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HÖLDER ANALYSIS AND GEOMETRY ON BANACH SPACES: HOMOGENEOUS HOMEOMORPHISMS AND COMMUTATIVE GROUP STRUCTURES, APPROXIMATION AND TZAR'KOV'S PHENOMENON PART II

S.S. Ajiev

Dedicated to my father

Communicated by O.V. Besov

Key words: Hölder classification of spheres, Hölder-Lipschitz mappings, approximation of uniformly continuous mappings, Tsar'kov's phenomenon, Mazur mappings, Lozanovskii factorisation, homogeneous right inverses, metric projection, asymmetric uniform convexity and smoothness, dimension-free estimates, λ-horn condition, local unconditional structure, Nikol'skii-Besov, Lizorkin-Triebel, Sobolev, non-commutative L_p and IG-spaces, Banach lattices, wavelets, three-space problem, Kalton-Pełczyńki decomposition, bounded extension of Hölder mappings, Markov type and cotype, complemented subspaces, UMD

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Abstract. In an explicit quantitative and often precise manner, we construct the homogeneous Hölder homeomorphisms and study the approximation of uniformly continuous mappings by the Hölder-Lipschitz ones between the pairs of abstract and concrete metric and (quasi) Banach spaces including, in particular, Banach lattices, general non-commutative L_p-spaces, the classes IG and IG_+ of independently generated spaces (for example, non-commutative-valued Bochner-Lebesgue spaces) and anisotropic Sobolev, Nikol'skii-Besov and Lizorkin-Triebel spaces of functions on an open subset or a class of domains of an Euclidean space defined with underlying mixed L_p-norms in terms of differences, local approximations by polynomials, wavelet decompositions and systems of closed operators, such as holomorphic functional calculus and Fourier multipliers of smooth Littlewood-Paley decompositions. Our approach also allows to treat both the finite (as in the initial and/or boundary value problems in PDE) and infinite l_p-sums of these spaces, their duals and "Bochnerizations". Many results are automatically extended to the setting of the function spaces with variable smoothness, including the weighted ones. The sharpness of the approximation results, shown for the majority of the pairs under some mild conditions and underpinning the corresponding sharpness of the Hölder continuity exponents of the homogeneous homeomorphisms, indicates that the range of the exponents is often a proper subset of $(0, 1]$, that is the presence of Tsar'kov's phenomenon. We also consider the approximation by the mappings taking the values in the convex envelope of the range of the original approximated mapping. Negative results on the absence of uniform embeddings of the balls of some function spaces, particularly including BMO, VMO, Nikol'skii-Besov and Lizorkin-Triebel spaces with $q = \infty$ and their VMO-like separable subspaces, into any Hilbert space are established. Relying on the solution to the problem of global Hölder continuity of metric projections and the existence of Hölder continuous homogeneous right inverses of closed surjective operators and retractions onto closed convex subsets, as well as our results on the bounded extendability of Hölder-Lipschitz mappings and re-homogenisation technique, we develop and employ our key explicit quantitative tools, such as the global (on arbitrary bounded subsets) Hölder continuity of duality mappings and the Lozanovskii factorisation, the answer to the three-space problem for the Hölder classification of infinite-dimensional spheres, the Hölder continuous counterpart of the Kalton-Pełczyński decomposition method, the Hölder continuity of the homogeneous homeomorphism induced by the complex interpolation method and such counterparts of the classical Mazur mappings as the abstract and simple Mazur ascent and complex Mazur descent. Important role is also played by the study of the local unconditional structure and other complementability results, as well as the existence of equivalent geometrically friendly norms.

Introduction

This is the second part of the article. The content of the first part [1] is briefly described below.

The first step towards the application of quantitative methods based on the quasi-Euclidean approach developed in [7, 9, 10, 13] is the choice, if necessary, of a geometrically-friendly equivalent norm in a space under consideration. Thus, in Section 2 we defined and divided into six Γ-groups all the parameterised spaces under consideration, described subfamilies of equivalent norms on some of them and relations between different classes of spaces and provided a quantitative description of their asymmetric uniform convexity and uniform smoothness. A large class of auxiliary IG-spaces, including, in particular, l_p-sums of L_p-spaces with mixed norm (and other IG-spaces), was introduced, studied and employed in [7, 8, 9, 10, 13]. The class IG_+ extends IG including also the l_p-sums and "Bochnerizations" of the Lebesgue and sequence spaces of functions (possibly, on a discrete set) with values in noncommutative L_p-spaces [40].

Section 3 contains elementary properties of Hölder-Lipschitz mappings and the auxiliary results (including some involving the matter of sharpness) on the existence of either ordinary or Hölder continuous (globally on arbitrary bounded subsets) homogeneous inverses for closed linear surjections between Banach spaces. We also introduce the notions related to the Hölder equivalence of spheres of abstract spaces. Moreover, Lemma 3.2 constitutes the answer to the three-space problem for our classification (see the equivalence relation $\overset{7}{\sim}$ in Section 1), while Theorem 3.4 is the Hölder continuous counterpart of Kalton's nonlinear version of A. Pełczyński's decomposition method.

Section 4 contains the definitions and properties of our relatively abstract but occasionally sharp key explicit quantitative tools: the global (on arbitrary bounded subsets)

Hölder continuity of duality mappings and the Lozanovskii factorisation and the Hölder continuity of the homogeneous homeomorphism induced by the complex interpolation method. The latter mapping and its uniform continuity are due to M. Daher [24] and N.J. Kalton [17].

In Section 5, we employed the latter key abstract tool and developed a re-homogenisation technique to construct and study our counterparts of the Mazur mapping that we call the abstract and simple Mazur ascents and complex Mazur descent. Their compositions appeared to be the Hölder homeomorphisms between the spheres of the pairs of compatible IG_{0+}-spaces that are sharp in the setting of the IG_0-spaces and occasionally sharp in the setting of the IG_{0+}-spaces.

Part II of the paper starts with Section 6 which contains main results of the paper on homogeneous Hölder homeomorphisms in a form that permits to trace the constants. We start with complete description of Banach lattices that are in the same equivalence class with Hilbert spaces and proceed by employing our abstract and constructive tools of Sections 4 and 5 to provide quantitative Hölder classification of the spheres of all the spaces under consideration with respect to the spheres of Hilbert spaces, including also some spaces that are not equivalent to a Hilbert space. Indeed, relying on the solution to Smirnov's problem due to P. Enflo [26] and our results [4, 5, 12] on the finite representability of c_0 in (anisotropic) $BMO(G)$, $VMO(G)$, $BMO(G) \cap L_\infty(G)$, $VMO \cap L_\infty(G)$, Nikol'skii (i.e. Nikol'skii-Besov with $q = \infty$) and corresponding Lizorkin-Triebel spaces, as well as their VMO-like subspaces, we show that the unit balls of these spaces cannot be uniformly embedded into any separable or nonseparable Hilbert space.

In Section 7, we introduce commutative homogeneous Hölder group structures (compatible with the norm and the existing linear structure) on all our spaces under consideration, even on those that do not admit any C^*-algebra structure.

Section 8 contains various results related to complementability of subspaces of abstract and specific Banach spaces, including the existence of certain complemented subspaces, that are employed either directly in the second group of the main results in Section 8, or via some key auxiliary results that are either our counterpart of the Kalton-Pełczyński decomposition method in Section 3, or the presence of Tsar'kov's phenomenon (our main sharpness tool) in Section 11.

Section 9 comprises, in an explicit and quantitative form relying on the asymmetric uniform convexity and smoothness and Markov type and cotype, the basic auxiliary properties of abstract and specific Hölder-Lipschitz mappings employed in our approaches to the second main task of the article: globally Hölder-continuous retractions and metric projections onto closed convex subsets of Banach spaces and the bounded extendability of Hölder-Lipschitz mappings between Banach spaces.

The second group of the main results that are on the approximation of uniformly continuous mappings is contained in Section 10, where we utilise all our key tools developed in the previous sections, as well as the sharpness tools of Section 11. We first establish the approximation results in abstract and semi-abstract settings of mappings from metric, quasi-Banach and IG-spaces into quasi-Banach and IG-spaces, and, then, apply some of these results, as well as our other tools, to treat the approximation of the uniformly continuous mappings between the pairs of either abstract Banach lattices, or our Γ-groups of the specific spaces under consideration.

