) follows from (7.11) and (7.7).

Step 2. Let Ψ be a functor of the real method. Then using Theorem 5.6.2 of [B-L] we obtain from (7.9)

$$\|f\|_{\left(S^{\overline{s}}_{p,\overline{q}}B, S^{\overline{m}}_{p,\overline{r}}B\right)_{\theta,q^*}} \sim \left\|\left\{F^{-1}F_1\varphi_k\left(\xi_1\right)Ff\left(\xi_1,\xi_2\right)\right\}\right\|_{l^{s^*}_{q^*}\left((A_0,A)_{\theta,q^*}\right)}, \tag{7.12}$$

where $A_0 = B^{s_2,2}_{p,q_2}(R_2)$, $A_1 = B^{m_2,2}_{p,r_2}(R_2)$. Using Theorem 2.3(i), Corollary 7.1 and Theorem 7.6 we get (ii).
Step 3. If in (7.12), we use as in *Step* 1, $A_0 = B^{s_2,2}_{p_0,q_2}(R_2)$, $A_1 = B^{m_2,2}_{p_1,r_2}(R_2)$, then we need to describe the interpolation space $(A_0,A_1)_{\theta,q^*}$. The additional assumption $p^* = q^*$ allows us to apply Theorem 2.4(ii). Then we obtain (iii). □

Remark 7.4. *In the same way we can also obtain other interpolation formulas using other interpolations formulas for B-spaces (see Theorem 2.3). If we assume that $A_0 = A_1 = A$ in (7.7) then using interpolation theorems for the spaces $l^s_q(A)$ (see [T-2, B-L]) we can obtain other interpolation formulas for spaces of functions of mixed smoothness.*

8 Traces

Let, as in Section 1, $\Re$ be a convex polyhedron (with vertices $(0,\ldots,0)$,$\alpha^j \in Z^+_n, j = 1,..,M$) such that $\Re$ has vertices on the each co-ordinate axes different from $(0,\ldots,0)$. We put (see (1.1))

$$\nu(\xi) = \left(1+\mu^2(\xi)\right)^{1/2} = \left(1+\sum_{j=1}^{M}\xi^{2\alpha^j}\right)^{1/2}. \tag{8.1}$$

We denote the co-ordinates of the vertices of the vertices of polyhedron $\Re$ by $m_0, m_1, \ldots, m_N$ where $0 = m_0 < m_1 < \ldots < m_N$. Assume that the points $(0, m_j)$, $j = 0,1,\ldots,N$ belong to $\Re$. We consider the hyper-planes

$$P_j = \{x = (x_1,\ldots x_n) \in R_n; x_n = m_j\}, j = 0,1,\ldots,N. \tag{8.2}$$

Let us denote by $\Re_j$ the projections onto $R_{n-1} = \{x \in R_n; x_n = 0\}$ of the cross-sections of the polyhedron $\Re$ by the hyper-planes P_j $(j = 0,1,\ldots,N)$. According to (8.1) the polyhedrons $\Re_j$ have their own functions $\nu_j(\xi')$, $\xi' = (\xi'_1,\ldots,\xi_{n-1})$, $j = 0,1,\ldots,N$. Then for the function $\nu(\xi)$ of the polyhedron $\Re$ we have

$$\nu(\xi) \sim \left(\sum_{j=0}^{N}\xi_n^{2m_j}\nu_j^2(\xi')\right)^{1/2}. \tag{8.3}$$

We formulate two lemmas, which were proved in [B-4].

Lemma 8.1. *Let $\Re$ be a convex polyhedron of considered type, $1 < p < \infty$, $\theta_j = \frac{1}{pm_j}$, $j = 0,1,\ldots,N$. Then there exists a positive number c_1 such that the following inequality*

$$\nu_0^{1-\theta_j}(\xi')\,\nu_j^{\theta_j}(\xi') \leqslant c_1\nu_0^{1-\theta_1}(\xi')\,\nu_1^{\theta_1}(\xi')$$

holds for all $\xi' \in R_{n-1}$ and all $j = 0,1,\ldots,N$.

With the help of Lemma 8.1 the following lemma is proved (in [B-4]).

