EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 1, Number 4 (2010), 20 – 31

PROPERTIES OF ITERATED NORMS IN NIKOL'SKII-BESOV TYPE SPACES WITH GENERALIZED SMOOTHNESS

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Communicated by V.I. Burenkov

Key words: iterated norms, Nikol'skii-Besov spaces, generalized smoothness.

AMS Mathematics Subject Classification: 54C35, 35B65.

Abstract. In this paper we consider iterated norms of Nikol'skii–Besov type in the spaces $\mathcal{B}^{\varphi}_{\theta}(\cdots(\mathcal{B}^{\varphi}_{\theta}(L_p(\Omega)))\cdots)$ with generalized smoothness φ and study their properties.

1 Introduction

In the works of V.I. Burenkov [3], [4], the iterated norms of Nikol'skii–Besov type and the spaces $\mathcal{B}^{l_k}_{\theta}(...\mathcal{B}^{l_1}_{\theta}(L_p(\Omega))...)$, where $l = (l_1, ..., l_n),\ l_j > 0,\ 1 \le \theta \le \infty$ and Ω is an open set in $\mathbb{R}^n$, were introduced. Under certain assumptions, it was proved that this space coincides with $B^{l_1+...+l_k}_{p,\theta}(\Omega)$. Using this theorem and the lemma on fractional differentiation of inequalities it was proved that if for some $l > 0$ and for each $\delta > 0$ there exists c_δ such that for all classical solutions of a linear equation $\mathcal{P}(\mathcal{D})u = 0$ in Ω with constant coefficients the inequality

$$\|u\|_{B^{l}_{p,\ \theta}(G_\delta)} \le c_\delta \|u\|_{L_p(G)}$$

holds for every open parallelepiped G with faces parallel to the coordinate planes satisfying $\overline{G} \subset \Omega$, then every classical solution of the equation $\mathcal{P}(\mathcal{D})u = 0$ is infinitely differentiable in Ω. (Here $G_\delta = \{x \in G : dist(x, \partial G) > \delta\}$.) To prove this, after $(k-1)$–fold fractional differentiation of the given inequality it was established that for all $\mu \ge 0$

$$\|f\|_{\underbrace{\mathcal{B}^{l}_{\theta}(\cdots(\mathcal{B}^{l}_{\theta}}_{k}(L_p(G_{\mu+\delta})))\cdots)} = \|f\|_{\mathcal{B}^{l}_{\theta}(\underbrace{\mathcal{B}^{l}_{\theta}(\cdots(\mathcal{B}^{l}_{\theta}}_{k-1}(L_p(G_\mu)))\cdots))}.$$

Then, using the theorem on iterated norms, it was deduced that

$$u \in \bigcap_{k=1}^{\infty} \underbrace{\mathcal{B}^{l}_{\theta}(\cdots(\mathcal{B}^{l}_{\theta}}_{k}(L_p(G_\delta)))\cdots) = \bigcap_{k=1}^{\infty} B^{kl}_{p,\ \theta}(G_\delta) \subset C^{\infty}(G_\delta).$$

In this paper we consider iterated norms of Nikol'skii–Besov type in the spaces $\mathcal{B}^{\varphi}_{\theta}(\cdots(\mathcal{B}^{\varphi}_{\theta}(L_p(\Omega)))\cdots)$ when the numerical index of smoothness l is replaced by a function φ belonging to a certain class of functions $\Phi(\sigma,\theta)$ and such that for any integer $k \geq 2$,

$$\|f\|_{\underbrace{\mathcal{B}^{\varphi}_{\theta}(\cdots(\mathcal{B}^{\varphi}_{\theta}(}_{k} L_p(\Omega)))\cdots)} = \|f\|_{\mathcal{B}^{\varphi}_{\theta}}\underbrace{(\mathcal{B}^{\varphi}_{\theta}(\cdots(\mathcal{B}^{\varphi}_{\theta}(}_{k-1} L_p(\Omega)))\cdots))},$$

and study their properties.

2 Definition of Nikol'skii-Besov type spaces with generalized smoothness

Let $\Omega \subset \mathbb{R}^n$ be open and $\forall \delta > 0$ $Z(\Omega_\delta)$ be a semi-normed space of functions defined in Ω_δ. Further let $1 \leq \theta < \infty$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ where $\sigma_j \in \mathbb{N}$ and let a function $\varphi = (\varphi_1, \cdots, \varphi_n)$ belong to the class of functions $\Phi(\sigma,\theta)$ which means that for each $j = 1, ..., n$

1. $\varphi_j(h) > 0$ for all $h > 0$,
2. φ_j increases monotonously,
3. $\varphi_j(h) \to 0$ as $h \to 0^+$,
4. φ_j possesses S_{σ_j} property, i.e., $\exists m_j \in (0, \sigma_j)$ such that $\varphi_j(t)t^{-m_j}$ is almost decreasing, that is $\exists c \geq 1$ such that $\varphi_j(t)t^{-m_j} \leq c\varphi_j(s)s^{-m_j}$ for all $0 < s \leq t < \infty$,
5. $\left\|\varphi_j^{-1}(h)h^{\sigma-\frac{1}{\theta}}\right\|_{L_\theta(0,1)} < \infty$.