In Section 11, we benefit from some uniform complementability results given in Section 8 (see also [12]) by detecting the presence of Tsar'kov's phenomenon for the majority of the pairs of the specific spaces under consideration.

The numbering of the equations is used sparingly. Since the majority of references inside every logical unit are to the formulas inside the unit, equations are **numbered independently** inside every proof of a corollary, lemma and theorem, or a definition (if there are any numbered formulas). The number of the corresponding logical unit does not accompany the number of the formula in the references inside this unit.

6 Main results: Hölder classification of spheres

This section contain the explicit quantitative (and occasionally sharp) Hölder classification of the spheres of both abstract Banach lattices and the spaces from the groups $\{\Gamma_i\}_{i=0}^5$. In particular, we show that the Hölder and uniform classification of the spheres of lattices coincide and reveal spaces from $\bigcup_{i=0}^5 \Gamma_i$ that are not in the same equivalence classes with the separable and nonseparable Hilbert spaces.

6.1 Hölder classification of abstract lattices

As we have seen in Remark 2.6, a), every (Banach) lattice X can be transformed into a p-convex lattice $X^{(p)}$ [31] with better properties, and there exists an abstract counterpart $\phi_p : X \to X^{(p)}$ of the Mazur mapping with the same properties. These properties are the subject of the next result in [17].

Theorem 6.1. (Proposition 9.3 in [17]). *For $p \in (1, \infty)$, let X and $X^{(p)}$ be a Banach lattice with weak unit and its p-convexification. Then $X \overset{(1/p,1)}{\longleftrightarrow} X^{(p)}$.*

After this preparation we can establish one of the main results of this section. It is the sharp (i.e. under the same conditions) Hölder version of the corresponding result due to E. Odell and Th. Schlumprecht [36] (Theorem 9.7 in [17]) that states that the sphere of a Banach lattice is uniformly homeomorphic to the unit sphere of a Hilbert space. We refer to [17] and [31] for the definitions and details of the proof that are not explained here.

Let us recall (see [31]) that, if a Banach lattice X does not contain an isomorphic copy of c_0, then it is order complete and order continuous. In the presence of the order continuity, every subspace of X contains a subspace with unconditional basis (1.e.9 in [31]). Separable and function Banach lattices possess weak units. Order continuous Banach lattices with a weak unit allow the function representation [31].

Theorem 6.2. *Let X be a Banach lattice with a weak unit. Then we have*

$$X \overset{(1/4,\beta)}{\longleftrightarrow} H$$

for some $\beta \in (0, 1]$ and a Hilbert space H if, and only if, X does not contain $l_\infty(I_n)$ uniformly.

Proof. We focus on the differences with the proof of Theorem 9.7 in [17]. The existence of subspaces that are uniformly (in n) isomorphic to $l_\infty(I_n)$ rules out even a uniform embedding of B_X into a Hilbert space according to P. Enflo [26].

The existence of a weak unit and the absence of the above mentioned subspaces allow a representation (order isomorphic and linear isometric) of X as a lattice of functions. The absence of the copies also implies that X is q-concave for some finite q. Then, according to [31], its 2-convexification $X^{(2)}$ is 2-convex and $2q$-concave and, therefore, admits an equivalent (lattice) norm that makes $X^{(2)}$ 2-uniformly smooth and $2q$-uniformly convex. The latter conditions are equivalent to the $(2, h_s)$-uniform smoothness and $(2q, h_c)$-uniform convexity. Eventually, Theorems 6.1 and 4.2 and the properties of the Mazur mappings between L_1 and L_2 lead to the following chain

$$X \overset{(1/2,1)}{\longleftrightarrow} X^{(2)} \overset{(1,(2q)^{-1})}{\longleftrightarrow} L_1 \overset{(1/2,1)}{\longleftrightarrow} L_2.$$

This chain finishes the proof with $\beta = 1/2q$ thanks to the transitivity of the Hölder homeomorphisms. □

Eventually, the combination of Theorems 3.4, 4.1, 4.2 and 6.2 implies the following result that includes the qualitative versions of all positive results in Section 6.2 that do not involve noncommutative spaces. It is also our Hölder counterpart of Corollary 9.11 in [17].

Theorem 6.3. *Let X be a superreflexive Banach lattice with a weak unit. Assume also that Y is either a subspace or a quotient of X. Then the unit sphere S_Y is Hölder homeomorphic to the unit sphere of a Hilbert space.*

Proof. As a superreflexive lattice, X is q-convex and p-concave for some $p, q \in (1, \infty)$ and, thus, can be renormed to be both (q, h_s)-uniformly smooth and (p, h_c)-uniformly convex. Hence, due to Theorem 4.2 for both X and L_2, we have

$$X \overset{(\frac{q-1}{2},\frac{1}{p})}{\longleftrightarrow} L_2 \overset{(\frac{1}{q'},\frac{p'-1}{2})}{\longleftrightarrow} X^*. \tag{1}$$

If Y is a subspace, then it has a subspace with an unconditional basis thanks to Proposition 1.*c*.9 in Lindenstrauss and Tzafriri [31]. The latter is renormed into a sequence lattice (1-unconditional basis) and, therefore, its unit sphere is Hölder homeomorphic to the unit sphere of a Hilbert space due to Theorem 6.2. The application of Theorem 3.4 shows that the same is true for Y.

If Y is a quotient, then Y^* is a subspace of the superreflexive lattice X^* and the above argument applies. Since Y is also (after renorming of X) (q, h_s)-uniformly smooth and (p, h_c)-uniformly convex, its unit sphere is Hölder homeomorphic to the unit sphere of Y^* according to Theorem 4.1. □

We also need the following lemma.

Lemma 6.1. *For $2 \in [q, p] \subset (1, \infty)$, let X be a Banach lattice that does not contain an isomorphic copy of c_0 and contains a subspace Y possessing the Rademacher type q and cotype p. Then Y contains a subspace Z with unconditional basis, such that, for every $\varepsilon > 0$, there exists a $(q_\varepsilon, h_{s,\varepsilon})$-uniformly smooth and $(p_\varepsilon, h_{c,\varepsilon})$-uniformly convex Banach (sequence) lattice Z_ε with $q_\varepsilon \in (q - \varepsilon, q]$, $p_\varepsilon \in [p, p + \varepsilon)$ and non-trivial $h_{s,\varepsilon}$ and $h_{c,\varepsilon}$ that is isomorphic to Z.*

Proof. The no-c_0 condition means that X is order continuous, and, hence, Y contains a subspace Z with an unconditional basis. Therefore, Z is isomorphic (i.e. can be renormed) to a sequence Banach lattice Z_0 possessing the Rademacher type q and cotype p. This condition implies that, for every $\varepsilon > 0$, there are q_ε and p_ε satisfying the conditions of the lemma, such that Z_0 is q_ε-convex and p_ε-concave. Hence, Z_0 is isomorphic to a $(q_\varepsilon, h_{s,\varepsilon})$-uniformly smooth and $(p_\varepsilon, h_{c,\varepsilon})$-uniformly convex Banach (sequence) lattice Z_ε that we are looking for. □

6.2 Hölder classification of Γ-groups

In this section, we establish the explicit Hölder classification of the spaces from the groups Γ_i estimating the parameters $\alpha(X, H)$ and $\alpha(H, X)$ for the spaces X under consideration. One uses combinations of our major tools developed in the previous sections, along with the sharpness tools in Section 11.

Theorem 6.4. *For $n \in \mathbb{N}$ and $i \in \{1, 2, 3\}$, let $X_i \subset \Gamma_i$. That is we assume that:*
(i) X_1 has the form $X^{s_1}_{p_1,q_1,a_1}(G_1)$ or $\tilde{X}^{s_1,A_1}_{p_1,q_1,a_1}(G_1)$, or $X^{s_1}_{p'_1,q'_1,a'_1}(G_1)^*$, or $\tilde{X}^{s_1,A_1}_{p'_1,q'_1,a'_1}(G_1)^*$ with an admissible a_1, where every component of a_1 is in the convex envelope of q_1 and $\{p_{1j}\}^n_{j=1}$,
(ii) X_2 has the form $X^{s_2}_{p_2}(G_2)$, or $X^{s_2}_{p'_2}(G_2)^*$,
and
(iii) X_3 has the form $X^{s_3}_{p_3,q_3}(\mathbb{R}^n)$, or $X^{s_3}_{p'_3,q'_3}(\mathbb{R}^n)^*$.