Lemma 8.2. *Let $\Re$ be a convex polyhedron of considered type. Then there exists a positive number c_2 such that*

$$\nu(\xi) \leqslant c_2 \left(\nu_0^2(\xi') + \xi_n^{2m_N} \nu_0^2(\xi') \left(\frac{\nu_1(\xi')}{\nu_0(\xi')} \right)^{\frac{2m_N}{m_1}} \right)^{1/2}, \xi' \in R_{n-1}. \tag{8.4}$$

Let us consider the trace operator:

$$(Trf)(x') = f(x', 0), f \in S(R_n).$$

Theorem 8.1. *Let $\Re$ be a convex polyhedron of considered type, $1 < p < \infty$, $\theta = \frac{1}{pm_1}$. Then the trace operator is a retract from $H_p^1(\nu; R_n)$ onto $\boldsymbol{B} \equiv I_{\nu_0^{1-\theta}\nu_1^{\theta}}\left[B_{p,p}^0\left(\frac{\nu_0}{\nu_1}\right); R_{n-1}\right]$, where the mentioned H and B-spaces are defined in Definition 2.1 and Definition 2.2.*

Proof. We denote by $\Re^*$ the part of the polyhedron $\Re$ between hyper-planes P_0 and P_1. We denote the appropriate function (see (8.1)) by $\nu^*(\xi)$:

$$\nu^*(\xi) = \left[\nu_0^2(\xi') + \xi_n^{2m_1}\nu_1^2(\xi')\right]^{1/2}. \tag{8.5}$$

The function in the right-hand side of inequality (8.4) we denote by $\nu^{**}(\xi)$:

$$\nu^{**}(\xi) = \left(\nu_0^2(\xi') + \xi_n^{2m_N} \nu_0^2(\xi') \left(\frac{\nu_1(\xi')}{\nu_0(\xi')} \right)^{\frac{2m_N}{m_1}} \right)^{1/2}. \tag{8.6}$$

From (8.3), (8.4) we have

$$\nu^*(\xi) \leqslant \nu(\xi) \leqslant c_2 \nu^{**}(\xi), \xi \in R_n. \tag{8.7}$$

Inequalities (8.7) (see [V-P]) are equivalent to the embeddings

$$H_p^1(\nu^{**}; R_n) \subset H_p^1(\nu; R_n) \subset H_p^1(\nu^*; R_n) \tag{8.8}$$

for the appropriate H-spaces from Definition 2.1. From Theorem 2.1 we have that both pairs $\left\{H_p^1(\nu_0; R_{n-1}), H_p^1(\nu_1; R_{n-1})\right\}$ and $\left\{H_p^1(\nu_0; R_{n-1}), H_p^1\left(\nu_0^{1-\frac{m_N}{m}}\nu_1^{\frac{m_N}{m_1}}; R_{n-1}\right)\right\}$ are quasi-linearizable. Hence we can apply Theorem 1.8.5 of [T-2] for the spaces $H_p^1(\nu^*; R_n)$ and $H_p^1(\nu^{**}; R_n)$ with functions $\nu^*(\xi)$ and $\nu^{**}(\xi)$ from (8.5),(8.6). From Theorem 1.8.5 of [T-2] (see also Lemma 2.9.1 and Theorem 2.9.1 of [T-2]) we obtain that the trace operator is a retract

$$\text{from } H_p^1(\nu^*; R_n) \quad \text{onto} \quad \left(H_p^1(\nu_0; R_{n-1}), H_p^1(\nu_1; R_{n-1})\right)_{\theta,p} \tag{8.9}$$

and

$$\text{from } H_p^1(\nu^{**}; R_n) \quad \text{onto} \quad \left(H_p^1(\nu_0; R_{n-1}), H_p^1\left(\nu_0^{1-\frac{m_N}{m_1}}\nu_1^{\frac{m_N}{m_1}}; R_{n-1}\right)\right)_{\theta_N,p}, \tag{8.10}$$

where $\theta_N = 1/pm_N$.

Applying Theorem 4.2 and Proposition 2.1 after simple calculations we can see that both of the interpolation spaces in (8.9) and (8.10) are equal to $\mathbf{B}$. Now the proof is complete due to (8.8). □

Remark 8.1. *Theorem 8.1 shows that space of traces of the H-space generated by the polyhedron $\Re$ depends only on $\Re^*$(the part of $\Re$ between hyper-plans P_0 and P_1 from* (8.2)*). Other parts of $\Re$ do not play any role.*

Remark 8.2. *To find the space of traces for the space $H_p^1(\nu; R_n)$ as we see in Theorem 8.1, we need to write down the functions $\nu_0(\xi')$ and $\nu_1(\xi')$. From* (8.3) *we have*

$$\nu^2(\xi) \sim \sum_{j=0}^{N} \xi_n^{2m_j} \nu_j^2(\xi').$$

The functions $\nu_j^2(\xi')$ can be found as the coefficients of the polynomial. Then we have

$$\nu_0^2(\xi') \sim \nu^2(\xi', 0), \nu_1^2(\xi') \sim \frac{\partial^{2m_1}\nu^2}{\partial \xi_n^{2m_1}}(\xi', 0).$$

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