Definition 4. *We say that a function* $f \in B^{\varphi}_{\theta}\big(Z(\Omega)\big) \equiv \mathcal{B}^{\varphi,\sigma;H}_{\theta}\big(Z(\Omega)\big)$ *if* $f \in L^1_{loc}(\Omega)$ *and*

$$\|f\|_{\mathcal{B}^{\varphi}_{\theta}(Z(\Omega))} := \|f\|_{Z(\Omega)} + \sum_{j=1}^{n} \|f\|_{\beta^{\varphi_j}_{\theta,j}(Z(\Omega))} < \infty, \tag{2.1}$$

where

$$\|f\|_{\beta^{\varphi_j}_{\theta,j}}(Z(\Omega)) := \left\|\varphi_j^{-1}(h)\right\| \Delta^{\sigma_j}_{h,j} f \left\|_{Z(\Omega_{\sigma_j h})}\right\|_{L^*_\theta(0,H)}, \tag{2.2}$$

$\Delta^{\sigma_j}_{h,j} f$ *is the difference of* f *of order* σ_j *with respect to* x_j *with step* h*, and* $L^*_\theta(0,H)$ *with* $1 \leq \theta < \infty$ *denotes the space of all functions* g *of one variable, measurable on* $(0,H)$), *for which*

$$\|g\|_{L^*_\theta(0,H)} := \left(\int_0^H |g(h)|^\theta \frac{dh}{h}\right)^{1/\theta} < \infty \qquad (L^*_\infty(0,H) = L_\infty(0,H)).$$

This definition is obtained from the well-known definition of the anisotropic Nikol'skii-Besov space $B^l_{p,\theta}(\Omega)$ ([6], [2], [5]) where the norm $\|\cdot\|_{L_p(\Omega_{\sigma_j h})}$ replaced by $\|\cdot\|_{Z(\Omega_{\sigma_j h})}$ and numerical index of smoothness l by a function $\varphi \in \Phi(\sigma,\theta)$. Thus $\mathcal{B}^{\varphi}_{\theta}\big(L_p(\Omega)\big) \equiv B^{\varphi}_{p,\theta}(\Omega)$.

Following the procedures in [4] and substituting in (2.1) and (2.2) $\varphi := \varphi_2 = (\varphi_{21},\cdots,\varphi_{2n})$, $\sigma := \sigma_2 = (\sigma_{21},\cdots,\sigma_{2n})$, $H := H_2$ and $Z(\Omega_\delta) := \mathcal{B}^{\varphi_1}_{p,\,\theta}\big(L_p(\Omega_\delta)\big) \equiv B^{\varphi_1}_{p,\,\theta}(\Omega_\delta)$ with the parameters $\sigma_1 = (\sigma_{11},\cdots,\sigma_{1n})$ and H_1, we obtain the norm of the form

$$\|f\|_{\mathcal{B}^{\varphi_2}_{\theta}\left(\mathcal{B}^{\varphi_1}_{\theta}\left(L_p(\Omega)\right)\right)}.$$

Continuing this process, we obtain the following norms which we naturally call *iterated norms* for all integers $k \geq 2$,

$$\|f\|_{\underbrace{\mathcal{B}^{\varphi_k}_{\theta}(\cdots(\mathcal{B}^{\varphi_1}_{\theta}(}_{k} L_p(\Omega)))\cdots)} = \|f\|_{\mathcal{B}^{\varphi_k}_{\theta}\underbrace{(\mathcal{B}^{\varphi_{k-1}}_{\theta}(\cdots(\mathcal{B}^{\varphi_1}_{\theta}(}_{k-1} L_p(\Omega)))\cdots))} \tag{2.3}$$

with the corresponding spaces $\mathcal{B}^{\varphi_k}_{\theta}\big(\cdots\big(\mathcal{B}^{\varphi_1}_{\theta}\big(L_p(\Omega)\big)\big)\cdots\big)$, which we call *iterated Nikol'skii–Besov spaces with generalized smoothness.*

In particular, taking $\varphi_1 = \cdots = \varphi_k = \varphi$ for all integers $k \geq 2$, we obtain the iterated norm

$$\|f\|_{\underbrace{\mathcal{B}^{\varphi}_{\theta}(\cdots(\mathcal{B}^{\varphi}_{\theta}(}_{k} L_p(\Omega)))\cdots)} = \|f\|_{\mathcal{B}^{\varphi}_{\theta}\underbrace{(\mathcal{B}^{\varphi}_{\theta}(\cdots(\mathcal{B}^{\varphi}_{\theta}(}_{k-1} L_p(\Omega)))\cdots))} \tag{2.4}$$

and the corresponding spaces $\mathcal{B}^{\varphi}_{\theta}\big(\cdots\big(\mathcal{B}^{\varphi}_{\theta}\big(L_p(\Omega)\big)\big)\cdots\big)$.

3 Properties of iterated norms

Lemma 2. (On fractional differentiation of an inequality). *Let $\mu_0 > 0$ and $\Omega \subset \mathbb{R}^n$ be an open set, and for each $\mu \in [0,\mu_0)$ let a set of functions $T(\Omega_\mu)$ and seminormed function spaces $X(\Omega_\mu)$ and $Y(\Omega_\mu)$ be defined such that*

$$T(\Omega_\mu) \cap X(\Omega_\mu) \subset T(\Omega_\mu) \cap Y(\Omega_\mu) \tag{3.1}$$

and

$$\|f\|_{X(\Omega_\mu)} \leq \|f\|_{Y(\Omega_\mu)} \qquad \forall f \in T(\Omega_\mu) \cap Y(\Omega_\mu) \tag{3.2}$$

(if $\Omega_\mu = \emptyset$ for some μ, we assume that $\|f\|_{X(\Omega_\mu)} := \|f\|_{Y(\Omega_\mu)} := 0$).

Further let $1 \leq \theta \leq \infty$, $\sigma = (\sigma_1,\cdots,\sigma_n)$, $\sigma_j \in \mathbb{N}$, and let

$$0 < H < \mu_0(\max_{1\leq j\leq n} \sigma_j)^{-1}. \tag{3.3}$$

Then

$$\forall \mu \in \Big[0, \mu_0 - H \max_{1\leq j\leq n} \sigma_j)\Big) \tag{3.4}$$

and $\forall f \in T(\Omega_\mu) \cap \mathcal{B}_\theta^\varphi(Y(\Omega_\mu))$ *such that* $\forall j \in \{1, 2, ..., n\}$

$$\Delta_{h,j}^{\sigma_j} f \in T(\Omega_{\mu+\sigma_j h}) \qquad \forall h \in (0, H) \tag{3.5}$$

the following inequality is satisfied

$$\|f\|_{\mathcal{B}_p^\varphi(X(\Omega_\mu))} \le \|f\|_{\mathcal{B}_p^\varphi(Y(\Omega_\mu))}. \tag{3.6}$$