Then, if $G_i \subset \mathbb{R}^n$ satisfies the C-flexible λ-horn condition for $i = 1, 2$, we have

$$a)\ \min\left(\alpha(X_i, l_2), \alpha(l_2, X_i)\right) \geq \delta_i = \frac{\min\left(p_{i\min}, q_i, 2\right)}{\max\left(p_{i\max}, q_i, 2\right)} \text{ for } i \in \{1, 3\},$$

where $\alpha(X_i, l_2) = \delta_i$ if $\max(p_{i\max}, q_i) \leq 2$, and $\alpha(l_2, X_i) = \delta_i$ if $\min(p_{i\min}, q_i) \geq 2$;

$$b)\ \min\left(\alpha(X_2, l_2), \alpha(l_2, X_2)\right) \geq \delta_2 = \frac{\min\left(p_{2\min}, 2\right)}{\max\left(p_{2\max}, 2\right)},$$

where $\alpha(X_2, l_2) = \delta_2$ if $\max(p_{2\max}, 2) \leq 2$, and $\alpha(l_2, X_2) = \delta_2$ if $\min(p_{2\min}, 2) \geq 2$.

Remark 6.1. *a*) It is possible to show that, under the above conditions of sharpness, the optimal value δ_i of $\alpha(X_i, l_2)$ and $\alpha(l_2, X_i)$ for $i \in I_3$ is achieved (i.e. $X_i \overset{(\delta_i,\delta_i)}{\longleftrightarrow} l_2$) if 2 is among the parameters $\{p_{ij}, q_i\}_{j \in I_n}$. This result will appear in a separate paper because it requires real/harmonic analysis tools and exposes the limitations of Theorem 4.3.

b) The choice of an admissible a_1 that will not affect the maximum and minimum of q_1 and all the components of p_1 is possible according to [3].

c) Note that the application of Theorem 4.3 in the proof permits to estimate, explicitly, the behavior of the Hölder norms of the homeomorphisms involved.

Proof. Regarding the complex interpolation method, we use the notation from [19]. In the case of every $i = 1, 2$ or 3, we apply Theorem 4.3. Indeed, it is easily checked (particularly, with the aid of the point of view in §2.2) that taking all the parameters (every component of the vector parameters) of X_i to be equal 2 makes it a Hilbert space that we denote $H_i = X_{i\bar{2}}$. O.V. Besov [20] has shown that the classes of reflexive spaces $B^s_{p,q,1}(G)$, $L_{p,q,1}(G)$ and $W^s_p(G)$ are closed with respect to the complex interpolation if the domain G satisfies the C-flexible λ-horm condition. For example,

$$\left(B^{s_0}_{p_0,q_0,1}(G), B^{s_1}_{p_1,q_1,1}(G)\right)_{[\theta]} \asymp B^{s_\theta}_{p_\theta,q_\theta,1}(G) \text{ and } \left(L^{s_0}_{p_0,q_0,1}(G), L^{s_1}_{p_1,q_1,1}(G)\right)_{[\theta]} \asymp L^{s_\theta}_{p_\theta,q_\theta,1}(G),$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \text{ and } s_\theta = (1-\theta)s_0 + \theta s_1. \tag{1}$$

At the same time, it is demonstrated in [2, 3] that the corresponding spaces defined in terms of the local approximation by polynomials are isomorphic to the corresponding spaces defined in terms of the averaged differences, and that the constant $a = \bar{1}$ is in the admissible range for X_1, and every a in the admissible range delivers an equivalent norm. Thanks to the duality theorem for the complex interpolation method [19] and this renorming, (1) remain valid for the dual spaces.

The interpolation properties of the spaces with wavelet norms of functions on $\mathbb{R}^n$ are well-known. For example, $\mathbb{R}^n$ satisfies the C-flexible λ-horn condition and they are isomorphic to the spaces in [20].

Hence, we can always find spaces $A_0, A_1 \in \Gamma_i$ for $i \in \{1, 2, 3\}$ with the parameters of A_0 and A_1 being arbitrary close to X_i and H_i correspondingly, such that

$$X_i \asymp (A_0, A_1)_{[\varepsilon]} \text{ and } H_i \asymp (A_0, A_1)_{[1-\varepsilon]}. \tag{2}$$

The application of Theorem 4.3 with $r = 2$ reduces the problem to computing the (q, h_s)-uniform smoothness and (p, h_c)-uniform convexity parameters for the l_2-sum

$$Y_i = L_2\left(\mathbb{R}, l_2\left(\{0, 1\}, \{A_0, A_1\}\right)\right) = L_2\left(\mathbb{R}, (A_0 \oplus A_1)_2\right).$$

According to Section 2.2 X_i, if it is not defined as a dual, is a subspace of an IG space Z_i featuring the same parameters. Hence, Y_i itself is a subspace of the IG space

$$E_i = l_2\left(\{0, 1\}, \{Z_i, H_i\}\right) = (Z_i \oplus H_i)_2 .$$

Since E_i has the same set of parameters as X_i plus $\{2\}$, we just combine Theorems 2.2 and 4.3 to establish the (α, α)-Hölder equivalence where α can be chosen arbitrary close to the proposed lower bound for $\alpha(X_i, H_i)$ and $\alpha(H_i, X_i)$. Similarly, if X_i is defined as a dual, we deal with the quotients covered by Theorem 2.2 as well.

The sharpness follows immediately by Lemma 11.1 and Theorem 10.6, finishing the proof. □

The next theorem, dealing with function spaces on arbitrary open subsets of an Euclidean space, shows that the irregularity of the domain of the functions from a function space is likely to worsen the homogeneous Hölder homeomorphism with l_2.

Theorem 6.5. *For $n \in \mathbb{N}$ and $i \in \{1,2,4\}$, let $X_i \subset \Gamma_i$. That is we assume that:*
(i) X_1 has the form $X^{s_1}_{p_1,q_1,a_1}(G_1)$ or $\tilde{X}^{s_1,A_1}_{p_1,q_1,a_1}(G_1)$, or $X^{s_1}_{p'_1,q'_1,a'_1}(G_1)^*$, or $\tilde{X}^{s_1,A_1}_{p'_1,q'_1,a'_1}(G_1)^*$ with an admissible a_1, where every component of a_1 is in the convex envelope of q_1 and $\{p_{1j}\}_{j=1}^n$,
(ii) X_2 has the form $X^{s_2}_{p_2}(G_2)$, or $X^{s_2}_{p'_2}(G_2)^*$,
and
(iii) X_4 has the form $X^{s_4}_{p_4,q_4}(G_4)$, or $X^{s_4}_{p'_4,q'_4}(G_4)^*$.
Then we have

$$a)\ \min\left(\alpha(X_i,l_2),\alpha(l_2,X_i)\right) \geq \frac{\min\left(p_{i\min},q_i,2\right)^5}{\max\left(p_{i\max},q_i,2\right)^6}\frac{\min\left(p_{i\min},q_i,2\right)-1}{2} \quad \text{for } i \in \{1,4\};$$

$$b)\ \min\left(\alpha(X_2,l_2),\alpha(l_2,X_2)\right) \geq \frac{\min\left(p_{2\min},2\right)^5}{\max\left(p_{2\max},2\right)^6}\frac{\min\left(p_{2\min},2\right)-1}{2}.$$

Remark 6.2. *a*) The group Γ_4 is very large and some of its classes are known to be isomorphic even to the spaces in Γ_3.

b) In the case of Part *a*), the estimates can be improved, at least, up to

$$\min\left(\alpha(X_i,l_2),\alpha(l_2,X_i)\right) \geq \delta_i^3,$$

where δ_i is defined in Theorem 6.4, but it requires an additional real/harmonic analysis argument.