Proof. Let $\mu \in \left[0, \mu_0 - H \max\limits_{1\le j\le n} \sigma_j\right]$ and $f \in T(\Omega_\mu) \cap \mathcal{B}_\theta^\varphi(Y(\Omega_\mu))$. Then by (3.5), for all $j \in \{1, ..., n\}$ and all $h \in (0, H)$ we have that $\Delta_{h,j}^{\sigma_j} f \in T(\Omega_{\mu+\sigma_j h})$. Moreover, by the definition of the spaces $\mathcal{B}_\theta^\varphi(Y(\Omega_\mu))$ for almost all $h \in (0, H)$ we have that $\Delta_{h,j}^{\sigma_j} f \in Y(\Omega_{\mu+\sigma_j h})$. Consequently, by (3.2), for almost all $h \in (0, H)$

$$\|\Delta_{h,j}^{\sigma_j} f\|_{X(\Omega_{\mu+\sigma_j h})} \le \|\Delta_{h,j}^{\sigma_j} f\|_{Y(\Omega_{\mu+\sigma_j h})}.$$

Multiplying both sides of this inequality by $\varphi_j^{-1}(h)$ and then applying the $L_\theta^*(0, H)$ norm, we obtain that

$$\|f\|_{\beta_{\theta,j}^{\varphi_j}(X(\Omega_\mu))} = \|\varphi_j^{-1}(h)\|\Delta_{h,j}^{\sigma_j} f\|_{X(\Omega_{\mu+\sigma_j h})}\|_{L_\theta^*(0,H)} \le$$

$$\le \|\varphi_j^{-1}(h)\|\Delta_{h,j}^{\sigma_j} f\|_{Y(\Omega_{\mu+\sigma_j h})}\|_{L_\theta^*(0,H)} = \|f\|_{\beta_\theta^{\varphi_j}(Y(\Omega_\mu))}.$$

Combining these inequalities with (3.2), we obtain (3.6). □

Remark 10.1. *We will use Lemma 2 in two cases:*

1. $T(\Omega_\mu)$ *is the set of all measurable functions in* Ω_μ. *In this case condition* (3.5) *is satisfied in an obvious way and the condition* $f \in T(\Omega_\mu)$ *can be omitted from the statement of Lemma 2.*

2. $T(\Omega_\mu) = \{f \in C^N(\Omega_\mu) : (\mathcal{P}f)(x) = 0\ \forall x \in \Omega_\mu\}$, *where the operator* $\mathcal{P} = \mathcal{P}(D) := \sum\limits_{|\alpha|\le N} a_\alpha D^\alpha$ *is a linear differential operator of degree* N *with constant coefficients* $a_\alpha \in \mathbb{R}$, *with* $\alpha \in \mathbb{N}_0^n$ *a multi-index,* $|\alpha| = \alpha_1 + ... + \alpha_n$, *and* $D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$. *In this case condition* (3.5) *is satisfied, because* $\Delta_{h,j}^{\sigma_j} f \in C^N(\Omega_{\mu+\sigma_j h})$ *and*

$$(\mathcal{P}(\Delta_{h,j}^{\sigma_j} f))(x) = (\Delta_{h,j}^{\sigma_j}(\mathcal{P}f))(x) = 0 \ \ \forall x \in \Omega_{\mu+\sigma_j h},$$

   since $\mathcal{P}$ *is an operator with constant coefficients.*

Corollary 2. *Let conditions* (3.2) *and* (3.3) *be satisfied, let* $1 \le \theta \le \infty$, $k \in \mathbb{N}$, $\sigma_s = (\sigma_{s1}, \dots, \sigma_{sn})$, $\sigma_{sj} \in \mathbb{N}$, $j = 1, \dots, n$, $s = 1, \dots, k$, *and*

$$\sum_{s=1}^k H_s \max_{1\le j\le n} \sigma_{sj} < \mu_0. \tag{3.7}$$

Then

$$\forall \mu \in \left[0, \mu_0 - \sum_1^k H_s \max_{1\le j\le n} \sigma_{sj}\right) \tag{3.8}$$

and $\forall f \in T(\Omega_\mu) \cap \mathcal{B}_\theta^{\varphi_k}\big(\cdots\big(\mathcal{B}_\theta^{\varphi_1} Y(\Omega_\mu)\big)\cdots\big)$ *which are such that* $\forall j_s \in \{1, ..., n\}$

$$\Delta_{h_1,j_1}^{\sigma_{1,j_1}} ... \Delta_{h_k,j_k}^{\sigma_{k,j_k}} f \in T(\Omega_{\mu+\sum_1^k \sigma_{sj} h_s}) \qquad \forall h_s \in (0, H_s), \tag{3.9}$$

the following inequality is satisfied

$$\|f\|_{\mathcal{B}_\theta^{\varphi_k}(\cdots(\mathcal{B}_\theta^{\varphi_1}(X(\Omega_\mu)))\cdots)} \leq \|f\|_{\mathcal{B}_\theta^{\varphi_k}(\cdots(\mathcal{B}_\theta^{\varphi_1}(Y(\Omega_\mu)))\cdots)}. \tag{3.10}$$

Corollary 2 can be proved by induction. In particular, if $\varphi_k = ... = \varphi_1 = \varphi$, then

$$\|f\|_{\underbrace{\mathcal{B}_\theta^\varphi(\cdots(\mathcal{B}_\theta^\varphi(}_{k} X(\Omega_\mu)))\cdots)} \leq \|f\|_{\underbrace{\mathcal{B}_\theta^\varphi(\cdots(\mathcal{B}_\theta^\varphi(}_{k} Y(\Omega_\mu)))\cdots)}. \tag{3.11}$$

Remark 10.2. *If* $\mu_0 = \infty$, *then Lemma 2 and its corollaries become simpler: it is not required to impose bounds on* H *(consequently on* H_s*), and inequalities* (3.1), (3.2), (3.6), (3.10) *and* (3.11) *are satisfied for all* $\mu \geq 0$.

Remark 10.3. *If, instead of* (3.2), *we consider the more general inequality*

$$\|f\|_{X(\Omega_\mu)} \leq \sum_{r=1}^{r_0} c_r \|f\|_{Y_r(\Omega_\mu)} \tag{3.12}$$

where $c_r \geq 0$ *and* $Y_r(\Omega_\mu)$ *is a seminormed space (for* $r \in \{1, ..., r_0\}$*), then the analogous statements are valid. It is necessary to consider the semi-norm*

$$\|f\|_{Y(\Omega_\mu)} := \sum_{r=1}^{r_0} c_r \|f\|_{Y_r(\Omega_\mu)}$$

and to take into account that, by definition,

$$\|f\|_{\mathcal{B}_\theta^\varphi(Y(\Omega_\mu))} \leq \sum_{r=1}^{r_0} c_r \|f\|_{\mathcal{B}_\theta^\varphi(Y(\Omega_\mu))}.$$

In the following lemma we will introduce on the space $\mathcal{B}_\theta^{\varphi_k}(...(\mathcal{B}_\theta^{\varphi_1}(L_p(\Omega)))...)$ an equivalent norm which is more convenient for estimation.

We set

$$\|f\|^*_{\mathcal{B}_\theta^{\varphi_k}(...(\mathcal{B}_\theta^{\varphi_1}(L_p(\Omega)))...)} := \|f\|_{L_p(\Omega)} + \sum_{r=1}^{k} \sum_{\substack{i_1,...,i_r=1 \\ i_1<i_2<\cdots<i_r}}^{k} \sum_{j_1,...,j_r=1}^{n} \|f\|_{\beta_{\theta,j_r}^{\varphi_{i_r},j_r}(...(\beta_{\theta,j_1}^{\varphi_{i_1},j_1}(L_p(\Omega))...)}. \tag{3.13}$$

Lemma 3. *For all considered values of the parameters*

$$\|f\|^*_{\mathcal{B}_\theta^{\varphi_k}(...(\mathcal{B}_\theta^{\varphi_1}(L_p(\Omega)))...)} \sim \|f\|_{\mathcal{B}_\theta^{\varphi_k}(...(\mathcal{B}_\theta^{\varphi_1}(L_p(\Omega)))...)}, \tag{3.14}$$

where $\sim$ *denotes the equivalence of norms.*

Proof.

$$\begin{aligned}\|f\|_{\mathcal{B}^{\varphi_2}_\theta(\mathcal{B}^{\varphi_1}_\theta(L_p(\Omega)))} &= \|f\|_{\mathcal{B}^{\varphi_1}_\theta(L_p(\Omega))} + \sum_{j_2=1}^n \Big\| \varphi^{-1}_{2,j_2}(h_2)\|\Delta^{\sigma_{2,j_2}}_{h_2,j_2} f\|_{\mathcal{B}^{\varphi_1}_\theta(L_p(\Omega))}\Big\|_{L^*_\theta(0,H_2)} \\ &= \|f\|_{L_p(\Omega))} + \sum_{j_1=1}^n \Big\| \varphi^{-1}_{1,j_1}(h_1)\|\Delta^{\sigma_{1,j_1}}_{h_1,j_1} f\|_{L_p(\Omega_{\sigma_{1,j_1}h_1})}\Big\|_{L^*_\theta(0,H_2)} \\ &+ \sum_{j_2=1}^n \|\varphi^{-1}_{2,j_2}(h_2)\big(\Lambda_{0,j_2} + \sum_{j_1=1}^n \Lambda_{j_1,j_2}\big)\|_{L^*_\theta(0,H)},\end{aligned}$$

where

$$\Lambda_{0,j_2} := \|\Delta^{\sigma_{2,j_2}}_{h_2,j_2} f\|_{L_p(\Omega_{\sigma_{2,j_2}h_2})}$$

and

$$\Lambda j_1, j_2 := \Big\| \varphi^{-1}_{1,j_1}(h_1)\|\Delta^{\sigma_{1,j_1}}_{h_1,j_1}\Delta^{\sigma_{2,j_2}}_{h_2,j_2} f\|_{L_p(\Omega_{\sigma_{1,j_1}h_1+\sigma_{2,j_2}h_2})}\Big\|_{L^*_\theta(0,H_1)}.$$

Making use of the fact that for nonnegative functions φ_s

$$(n+1)^{\frac{1}{\theta}-1}\sum_{s=0}^n \|\varphi_s\|_{L^*_\theta(0,H_2)} \le \Big\|\sum_{s=0}^n \varphi_s\Big\|_{L^*_\theta(0,H_2)} \le \sum_{s=0}^n \|\varphi_s\|_{L^*_\theta(0,H_2)},$$

and taking into account that

$$\|\varphi^{-1}_{2,j_2}(h_2)\Lambda_{0,j_2}\|_{L^*_\theta(0,H_2)} = \|f\|_{\beta^{\varphi_{2,j_2}}_{\theta,j_2}(L_p(\Omega))},$$

and

$$\|\varphi^{-1}_{2,j_2}(h_2)\Lambda_{j_1,j_2}\|_{L^*_\theta(0,H_2)} = \|f\|_{\beta^{\varphi_{2,j_2}}_{\theta,j_2}\left(\beta^{\varphi_{1,j_1}}_{\theta,j_1}(L_p(\Omega))\right)},$$

we obtain that

$$\begin{aligned}\|f\|_{\mathcal{B}^{\varphi_2}_\theta(\mathcal{B}^{\varphi_1}_\theta(L_p(\Omega)))} \sim \|f\|_{L_p(\Omega)} + \sum_{j_1=1}^n \|f\|_{\beta^{\varphi_{1,j_1}}_{\theta,j_1}(L_p(\Omega))} + \sum_{j_2=1}^n \|f\|_{\beta^{\varphi_{2,j_2}}_{\theta,j_2}(L_p(\Omega))} \\ + \sum_{j_1=1}^n\sum_{j_2=1}^n \|f\|_{\beta^{\varphi_{2,j_2}}_{\theta,j_2}\left(\beta^{\varphi_{1,j_1}}_{\theta,j_1}(L_p(\Omega))\right)},\end{aligned}$$