Proof. We prove only Part *a*) because the proof of *b*) is almost identical (one should omit q and q'). According to §2.2, a non-dual X_i is a subspace of an IG-space Y_i with the same set of parameters (excluding smoothness) as X_i, and Corollary 5.1 implies the equivalence

$$Y_i \overset{(\alpha_0,\beta_0)}{\longleftrightarrow} l_2 \text{ with } \alpha_0 = \frac{\min\left(p_{i\min},q_i,2\right)}{2}, \beta_0 = \frac{2}{\max\left(p_{i\max},q_i,2\right)}. \tag{1}$$

The reflexivity of Y_i precludes it from possessing isomorphic copies of c_0, and Theorem 2.2 provides its Rademacher type and cotype sets. Therefore, we combine Lemma 6.1 with our quantitative Lozanovskii factorization in Theorem 4.2 to find, for an arbitrary small $\varepsilon > 0$, a subspace Z_ε of X_i, such that

$$Z_\varepsilon \overset{(\alpha_1,\beta_1)}{\longleftrightarrow} l_2 \text{ with } \alpha_1+\varepsilon = \frac{\min\left(p_{i\min},q_i,2\right)-1}{2}, \beta_1+\varepsilon = \frac{1}{\max\left(p_{i\max},q_i,2\right)}. \tag{2}$$

Now we finish the proof in the non-dual setting by means of applying Theorem 3.4 and letting $\varepsilon \to 0$.

If X_i is a dual space, then, thanks to the reflexivity of X_i [2, 13], X_i^* is a subspace of an IG-space Z_i, with the same set of parameters (excluding smoothness) as X_i^*, and, therefore, X_i itself is isometric to a quotient of the IG-space Z_i^*. Corollary 5.1 implies the equivalence

$$Z_i^* \overset{(\alpha_0,\beta_0)}{\longleftrightarrow} l_2. \tag{3}$$

Applying Lemma 6.1 to X_i^* as a subspace of the lattice Z_i, we find, for an arbitrary small $\varepsilon > 0$, a subspace W_ε of X_i^* that is isomorphic to a $(q_\varepsilon, h_{s,\varepsilon})$-uniformly smooth and $(p_\varepsilon, h_{c,\varepsilon})$-uniformly convex Banach (sequence) lattice $\widetilde{W}_\varepsilon$ with

$$p_\varepsilon = \max\left(p'_{i\,\min}, q'_i, 2\right) + \varepsilon \text{ and } q_\varepsilon = \min\left(p'_{i\,\max}, q'_i, 2\right) - \varepsilon. \tag{4}$$

Thus, W_ε^* is isometric to a quotient of X_i and isomorphic to the lattice $\widetilde{W}_\varepsilon^*$. Moreover, thanks to the duality Theorem 4.5 from [13] (see also [10]), the lattice $\widetilde{W}_\varepsilon^*$ is $\left(p'_\varepsilon, h_{c,\varepsilon}^{1-p'_\varepsilon}\right)$-uniformly smooth and $\left(q'_\varepsilon, h_{s,\varepsilon}^{1-q'_\varepsilon}\right)$-uniformly convex. Theorem 4.2 leads to

$$W_\varepsilon^* \overset{(\alpha_\varepsilon,\beta_\varepsilon)}{\longleftrightarrow} l_2 \quad \text{with} \quad \alpha_\varepsilon = \frac{p'_\varepsilon - 1}{2}, \beta_\varepsilon = \frac{1}{q'_\varepsilon}. \tag{5}$$

Now we finish the proof by applying Theorem 3.4 to Z_i^*, X_i and W_ε^* and tending ε to 0. □

The following theorem shows that not all (even separable) Nikol'skii-Besov and Lizorkin-Triebel spaces are in the same equivalence class with a Hilbert space with respect to our Hölder classification of spheres.

Theorem 6.6. *Let $X \in \Gamma_1([1,\infty])$ be not defined as a dual and its $p_{\max} = \infty$. For $n \in \mathbb{N}$, $p \in [1,\infty]^n$, $a \in (0,\infty]^n$, $s \in [0,\infty)$, $D = \hat{D} \subset \mathbb{N}_0^n$, $|D| < \infty$, assume also that either $s = 0$ and $p = (\infty,\dots,\infty)$, or $s > \max\left(0, (\gamma_a, 1/p - 1/a)\right)$. Let also*

$$Y \in \Big\{ \overset{\circ}{B}{}^s_{p,\infty,a}(G), \overset{\circ}{b}{}^s_{p,\infty,a}(G), B^s_{p,\infty,a}(G), b^s_{p,\infty,a}(G), \overset{\circ}{L}{}^s_{p,\infty,a}(G), \overset{\circ}{l}{}^s_{p,\infty,a}(G), L^s_{p,\infty,a}(G), l^s_{p,\infty,a}(G),$$
$$\overset{\circ}{\widetilde{B}}{}^{s,D}_{p,\infty,a}(G), \overset{\circ}{\widetilde{b}}{}^{s,D}_{p,\infty,a}(G), \widetilde{B}^{s,D}_{p,\infty,a}(G), \widetilde{b}^{s,D}_{p,\infty,a}(G), \overset{\circ}{\widetilde{L}}{}^{s,D}_{p,\infty,a}(G), \overset{\circ}{\widetilde{l}}{}^{s,D}_{p,\infty,a}(G), \widetilde{L}^{s,D}_{p,\infty,a}(G), \widetilde{l}^{s,D}_{p,\infty,a}(G) \Big\}.$$

Then the unit balls B_X and B_Y of X and Y cannot be uniformly embedded into a Hilbert space.

Remark 6.3. Note that Theorem 6.6 covers the (anisotropic) spaces

$$Y \in \left\{ BMO^{\lambda_a}(G), VMO^{\lambda_a}(G), BMO^{\lambda_a}(G) \cap L_\infty, VMO^{\lambda_a}(G) \cap L_\infty \right\}.$$

Proof. It was shown in [4] and [13] that Y contains an isomorphic copy of c_0, while X contains $l_\infty(I_n)$ uniformly according to [13] (see also [5] for the application to strengthening and extending a result due to G. M. Fichtenholtz and L. V. Kantorovich [27] on the non-complementability of $C([0,1])$ in $L_\infty([0,1])$, such as that $VMO(G)$ is not complemented in $BMO(G)$). Thanks to Enflo's result in [26], this means that the unit balls of all these spaces are not even uniformly embedded into any Hilbert space. □

Theorem 6.7. *Let $X \in IG_+$ and $H = X_{\bar{2}}$, where the function $\bar{2}$ is constant 2 on $\mathcal{V}(X)$. For $p \in (1,\infty)$, assume that*

$$\gamma = \frac{\min(p,2)}{\max(p,2)} \quad \text{and} \quad \delta = \frac{\min\left(p_{\min}(X), 2\right)}{\max\left(p_{\max}(X), 2\right)}.$$

Then we have

a) $\min\left(\alpha(X,H), \alpha(H,X)\right) \geq \delta$ *and, if* $X \in IG_{0+}$, $X \overset{(\delta,\delta)}{\longleftrightarrow} H$.

If X contains an (isomorphic) complemented copy of l_p, we also have

b) $X \overset{(\delta^2\gamma,\delta^2\gamma)}{\longleftrightarrow} H$ *and, if* $X \in IG$, $X \overset{(\delta\gamma,\delta\gamma)}{\longleftrightarrow} H$.

Remark 6.4. *a*) Note that, if one of the elements of the tree $T(X)$ is l_2, $lt_{p,q}(\mathbb{Z}^n \times \mathbb{R})$, or $lt_{p,q}(\mathbb{Z}^n \times \mathbb{R})^*$ with $2 \in \{p, q\}$, then X contains an isometric complemented copy of l_2 (see [13]). Moreover, if X is not purely a sequence space (i.e. one of the elements of the tree $T(X)$ is $L_p(\Omega, \mu)$ with not purely atomic μ), then X contains a complemented copy of l_2 as well. Indeed, X contains (an isometric copy of) $L_p(\Omega, Y)$ for some $Y \in IG$, whose tree is the subtree of X grown from $L_p(\Omega, \mu)$, while the latter obviously contains $L_p(\Omega, \mu)$ itself, where we can construct a Rademacher system and use the Khinchin isomorphism to show the complementability of its span in $L_p(\Omega, \mu)$.

b) We can substitute the condition $X \in IG$ in Part *b*) of the preceding theorem with $X \in IG_{0+}$ and $P_{nc} = P_I$ (see Theorem 5.2 and Corollary 5.4) and employ Corollary 5.5 instead of Corollary 5.2 in the proof.