which is (3.14) for $k=2$. The case $k>2$ follows by induction. □

We notice further that for all considered values of the parameters

$$\begin{aligned}\|f\|_{\beta^{\varphi_2}_\theta(\beta^{\varphi_1}_\theta(L_p(\Omega))} &= \Big\|\big\|\varphi_2^{-1}(h)\varphi_1^{-1}(\eta)\|\Delta^{\sigma_1}_\eta\Delta^{\sigma_2}_h f\|_{L_p(\Omega_{\sigma_1\eta+\sigma_2})}\big\|_{L^*_\theta(0,H_1)}\Big\|_{L^*_\theta(0,H_2)} \\ = \Big\|\big\|\varphi_1^{-1}(\eta)\varphi_2^{-1}(h)\|\Delta^{\sigma_2}_h\Delta^{\sigma_1}_\eta f\|_{L_p(\Omega_{\sigma_1\eta+\sigma_2})}\big\|_{L^*_\theta(0,H_1)}\Big\|_{L^*_\theta(0,H_2)} &= \|f\|_{\beta^{\varphi_1}_\theta(\beta^{\varphi_2}_\theta(L_p(\Omega))}. \quad (3.15)\end{aligned}$$

In the rest of this section, we shall mostly consider the one-dimensional case. Everywhere below, till Lemma 6 $n:=1$ and $G:=(a,b)$, where $-\infty \le a \le b \le \infty$.

Lemma 4. *Let* $1 \le p, \theta \le \infty$, $\sigma_1, \sigma_2 \in \mathbb{N}$, $\varphi_1 \in \Phi(\sigma_1, \theta)$, $\varphi_2 \in \Phi(\sigma_2, \theta)$ *and* $0 < H \le \infty$. *Then*

$$\|f\|_{\beta_\theta^{\varphi_1\varphi_2,\ \sigma_1+\sigma_2;H}(L_p(G)))} \le c_1 \|f\|_{\beta_\theta^{\varphi_2,\sigma_2;H}(\mathcal{B}_\theta^{\varphi_1,\sigma_1;H}(L_p(G)))} \tag{3.16}$$

and c_1 *depends only on* $\min\{\sigma_1, \sigma_2\}$.

Proof. Taking into account (3.15), without loss of generality, let $\sigma_1 \le \sigma_2$. We use the inequality

$$\|\Delta_h^\sigma f\|_{L_p(G_{\sigma h})} \le A(\sigma)\frac{1}{h}\int_0^h \|\Delta_\eta^\sigma f\|_{L_p(G_{\sigma\eta})} d\eta, \tag{3.17}$$

where $A(\sigma) > 0$ depends only on σ and is a monotonously increasing function (see [4], [5] and the references therein).

From (3.17) it follows that

$$\|\Delta_h^\sigma f\|_{L_p(G_{\sigma h})} \le A(\sigma)\left(\frac{1}{h}\int_0^h \|\Delta_\eta^\sigma f\|^\theta_{L_p(G_{\sigma\eta})} d\eta\right)^{1/\theta}$$

$$\le A(\sigma)\left(\int_0^h \|\Delta_\eta^\sigma f\|^\theta_{L_p(G_{\sigma\eta})} \frac{d\eta}{\eta}\right)^{1/\theta} = A(\sigma)\| \|\Delta_h^\sigma f\|_{L_p(G_{\sigma h})}\|_{L_\theta^*(0,h)}.$$

Furthermore,

$$\begin{aligned}\|\Delta_h^{\sigma_1+\sigma_2} f\|_{L_p(G_{(\sigma_1+\sigma_2)h})} &= \|\Delta_h^{\sigma_1}(\Delta_h^{\sigma_2} f)\|_{L_p((G_{(\sigma_2 h)\sigma_1 h}))} \\ &\le A(\sigma_1)\left\| \|\Delta_\eta^{\sigma_1}\Delta_h^{\sigma_2} f\|_{L_p((G_{(\sigma_2 h)\sigma_1\eta}))}\right\|_{L_\theta^*(0,H)} \\ \le A(\sigma_1)\varphi_1(h) &\left\| \varphi_1^{-1}(\eta)\|\Delta_\eta^{\sigma_1}\Delta_h^{\sigma_2} f\|_{L_p((G_{(\sigma_1\eta+\sigma_2 h)}))}\right\|_{L_\theta^*(0,H)},\end{aligned}$$

from which we get

$$\begin{aligned}\|f\|_{\beta_\theta^{\varphi_1\varphi_2,\sigma_1+\sigma_2;\,H}(G)} &= \left\|\varphi_1^{-1}(h)\varphi_2^{-1}(h)\|\Delta_\eta^{\sigma_1}(\Delta_h^{\sigma_2} f)\|_{L_p(G_{\sigma_1\eta+\sigma_2 h})}\right\| \\ &\le A(\sigma_1)\left\|\varphi_2^{-1}(h)\|\varphi_1^{-1}(\eta)\|\Delta_\eta^{\sigma_1}\Delta_h^{\sigma_2} f\|_{L_p(G_{\sigma_1\eta+\sigma_2 h})}\|_{L_\theta^*(0,H)}\right\|_{L_\theta^*(0,H)} \\ &= A(\sigma_1)\|f\|_{\beta_\theta^{\varphi_1,\,\sigma_1;\,H}(\beta_\theta^{\varphi_2,\,\sigma_2;\,H}(L_p(G)))}.\end{aligned}$$

We also note that $\varphi_1\varphi_2 \in \Phi(\sigma_1+\sigma_2, \theta)$. □

Corollary 3. *For any natural* k, $1 \le p, \theta \le \infty$, $\sigma \in \mathbb{N}$, $\varphi \in \Phi(\sigma, \theta)$ *and* $0 < H \le \infty$

$$\|f\|_{\beta_\theta^{\varphi^k,k\sigma;H}(L_p(G))} \le c_2^{k-1}\|f\|_{\underbrace{\beta_\theta^{\varphi,\sigma;H}(\ldots\beta_\theta^{\varphi,\sigma;H}}_{k}(L_p(G))\ldots)}, \tag{3.18}$$

where c_2 *depends only on* σ.

Proof. From (3.17) it follows that

$$\|f\|_{\beta_\theta^{\varphi^2,\,2\sigma;\,H}(L_p(G_\mu))} \le A(\sigma)\|f\|_{\beta_\theta^{\varphi,\,\sigma;\,H}(\beta_\theta^{\varphi,\,\sigma;\,H}(L_p(G_\mu))}$$

for all $\mu \ge 0$. By Lemma 2 with $T(G_\mu)$ being the set of all functions measurable on G_μ and $\mu_0 = \infty$

$$\|f\|_{\beta_\theta^{\varphi,\,\sigma;\,H}(\beta_\theta^{\varphi^2,\,2\sigma;\,H}(L_p(G_\mu)))} \le A(\sigma)\|f\|_{\beta_\theta^{\varphi,\,\sigma;\,H}(\beta_\theta^{\varphi,\,\sigma;\,H}(\beta_\theta^{\varphi,\,\sigma;\,H}(L_p(G_\mu))))}$$

for all $\mu \ge 0$. Using (3.17), we get that

$$\|f\|_{\beta_\theta^{\varphi^3,\,3\sigma;\,H}(L_p(G_\mu))} \le A(\sigma)^2\|f\|_{\beta_\theta^{\varphi,\,\sigma;\,H}(\beta_\theta^{\varphi,\,\sigma;\,H}(\beta_\theta^{\varphi,\,\sigma;\,H}(L_p(G_\mu))))},$$

then (3.18) follows by induction. □

Corollary 4. *Let $\Omega \subset \mathbb{R}^1$ be an arbitrary open set. Then, under the conditions of Lemma 4,*

$$\|f\|_{\beta_\theta^{\varphi_1\varphi_2,\sigma_1+\sigma_2;H}(L_p(\Omega)))} \le c_1\|f\|_{\beta_\theta^{\varphi_2,\sigma_2;H}(\mathcal{B}_\theta^{\varphi_1,\sigma_1;H}(L_p(\Omega))).} \qquad (3.19)$$

Proof. The proof follows by Lemma 4 as in [3], [4].

Lemma 5. *Let $1 \le p,\ \theta \le \infty,\ \sigma \in \mathbb{N},\ \varphi \in \Phi(\sigma,\theta),\ k \in \mathbb{N}$ and $0 < H < \infty$. Then*

$$\|f\|_{\underbrace{\beta_\theta^{\varphi,\sigma;\infty}(\ldots\beta_\theta^{\varphi,\sigma;\infty}}_{k}(L_p(G))\ldots)} \le c_3\sum_{r=0}^{k}\varphi(H)^{r-k}\|f\|_{\underbrace{\beta_\theta^{\varphi,\sigma;H}(\ldots\beta_\theta^{\varphi,\sigma;H}}_{r}(L_p(G))\ldots)}, \qquad (3.20)$$

where c_3 is independent of f and H.

Proof. The proof follows by Corollary 3.10 and relation (33) of [4].

Lemma 6. *Let $1 < p < \infty,\ 1 \le \theta \le \infty,\ \sigma \in \mathbb{N},\ \varphi \in \Phi(\sigma,\theta)$. Then*

$$\|f\|_{\mathcal{B}_\theta^{\varphi,\sigma;\infty}(\mathcal{B}_\theta^{\varphi,\sigma;\infty}(L_p(\mathbb{R}^n)))} \sim \|f\|_{\mathcal{B}_\theta^{\varphi^2,2\sigma;\infty}(L_p(\mathbb{R}^n))}. \qquad (3.21)$$

Proof. Let us consider the expansion

$$f = \sum_{k=0}^{\infty} f * v_k \qquad (3.22)$$

where $v_0 = \mathcal{F}^{-1}(\chi_{P_0})$ and for $k \in \mathbb{N}$ $v_k = \mathcal{F}^{-1}(\chi_{P_k \setminus P_{k-1}})$ ($\mathcal{F}$ denotes Fourier transform) and

$$P_k = \{\xi \in \mathbb{R}^n : |\xi_j| < \varphi_j^{(-1)}(2^k)\}.$$

Then, similarly to the appropriate argument argument in [3],

$$\|f\|_{\mathcal{B}_{p,\,\theta}^{\varphi,\,\sigma;\infty}(\mathbb{R}^n)} \sim \left[\sum_{k=0}^{\infty}\left(2^k\|f * v_k\|_{L_p(\mathbb{R}^n)}\right)^\theta\right]^{1/\theta}. \qquad (3.23)$$

We shall first show that

$$\|f\|_{\mathcal{B}^{\varphi,\sigma;\infty}_{p,\theta}\left(\mathcal{B}^{\varphi,\sigma;\infty}_{p,\theta}(\mathbb{R}^n)\right)} \sim \left[\sum_{k=0}^{\infty}\left(2^k\|f*v_k\|_{\mathcal{B}^{\varphi,\ \sigma;\infty}_{p,\theta}(\mathbb{R}^n)}\right)^{\theta}\right]^{1/\theta}. \tag{3.24}$$

Put

$$\|f\|_{\mathcal{B}^{\varphi,\sigma;\infty}_{p,\theta}\left(\mathcal{B}^{\varphi,\sigma;\infty}_{p,\theta}(\mathbb{R}^n)\right)} := \underbrace{\|f\|_{\mathcal{B}^{\varphi,\sigma;\infty}_{p,\theta}(\mathbb{R}^n)}}_{S_1} + \underbrace{\sum_{j=1}^{n}\|\varphi_j^{-1}(h)\|\Delta^{\sigma_j}_{h,j}f\|_{\mathcal{B}^{\varphi,\sigma;\infty}_{p,\theta}(\mathbb{R}^n)}\|_{L^*_\theta(0,\infty)}}_{S_2}. \tag{3.25}$$