Proof. It is straightforward from [19] that $A_0, A_1 \in IG_{0+}$ form a compatible pair if $T(A_0) = T(A_1)$ and, thus, the same parameter position set p, and we also have

$$(A_0, A_1)_{[\theta]} = A_{p_\theta} \text{ (isometry).} \tag{1}$$

Therefore, applying Corollary 4.1 to the pair $(X, X_{\bar{2}})$ and repeating the limiting argument from the proof of Theorem 4.5, we obtain the second relation in Part *a*).

In the case $A_0, A_1 \in IG_+$, we now combine (1) with Theorem 8.5 with the aid of Lemma 8.2 to obtain

$$(A_0, A_1)_{[\theta]} \asymp A_{p_\theta}, \tag{2}$$

where $A_{p_\theta} \in IG_+$ is the space with the same tree as A_0 and A_1 and the parameter position function

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_{A_0}} + \frac{1}{p_{A_1}}.$$

Similarly to the proof of Theorem 6.4, we choose Y_0 and Y_1, so that

$$p_X = p_\varepsilon \text{ and } p_{1-\varepsilon} = \bar{2}$$

and use Theorem 4.3 with $r = 2$, where we compute the convexity and smoothness exponents of $L_2(\mathbb{R}, l_2(\{0,1\}, \{A_0, A_1\}))$ with the aid of Theorem 2.2. Eventually the lower bound δ in Part *a*) is obtained by passing to the limit $\varepsilon \to 0$.

According to Lemma 8.2, $X \in IG_+$ ($X \in IG$) is either a complemented subspace of a quotient Z/Z_0 with Z_0 being complemented in Z, or a complemented subspace of Z, where $Z \in IG_{0+}$ ($Z \in IG_0$) has exactly the same range of the parameters (the image of the parameter position function: $I(X) = I(Z)$) as X. If X contains a complemented copy of l_p, we apply Theorem 3.4 either to the triple $l_2 \subset X \subset Z$ in the former case, or to the triple $l_2 \subset X_0 \subset Z$ in the latter case, where X_0 is chosen to be isomorphic to X in the following way. Choosing a complement Z_1 to Z_0 in Z (i.e. $Z = Z_0 \oplus Z_1$), we see that Z_1 is isomorphic to Z/Z_0 and, therefore, Z_1 contains a complemented subspace X_0 that is isomorphic to X. Let us note that X_0 is also complemented in Z. Thus, we obtain the first relation in Part *b*) combining this observation and the proof of the lower estimate in Part *a*) by means of Theorem 3.4.

For $X \in IG$, the parameters

$$\alpha = \frac{\min(p_{\min}(X), 2)}{2}, \ \beta = \frac{2}{\max(p_{\max}(X), 2)} \text{ in } Z \overset{(\alpha,\beta)}{\longleftrightarrow} Z_{\bar{2}}$$

(recall that the Hilbert $Z_{\bar{2}}$ has the tree $T(Z)$ but all the parameters are equal to 2, $p_{\min}(Z) = p_{\min}(X)$ and $p_{\max}(Z) = p_{\max}(Z)$) are provided by Corollary 5.2. Thus, Theorem 3.4 gives us $X \overset{(\delta,\delta)}{\longleftrightarrow} H$ with the desired $\delta = \alpha\beta$. □

Theorem 6.8. *For $p \in (1, \infty)$, $\beta = \min(p, 2)/\max(p, 2)$, let $X = L_p(\mathcal{M})$ and $H = L_2(\mathcal{M})$, where $\mathcal{M}$ is a von Neumann algebra, be infinite-dimensional. Then, for $\delta = \beta^3$, we have*

$$X \overset{(\delta,\delta)}{\longleftrightarrow} H.$$

Proof. Let Z be the space corresponding to X and described in Theorem 8.6. Then, extending the mutually inverse mappings $m_{p,2}$ and $m_{2,p}$ provided by Theorem 4.5 by continuity from the dense subset provided by Part 2) of Theorem 8.6 to Z, we obtain

$$Z \overset{(\beta,\beta)}{\longleftrightarrow} H, \tag{1}$$

where H has the same density character as $L_2(\mathcal{M})$. According to Lemma 8.4, X contains a 1-complemented copy of l_p. It is the property of the classical Mazur mapping [32] that

$$l_p \overset{(\alpha_1,\beta_1)}{\longleftrightarrow} l_2 \text{ for } \alpha_1 = \min(p/2, 1) \text{ and } \beta_1 = \min(2/p, 1). \tag{2}$$

We finish the proof by combining (1) and (2) with the aid of Theorem 3.4. □

7 Homogeneous Hölder group structures on Banach spaces

To rule out the existence of C^*-algebra structures (compatible with the norm and linear structure) for some spaces under the consideration, one can use the following result established by G. Pisier in the proof of Theorem 9.6 and on pages 128 and 129 in [39].

Theorem 7.1. ([39]). *Let A be a C^*-algebra, and let Y be of Rademacher cotype* 2. *Then any operator $T \in \mathcal{L}(A, Y)$ factors through a Hilbert space.*

Corollary 7.1. *Let an infinite-dimensional $X \in \Gamma_i((1, 2))$ for $i \in \{0, 1, 2, 3, 4, 5\}$, or $X = L_p(\mathcal{M})$ with $p \in (1, 2)$. Then X does not allow an introduction on it of a C^*-algebra structure compatible with the norm and the existing linear structure.*

Proof. Thanks to Theorems 2.2−2.8, X has the Rademacher cotype 2 (sharper type and cotype constant estimates for the spaces under consideration are in [10, 13]). Therefore, it is isomorphic to a Hilbert space thanks to Theorem 7.1 (i.e. the identity operator factors through a Hilbert space) if X admits a multiplication group structure making it a C^*-algebra. Since X does not have the Rademacher type 2, according to Theorem 8.11 from §8 in [13] (see also [12]), it is impossible. The absence of the Rademacher type 2 for many groups of the spaces under consideration is also implied by the existence of the copies of sequence spaces (and the related finite representability) discussed in Section 8, and the consideration of other spaces is similar. □

It appears to be still possible to introduce a multiplication group structure by means of sacrificing the distributivity and the Lipschits continuity.

Definition 7.1. *Let X be a Banach space and $\gamma \in (0, 1]$. We say that X admits a homogeneous γ-Hölder group structure if there exists a binary operation $* : X \times X \longrightarrow X$ and a constant $C > 0$, satisfying*

a) $\lambda a * \mu b = \lambda\mu(a * b)$ *for* $a, b \in X$ *and* $\lambda, \mu \in \mathbb{R}$;

b) $\|a * b\|_X \leq \|a\|_X \|b\|_X$ *for* $a, b \in X$;

c) $\|a_0 * b_0 - a_1 * b_1\|_X \leq C \max\limits_{i=0,1} (\|a_i\|_X, \|b_i\|_X)^{1-\gamma} (\|a_0 - a_1\| + \|b_0 - b_1\|)^{\gamma}$

for $a_i, b_i \in X$.

We say that X admits a homogeneous Hölder group structure if it admits a homogeneous γ-Hölder group structure for some λ.

Surprisingly, our infinite-dimensional spaces under consideration allow even commutative homogeneous Hölder group structures.

Theorem 7.2. *Let an infinite-dimensional Banach space X be either in Γ_i for some $i \in \{0, 1, 2, 3, 5\}$ or $L_p(\mathcal{M})$ with $p \in (1, \infty)$, or a Banach lattice with a weak unit that does not contain $l_\infty(I_n)$ uniformly (by $n \in \mathbb{N}$). Then X allows a commutative homogeneous Hölder group structure. In particular, $L_p(\mathcal{M})$ allows a commutative homogeneous $\delta^2/2$-Hölder group structure, where δ is defined in Theorem* 5.13.

Proof. Corollaries 5.2 and 5.5, Theorems $6.2, 6.4, 6.5, 6.7, 6.8$, the Hölder regularity properties of the classical Mazur mapping [32] (or Theorem 4.2 for L_2) and the Riesz-Fisher theorem imply, for some infinite index set I, the relations

$$X \overset{(\alpha,\beta)}{\longleftrightarrow} l_2(I) \overset{(1,1/2)}{\longleftrightarrow} l_1(I) \text{ and, thus, } X \overset{(\alpha,\beta/2)}{\longleftrightarrow} l_1(I).$$

The convolution operation makes $l_1(I)$ a commutative Banach algebra. Hence, defining, for $a, b \in X$, the multiplication by

$$a * b = \phi^{-1} (\phi(a) * \phi(b)),$$

where $\phi : X \longleftrightarrow l_1(I)$ is the homogeneous extension of a Hölder homeomorphism of spheres, we introduce a commutative homogeneous $\alpha\beta/2$-Hölder group structure on X. In the case of $L_p(\mathcal{M})$, one has $\alpha = \beta = \delta$ thanks to Theorem 6.8. □

The same argument relying on Theorem 6.2 imply the existence of the homogeneous Hölder group structure on a very large class of lattices.