Then, $S_1 \sim \left[\sum\limits_{k=0}^{\infty}\left(2^k\|f*v_k\|_{L_p(\mathbb{R}^n)}\right)^{\theta}\right]^{1/\theta}$ and

$$\begin{aligned}
S_2 &= \sum_{j=1}^{n}\left\|\varphi_j^{-1}(h)\|\Delta^{\sigma_j}_{h,j}f\|_{\mathcal{B}^{\varphi,\sigma;\infty}_{p,\theta}(\mathbb{R}^n)}\right\|_{L^*_\theta(0,\infty)} \\
&\sim \sum_{j=1}^{n}\left\|\varphi_j^{-1}(h)\left[\sum_{k=0}^{\infty}\left(2^k\|\Delta^{\sigma_j}_{h,j}(f*v_k)\|_{L_p(\mathbb{R}^n)}\right)^{\theta}\right]^{1/\theta}\right\|_{L^*_\theta(0,\infty)} \\
&= \sum_{j=1}^{n}\left\{\int_0^{\infty}\varphi_j^{-\theta}(h)\left[\sum_{k=0}^{\infty}\left(2^k\|\Delta^{\sigma_j}_{h,j}(f*v_k)\|_{L_p(\mathbb{R}^n)}\right)^{\theta}\right]\frac{dh}{h}\right\}^{1/\theta} \\
&\sim \left\{\sum_{k=0}^{\infty}\left[2^k\sum_{j=1}^{n}\left\|\varphi_j^{-1}(h)\|\Delta^{\sigma_j}_{h,j}(f*v_k)\|_{L_p(\mathbb{R}^n)}\right\|_{L^*_\theta(0,\infty)}\right]^{\theta}\right\}^{1/\theta}.
\end{aligned}$$

Then

$$S_1+S_2 \sim \left[\sum_{k=0}^{\infty}\left(2^k\|f*v_k\|_{L_p(\mathbb{R}^n)}\right)^{\theta}\right]^{1/\theta} +$$

$$+\left\{\sum_{k=0}^{\infty}\left[2^k\sum_{j=1}^{n}\left\|\varphi_j^{-1}(h)\|\Delta^{\sigma_j}_{h,j}(f*v_k)\|_{L_p(\mathbb{R}^n)}\right\|_{L^*_\theta(0,\infty)}\right]^{\theta}\right\}^{1/\theta}$$

$$\sim \left\{\sum_{k=0}^{\infty}2^{k\theta}\left[\|f*v_k\|^{\theta}_{L_p(\mathbb{R}^n)} + \left(\sum_{j=1}^{n}\left\|\varphi_j^{-1}(h)\|\Delta^{\sigma_j}_{h,j}(f*v_k)\|_{L_p(\mathbb{R}^n)}\right\|_{L^*_\theta(0,\infty)}\right)^{\theta}\right]\right\}^{1/\theta}$$

$$\sim \left\{\sum_{k=0}^{\infty}2^{k\theta}\left[\|f*v_k\|_{L_p(\mathbb{R}^n)} + \left(\sum_{j=1}^{n}\left\|\varphi_j^{-1}(h)\|\Delta^{\sigma_j}_{h,j}(f*v_k)\|_{L_p(\mathbb{R}^n)}\right\|_{L^*_\theta(0,\infty)}\right)\right]^{\theta}\right\}^{1/\theta}$$

$$\sim \left[\sum_{k=0}^{\infty}\left(2^k\|f*v_k\|_{\mathcal{B}^{\varphi,\sigma;\infty}_{p,\theta}(\mathbb{R}^n)}\right)^{\theta}\right]^{1/\theta}.$$

Since

$$\varphi_k * \varphi_m = \mathcal{F}^{-1}(\mathcal{F}(\varphi_k * \varphi_m)) = \mathcal{F}^{-1}(\mathcal{F}\varphi.\mathcal{F}\varphi_m) = \mathcal{F}^{-1}(\chi_{P+1\setminus P_k}\chi_{P_{m+1}\setminus P_m})$$

it follows that

$$\varphi_k * \varphi_m = 0 \quad \text{for} \quad k \neq m, \text{ and } \ \varphi_k * \varphi_k = \varphi_k.$$

Hence

$$\|f * v_k\|_{\mathcal{B}^{\varphi,\sigma;\infty}_{p,\,\theta}} \sim \left(\sum_{m=0}^{\infty}\left(2^k\|f * v_k * v_m\|_{L_p(\mathbb{R}^n)}\right)^{\theta}\right)^{1/\theta} = 2^k\|f * v_k\|_{L_p(\mathbb{R}^n)}.$$

And finally,

$$\left[\sum_{k=0}^{\infty}\left(2^k\|f * v_k\|_{\mathcal{B}^{\varphi,\sigma;\infty}_{p,\,\theta}(\mathbb{R}^n)}\right)^{\theta}\right]^{1/\theta} \sim \left[\sum_{k=0}^{\infty}\left(2^k\cdot 2^k\|f * v_k\|_{L_p(\mathbb{R}^n)}\right)^{\theta}\right]^{1/\theta}$$

$$= \left[\sum_{k=0}^{\infty}\left(2^{2k}\|f * v_k\|_{L_p(\mathbb{R}^n)}\right)^{\theta}\right]^{1/\theta} \sim \|f\|_{\mathcal{B}^{\varphi^2,2\sigma;\infty}_{p,\theta}(\mathbb{R}^n)}$$

which proves equivalence (3.21). The last equivalence follows by an argument similar to the one in [3], [4]. □

Corollary 5. *Let* $1 < p < \infty$, $1 \leq \theta \leq \infty$, $\varphi \in \Phi(\sigma,\theta)$, $k \in \mathbb{N}$. *Then*

$$\|f\|_{\underbrace{\mathcal{B}^{\varphi,\sigma;\infty}_{\theta}(\ldots\mathcal{B}^{\varphi,\sigma;\infty}_{\theta}}_{k}(L_p(\mathbb{R}^n))\ldots)} \sim \|f\|_{\mathcal{B}^{\varphi^k,k\sigma;\infty}_{\theta}(L_p(\mathbb{R}^n))}. \tag{3.26}$$

4 Main results

Theorem 12 (On iterated norms). *Let* $1 < p < \infty$, $1 \leq \theta < \infty$, $\sigma = (\sigma_1,\cdots,\sigma_n)$, $\sigma_j \in \mathbb{N}$, $\varphi = (\varphi_1,\cdots,\varphi_n) \in \Phi(\sigma,\theta)$, $k \in \mathbb{N}$, $\varphi^k = \left(\varphi_1^k,\cdots,\varphi_n^k\right)$, $H = (H_1,\ldots,H_n)$, $0 < H_j \leq \infty$, *and let* $G \subset \mathbb{R}^n$ *be an open parallelepiped with faces parallel to the coordinate planes. Then*

1. *holds true the inclusion*

$$\underbrace{\mathcal{B}^{\varphi,\sigma;H}_{\theta}(\cdots(\mathcal{B}^{\varphi,\sigma;H}_{\theta}}_{k}(L_p(G)))\cdots) \subset \mathcal{B}^{\varphi^k,k\sigma;H}_{p,\,\theta}(G), \tag{4.1}$$

2. *under the additional assumption that there exists bounded extension operator*

$$S : \mathcal{B}^{\varphi^k,k\sigma;H}_{p,\theta}\big(G\big) \to \mathcal{B}^{\varphi^k,k\sigma;\infty}_{p,\theta}\big(\mathbb{R}^n\big) \tag{4.2}$$

holds true the equality of spaces

$$\underbrace{\mathcal{B}^{\varphi,\sigma;H}_{\theta}(\cdots(\mathcal{B}^{\varphi,\sigma;H}_{\theta}}_{k}(L_p(G)))\cdots) = \mathcal{B}^{\varphi^k,k\sigma;H}_{p,\,\theta}(G) \tag{4.3}$$

with the equivalence of the norms.

Proof. 1. By Corollary 3 for any open parallelepiped with faces parallel to the coordinate planes

$$\|f\|_{\beta_{\theta,j}^{\varphi_j^k,k\sigma_j;H_j}(L_p(G))} \le c_2^{k-1}\|f\|_{\underbrace{\beta_{\theta,j}^{\varphi_j,\sigma_j;H}(\ldots\beta_{\theta,j}^{\varphi_j,\sigma_j;H}}_{k}(L_p(G))\ldots)}.$$

Summing up these inequalities and adding $\|f\|_{L_p(G)}$ to both sides we obtain

$$\begin{aligned}
\|f\|_{\mathcal{B}_{p,\theta}^{\varphi^k,k\sigma;H}(G)} &= \|f\|_{L_p(G)} + \sum_{j=1}^{n}\|f\|_{\beta_\theta^{\varphi^k,k\sigma;H}(L_p(G))} \\
&\le \|f\|_{L_p(G)} + \sum_{j=1}^{n} c_2^{k-1}\|f\|_{\underbrace{\beta_{\theta,j}^{\varphi_j,\sigma_j;H}(\ldots\beta_{\theta,j}^{\varphi_j,\sigma_j;H}}_{k}(L_p(G))\ldots)} \\
&\le \max\{1, c_2^{k-1}\}\|f\|^*_{\underbrace{\mathcal{B}_\theta^{\varphi,\sigma;H}(\cdots(\mathcal{B}_\theta^{\varphi,\sigma;H}}_{k}(L_p(G)))\cdots)},
\end{aligned}$$

which implies inclusion (4.1).

2. Due to condition (4.2), the proof of the inverse inclusion to (4.1)

$$\mathcal{B}_{p,\theta}^{\varphi^k,k\sigma;H}(G) \subset \underbrace{\mathcal{B}_\theta^{\varphi,\sigma;H}(\cdots(\mathcal{B}_\theta^{\varphi,\sigma;H}}_{k}(L_p(G)))\cdots)$$

reduces to the proof for the case $G = \mathbb{R}^n$ which is considered in Lemma 6.

This completes the proof of the theorem. □

5 Conclusions

1. If we set $\varphi_j(h) = h^{l_j}$ in this result, then we get the results obtained in the works of V.I. Burenkov [3], [4].
2. If φ satisfies additional condition: $\exists \varepsilon > 0$ such that $\frac{\varphi_j(h)}{h^\varepsilon} \uparrow$ on $(0, H]$ (for example $\varphi_j(h) = h^{\alpha_j}\ln^{\gamma_j}(\frac{2H}{h})$, $\alpha_j > 0$, $\gamma_j \in \mathbb{R}$), in which case the space $\mathcal{B}_\theta^\varphi(L_p(G))$ possesses a power reserve of smoothness, then the theorem yields increment of smoothness in the iterated norms, which after finite number of steps enables attaining any order of smoothness in the power scale.
3. If the function φ has only logarithmic character (for example $\varphi_j(h) = \ln^{\gamma_j}\left(\frac{2H}{h}\right)$, $\gamma_j \in \mathbb{R}$), then the iterated norms enable to increase smoothness only in the logarithmic scale.

Acknowledgments

Our special thanks to Prof. V.I. Burenkov, Prof. M.L. Goldman and Prof. S.E. Mikhailov for their valuable advice and suggestions. We also thank Addis Ababa University Office of Vice President for Research and Graduate Studies, and the Local Organizing Committee of 7^{th} ISAAC Congress for their financial support, which allowed to present these results at the 7^{th} ISAAC Congress in London in July, 2009.

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Received: 01.09.2010