Theorem 7.3. *Let X be a Banach lattice with a weak unit that does not contain $l_\infty(I_n)$ uniformly (by n). Then X allows a commutative homogeneous Hölder group structure.*

8 Complemented subspaces, copies of sequence spaces and local unconditional structure

The possession of the local unconditional structure by a Banach space indicates its local similarity to a lattice [25]. A Banach space X possesses the local unconditional structure simultaneously with its dual X^* [25]. Moreover, it happens if, and only if, X^{**} is isomorphic to a complemented subspace of a Banach lattice [25]. One of the traditional fine questions in functional analysis is the correlation between the local unconditional structure and the other properties of Banach space, and the last characterisation of this property, as well as the lattice property of the IG-spaces, relates it to the complementability matter.

As we see in Sections 3.2, 3.3, 6.2, 10 and 11, the sharpness of both the exponents and the constants of the Hölder-Lipschitz regularity strongly depends on the existence and complementability of certain subspaces in the spaces under consideration. In particular, the presence of the local unconditional structure appears to be intimately related to our problems at hand.

Remark 8.1. Let us note that the complementability of a subspace X in $Y = Y^{**}$ is equivalent to the existence of a Lipschitz retraction of Y onto X due to the linearisation properties of the Lipschitz mappings into reflexive subspaces (see Corollary 7.3 in [17]).

The next theorem follows from a celebrated result due to O.V. Besov in [20] on the interpolation of Nikol'skii-Besov and Lizorkin-Triebel spaces of function defined on a domain satisfying the C-flexible λ-horn condition.

Theorem 8.1. *Let $X \in \Gamma_1$ be a space of functions defined on a domain $G \subset \mathbb{R}^n$ satisfying the C-flexible λ-horn condition with an admissible parameter a and $\lambda = \gamma_a$. Then X is a complemented subspace of the corresponding IG-space Y with $I(Y)$ consisting of the same parameters as X, except for the components $\{a_i\}_{i=1}^n$ of a, and X^* is isomorphic to a complemented subspace of Y^*.*

Proof. In the case $X \in \{B^s_{p,q,1}(G), L^s_{p,q,1}(G)\}$, the conclusion of the theorem is established by O. V. Besov in [20]. For a wide class of the domains G satisfying flexible λ-horn condition (also introduced by Besov), the isomorphisms

$$B^s_{p,q,1}(G) \asymp B^s_{p,q,a}(G) \asymp \tilde{B}^{s,A}_{p,q,a}(G) \text{ and } L^s_{p,q,1}(G) \asymp L^s_{p,q,a}(G) \asymp \tilde{L}^{s,A}_{p,q,a}(G)$$

were established for every admissible a. Combining these results, we establish the statement for X. Lemma 6.1 finishes the proof, providing the statement regarding X^*. □

The next lemma is very helpful despite its simplicity.

Lemma 8.1. ([9, 13]). *Let X be a Banach space, and $P \in \mathcal{L}(X)$ be a projector onto its subspace $Y \subset X$. Assume also that $Q_Y : X \to \tilde{X} = X/\operatorname{Ker} P$ as the quotient map. Then we have*

$$\|Q_Y x\|_{\tilde{X}} \leq \|Px\|_X \leq \|P|\mathcal{L}(X)\| \|Q_Y x\|_{\tilde{X}} \text{ for every } x \in X.$$

In particular, the dual space $Y^* = X^*/Y^\perp$ and Y are isometric to P^*X^* and $\tilde{X}$ if, and only if, Y is 1-complemented in X, i.e $\|P|\mathcal{L}(X)\| = 1$.

These lemma and theorem immediately imply that some spaces under consideration possesses the local unconditional structure (see [25]).

Remark 8.2. As mentioned after Definition 2.2, the space $lt_{p,q}$ is isometric to a complemented subspace of $L_p(\mathbb{R}^n, l_q)$ for $p \in (1,\infty)^n$, $q \in (1,\infty)$ (see [9, 13] for the mixed norm case). In the case of scalar p and q, it an immediate consequence of the Fefferman-Stein inequality. Therefore, according to the previous lemma, the space $lt^*_{p,q}$ is not necessarily isometric to $lt_{p',q'}$ unless $p = \bar{q}$ but still isomorphic to $lt_{p',q'}$.

Relying on Theorem 2.1 and a celebrated result due to J. Bourgain [22] (extended in [6]), we have established [9] the following lemma allowing us to treat the whole class $\Gamma_0 = IG_+$.

Lemma 8.2. ([9, 13]). *Let $X \in IG_+$ ($X \in IG$). Then there exists $X \in IG_{0+}$ ($X \in IG_0$) with $I(X) = I(Y)$ and $T(X) \subset T(Y)$, such that X is a complemented subspace in the quotient Y/Z, where Z is a complemented subspace in Y. Moreover, if $lt^*_{p,q} \notin T(X)$, then X is a complemented subspace in Y.*

Corollary 8.1. *Let $X \in IG \cup \bigcup_{j=1}^{3} \Gamma_j$. Assume also that, if $X \in \Gamma_1$ is a space of functions defined on a domain $G \subset \mathbb{R}^n$ or its dual, the domain G satisfies the C-flexible λ-horn condition. Then X has the local unconditional structure.*

The proof of Corollary 8.1. If $X \in IG \cup \Gamma_3$, then it has the lattice structure of its own and, thus, has the local unconditional structure. If $X \in \Gamma_1 \cup \Gamma_2$ and is not defined as a dual, then it possesses the local unconditional structure because it is a complemented subspace of an IG-space (lattice; see [25]) due to Theorem 8.1. We finish the proof by noticing that, if $X \in \Gamma_1$ is defined as a dual, then it is isomorphic to a complemented subspace thanks to Lemma 8.1. □

Here we present the results on the existence of isomorphic copies of l_p-spaces in the various Sobolev, Nikol'skii-Besov and Lizorkin-Triebel spaces from Γ_i for $i \in I_3$. The combination of the succeeding theorems and lemmas with the next observation complements Dvoretzky's theorem for the spaces under consideration.

Remark 8.3. Let us recall that, for $p \in (1,\infty)$, $l_p(I_{2^n})$ contains a C_0-isomorphic and C_2-complemented copy of $l_2(I_n)$ for every $n \in \mathbb{N}$ thanks to the Hölder and Khinchin inequalities and Lemma 8.1.

Theorem 8.2. ([12, 13]). *Let $G \subset \mathbb{R}^n$, $p, a \in (1,\infty)^n$, $q, \varsigma \in (1,\infty)$, $s \in (0,\infty)$ and $r \in \{p_{\min}, p_{\max}, q, 2\}$. Assume also that*

$$Y \in \Big\{ B^s_{p,q,a}(G), \tilde{B}^{s,A}_{p,q,a}(G), L^s_{p,q,a}(G), \tilde{L}^{s,A}_{p,q,a}(G), b^s_{p,q,a}(G), \tilde{b}^{s,A}_{p,q,a}(G), l^s_{p,q,a}(G), \tilde{l}^{s,A}_{p,q,a}(G),$$

$$B^s_{p',q',a'}(G)^*, \tilde{B}^{s,A}_{p',q',a'}(G)^*, L^s_{p',q',a'}(G)^*, \tilde{L}^{s,A}_{p',q',a'}(G)^*,$$

$$b^s_{p',q',a'}(G)^*, \tilde{b}^{s,A}_{p',q',a'}(G)^*, l^s_{p',q',a'}(G)^*, \tilde{l}^{s,A}_{p',q',a'}(G)^* \Big\},$$

and a is in admissible range for Y. Then there are constants $C_0, C_1 > 0$, such that Y contains an C_0-isomorphic and C_1-complemented copy of $l_r(I_m)$ for every $m \in \mathbb{N}$.

Theorem 8.3. ([12, 13]). *Let* $Y \in \{W^s_p(G),\ W^s_{p'}(G)^*\}$ *for* $G \subset \mathbb{R}^n$, $p \in (1,\infty)^n$, $\varsigma \in (1,\infty)$, $s \in \mathbb{N}^n_0$ *and* $r \in \{p_{\min}, p_{\max}, 2\}$. *Then there are constants* $C_0, C_1 > 0$, *such that* Y *contains an* C_0*-isomorphic and* C_1*-complemented copy of* $l_r(I_m)$ *for every* $m \in \mathbb{N}$.

Theorem 8.4. ([12, 13]). *Let* $Y \in \{B^s_{p,q}(\mathbb{R}^n)_w,\ L^s_{p,q}(\mathbb{R}^n)_w,\ B^s_{p',q'}(\mathbb{R}^n)^*_w,\ L^s_{p',q'}(\mathbb{R}^n)^*_w\}$ *for* $p \in (1,\infty)^n$, $\varsigma \in (1,\infty)$, $s \in (0,\infty)$ *and* $r \in \{p_{\min}, p_{\max}, q, 2\}$. *Then* Y *contains an isometric* 1*-complemented copy of* $l_r(I_m)$ *for every* $m \in \mathbb{N}$.

Nevertheless, the spaces $lt_{p,q}$ and $lt^*_{p,q}$ contain an isometric and 1-complemented copies of l_p and l_q according to the next lemma.

Lemma 8.3. ([9, 13]). *Let* $p \in [1,\infty)^n$ *and* $q \in [1,\infty)$. *Then the spaces* $lt_{p,q}(\mathbb{R}^n)$ *and* $lt_{p',q'}(\mathbb{R}^n)^*$ *contain isometric* 1*-complemented copies of* $l_p(\mathbb{N}^n)$, $l_q(\mathbb{N})$ *and* $l_p\left(\mathbb{Z}^n, l_q(\mathbb{N})\right)$, *and the spaces* $lt_{p,q}(F)$ *and* $lt_{p',q'}(F)^*$ *contain isometric* 1*-complemented copies of* $l_q(\mathbb{N})$, $l_p\left(I_m, l_q(\mathbb{N})\right)$ *for every* $m \in \mathbb{N}^n$.

What follows is the counterpart of Lemma 8.3 for Schatten-von Neumann classes and general $L_p(\mathcal{M})$, where $\mathcal{M}$ is a von-Neumann algebra (with a normal semifinite faithful weight that always exists).

Lemma 8.4. ([9, 13]). *For* $p \in [1,\infty]$, *the space* S_p *contains* 1*-complemented copies of* S^n_p, $l_p(I_n)$ *and* l_p *for* $n \in \mathbb{N}$. *Moreover, an infinite-dimensional* $L_p(\mathcal{M})$ *contains a* 1*-complemented isometric copy of* l_p.

Remark 8.4. The existence of isomorphic copies of c_0 and l_∞ in different classes of function spaces was investigated in [4, 5], where stronger results than the counterparts of the celebrated non-complementability of $C([0,1])$ in $L_\infty([0,1])$ due to G.M. Fichtenholtz and L.V. Kantorovich [27] were established. The finite representability and the existence of the copies of l_p and other sequence spaces in the spaces under consideration was studied in [12, 13].

We also need the following quantitative version of the result due to M.S. Baouendi and G. Goulaouic [16, 41].

Theorem 8.5. *For* $p \in [1,\infty]$, $\theta \in (0,1)$, *let* (A_0, A_1) *be a compatible couple of Banach spaces, and let* B *be a complemented subspace of* $A_0 + A_2$, *whose projector* $P \in \mathcal{L}(A_0) \cap \mathcal{L}(A_1)$. *Then* $(B_0, B_1) = (A_0 \cap B, A_1 \cap B)$ *is also compatible, and we have*

a) $d_{BM}\left((B_0,B_1)_{\theta,p}, (A_0,A_1)_{\theta,p} \cap B\right) \le \|P|\mathcal{L}(A_0)\|^{1-\theta} \|P|\mathcal{L}(A_1)\|^{\theta}$;

b) $d_{BM}\left((B_0,B_1)_{[\theta]}, (A_0,A_1)_{[\theta]} \cap B\right) \le \|P|\mathcal{L}(A_0)\|^{1-\theta} \|P|\mathcal{L}(A_1)\|^{\theta}$.

While the lower estimates for $\|x|(B_0,B_1)_{\theta,p}\|$ and $\|x|(B_0,B_1)_{[\theta]}\|$ are provided by the definitions of the interpolation functors, the upper estimates follow from the exactness of these functors:

$$\max\left(\|P|\mathcal{L}\left((A_0,A_1)_{\theta,p}\right)\|, \left\|P|\mathcal{L}\left((A_0,A_1)_{[\theta]}\right)\right\|\right) \le \|P|\mathcal{L}(A_0)\|^{1-\theta} \|P|\mathcal{L}(A_1)\|^{\theta}.$$

The following theorem permits us to reduce the study of the properties of the general $L_p(\mathcal{M})$ (Haagerup L_p-spaces) to checking them for the L_p spaces of finite von Neumann algebras ($L_p(\mathcal{M},\tau)$ with n.f.f. τ).

Theorem 8.6. (Haagerup [40]). *Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful weight ϕ, $p \in (0,\infty)$), and let $L_p(\mathcal{M},\phi)$ be the associated Haagerup L_p-space. Then there are a $\min(p,1)$-Banach space Z, a directed family $\{(\mathcal{M}_i,\tau_i)\}_{i\in I}$ of finite von Neumann algebras and a family $\{J_i\}_{i\in I}$ of isometric embeddings $J_i : L_p(\mathcal{M}_i,\tau_i) \hookrightarrow Z$ satisfying*

1) $\operatorname{Im} J_i \subset \operatorname{Im} J_{i'}$ *for all* i, i' *with* $i \le i'$;
2) $\bigcup_{i\in I} \operatorname{Im} J_i$ *is dense in* Z;
3) $L_p(\mathcal{M},\phi)$ *is isometric to a subspace of* Z, *complemented if* $p \in [1,\infty)$.

9 Hölder-Lipschitz mappings: basic mappings and properties. II

This section is dedicated to the following important tools of the analysis of mappings between Banach spaces: the existence and the ordinary and explicit (an occasionally sharp) global Hölder continuity of retractions and metric projections onto closed convex subsets and the problem of the bounded extension of the Hölder-Lipschitz mappings from an arbitrary subset of a matric space into a Banach space to Hölder-Lipschitz mappings defined on the whole metric space with explicit and occasionally sharp bounds.

9.1 Retractions

Definition 9.1. *For a metric space Y and its subset X, a mapping $f : Y \to X$ is a retraction of Y onto X if $f(x) = x$ for every $x \in X$. The subset X is said to be a retract of Y.*

According to Part *b*) of the next lemma, $l_\infty(\Gamma)$ is an absolute 1-Lipschitz retract. It is Lemma 1.1 from [17].

Lemma 9.1. ([17]). *a*) *Every metric space is isometric to a subset of $l_\infty(X)$.*

b) *Let Y be a metric space, $Z \subset Y$, and ω be a nondecreasing subadditive function defined on $(0,\infty)$ with $\lim_{t\to 0} \omega(t) = 0$. Assume also that $f : Z \to l_\infty(\Gamma)$ satisfies $\omega(\cdot, f, Z) \le \omega$. Then there exists a uniformly continuous extension $F : Y \to l_\infty(\Gamma)$ of f with $\omega(\cdot, F, Y) \le \omega$.*

The next theorem is a particular setting of a bounded set A of the corresponding more general results in [7, 13]. The numerical constant from [7] was improved in [13] with the aid of [10].

Theorem 9.1. ([7, 13]). *For $p \in [2,\infty)$, let A be a closed convex bounded subset of a quasi-Banach space X that is isomorphic to a (p,h_c)-uniformly convex Banach space Z with $d_{BM}(X,Z) < d$ and $\sigma \in (0,\infty)$. Assume also that a metric space Y contains an isometric copy $\tilde{A}$ of A (endowed with the metric inherited from X), and $A_a\sigma$ is* *the σ-neighborhood of this copy in Y. Then there exists a retraction ϕ of A_σ onto $\tilde{A}$ satisfying*

$$\|\phi|H^{1/p}(Y,\tilde{A})\| \le d(8p)^{1/p}\left(\sup_{\mu\in(0,1/2]}(1-\mu)h_c(\mu)\right)^{-1/p}(r(A,A)+d(A))^{1/p'}.$$

Moreover, if X is a (p,h_c)-uniformly convex Banach space itself, one should take $d=1$ in this estimate.

9.2 Chebyshev sets and metric projection

Metric projection is a very important example of a retraction possessing better smoothness than the retractions considered in the previous subsections. In approximation theory it corresponds to the best approximation of a function by a function from a closed convex or linear subclass.

In this subsection, we describe the smoothness of the metric projections on closed convex subsets of either uniformly convex, or both uniformly convex and uniformly smooth spaces. We further provide retractions onto such subsets from the ambient space that is either uniformly convex, or both uniformly convex and uniformly smooth. These retractions possess either better smoothness, or better constants than their counterparts in the preceding subsection.

Definition 9.2. *A subset $D\subset X$ of a Banach space X is a Cheyshev set if the metric projection mapping $P_D:\ X\to D$ is well-defined by the relation*

$$\|x-P_Dx\|_X=\min_{y\in D}\|x-y\|_X,$$

that is, for every $x\in X$, there exists a unique $y=P_Dx$ minimizing the distance between x and D.

Note that, thanks to the Hahn-Banach theorem, $P_Dy=P_Dx$ if $y=\lambda x+(1-\lambda)P_Dx$ for some $\lambda\ge 0$.

Remark 9.1. *a*) While every closed convex subset of a reflexive and strictly convex Banach space is a Chebyshev set, there exist examples of such Banach spaces with discontinuous (in norm) metric projections on some Chebyshev sets (see [23, 46]). The necessary and sufficient condition on a Banach space for the continuity of the metric projections onto the closed convex subsets was found by L. P. Vlasov [47] (see Theorem 9.2 below). This condition was introduced by V.L. Shmul'yan in 1940. Every uniformly convex space satisfies this condition.

b) There exists an important characterization of inner product spaces due to Phelps [37]: a Banach space X with $\dim X>2$ is a Hilbert space if, and only if, the metric projection on every closed convex Chebyshev subset is 1-Lipschitz (nonexpansive).

c) The metric projections onto the balls of a strictly convex normed space X are 2-Lipschitz (see, for example, [7] and (3) in the proof of Lemma 5.3). This means that balls are too good subsets to distinguish the peculiar features of the (local) geometry of X from the point of view of the metric projection.

Theorem 9.2. (L.P. Vlasov [47]). *The metric projection onto every closed and convex subset of a Banach space X is single-valued and continuous (in norm) if, and only if, every subsequence $\{x_k\}_{k\in\mathbb{N}} \subset X$ with $\|x_n\|_X = 1$ for every n, satisfying the condition $\lim_{k\to\infty} f(x_k) = 1$ for some $f \in X^*$ with $\|f\|_{X^*} = 1$, is convergent in X.*

The uniform continuity of the set-valued metric projection was investigated by Berdyshev [18], while the same phenomenon for the (single-valued) metric projection in uniformly continuous and uniformly smooth spaces was studied by Björnestål [21], in the case of the metric projections onto subspaces, and by Benyamini and Lindenstrauss [17] in the case of the metric projections onto the closed convex subsets. In the latter case, the estimates for the local uniform continuity, that is for the modulus $\omega(t, P_D, x + r(x)B_X)$ with $r(x) \le Cd(x, D)$, were established in terms of the classical moduli of the uniform continuity and uniform smoothness. In some special case, global estimates of similar nature (that cannot be derived from the local ones) were established by Alber [14]. In this section, we present global estimates in the general setting of an arbitrary closed convex subset providing the same order of the Hölder regularity with explicit numerically friendly constants.

Since every Hilbert space is $(2,1)$-uniformly convex and smooth according to the Jacoby identity [9], even Part $b)$ of the last remark suggests that the global regularity of the metric projection could be higher if the space is not only (p, h_c)-uniformly convex but also (q, h_s)-uniformly smooth. It is the subject of the next theorem and corollary that are extracts from the corresponding results in [13] (their counterparts in [7] are less precise in the general setting but still lead to the same numerical estimates for the spaces under consideration).

According to Theorem 6.16 from [13], the Hölder-Lipschitz regularity exponent given in the next theorem and corollary are sharp for $X \in IG_+$ under the restriction that, if $p_{\min}(X) < 2$, X (Y) contains isometric 1-complemented copies of $\{l_{p_k}\}_{k\in\mathbb{N}}$ with $p_k \in I(X)$ for every $k \in \mathbb{N}$ and $\lim_{k\to\infty} p_k = p_{\min}(X)$, and, if $p_{\max}(X) > 2$, X contains isometric 1-complemented copies of $\{l_{q_k}\}_{k\in\mathbb{N}}$ with $q_k \in I(X)$ for every $k \in \mathbb{N}$ and $\lim_{k\to\infty} q_k = p_{\max}(X)$.

Theorem 9.3. ([7, 13]). *For $2 \in [q,p] \subset (1,\infty)$, let X be a (p, h_c)-uniformly convex and (q, h_s)-uniformly smooth Banach space and a closed convex $D \subset X$. Assume also that $A \subset X$ is a bounded subset of X and*

$$c_c = \sup_{\mu\in(0,1/2]} (1-\mu)h_c(\mu) \quad \text{and} \quad c_s = \inf_{\mu\in(0,1/2]} (1-\mu)^{1-q} h_s(\mu).$$

Then we have

$$\big\| P_D \big| H^{q/p}(A, D)\big\| \le \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \left(æ(A,D)^q + c_s c_c^{-q/p} d(A)^q\right)^{1/q-1/p}.$$

Moreover, if $p = q = 2$, we also have

$$\big\| P_D \big| H^1(X, D)\big\| \le \frac{c_s^{1/2}}{c_c},$$

that is P_D is $c_s^{1/2}/c_c$-Lipschitz.

Corollary 9.1. ([7, 13]). *For $2 \in [q,p] \subset (1,\infty)$, let X be a quasi-Banach space that is isomorphic to a (p,h_c)-uniformly convex and (q,h_s)-uniformly smooth Banach space Y with $d_{BM}(X,Y) < d$, and a closed convex $D \subset X$. Assume also that $A \subset X$ is a bounded subset of X and*

$$c_c = \sup_{\mu\in(0,1/2]} (1-\mu)h_c(\mu) \text{ and } c_s = \sup_{\mu\in(0,1/2]} (1-\mu)^{1-q}h_s(\mu).$$

Then there exists a retraction ψ_D of X onto D satisfying

$$\left\| \psi_D |\, H^{q/p}(A,D)\right\| \le d\left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p}\left(æ(A,D)^q + c_s c_c^{-q/p} d(A)^q\right)^{1/q-1/p}.$$

Moreover, if $p=q=2$, we also have

$$\left\| \psi_D |\, H^1(X,D)\right\| \le \frac{dc_s^{1/2}}{c_c},$$

that is P_D is $dc_s^{1/2}/c_c$-Lipschitz.

9.3 Hölder-Lipschitz mappings: bounded extension

This auxiliary section is dedicated to the extension problem for the Hölder-Lipschitz mappings from a subset of a metric or a Banach space into a Banach space. It is essentially an extract from [11, 13] where more information including complete proofs, background and applications to the pairs of the spaces under consideration are presented, a well as a Markov type and cotype counterpart of the Rademacher type and cotype theory (see also references therein).

Definition 9.3. *Assume that X is a metric space, Y is a Banach space, and $\alpha, d > 0$. Let $H^\alpha(X,Y)$ be the Banach space of all Y-valued continuous functions f defined on X with the finite norm:*

$$\|f|H^\alpha(X,Y)\| := \sup\left\{\|f(x)-f(y)\|_Y / d_X(x,y) : x,y \in X \text{ and } x \ne y\right\}.$$

We say that the pair (X,Y) possesses (d,α)-extension property if, for every subset $F \subset X$ (with the induced metric) and every $f \in H^\alpha(F,Y)$, there is an extension $\tilde f \in H^\alpha(X,Y)$ satisfying

$$\tilde f(x) = f(x) \text{ for } x \in F \quad \text{and} \quad \|\tilde f|H^\alpha(X,Y)\| \le d\|f|H^\alpha(F,Y)\|.$$

Let $S_b(X,Y) \subset (0,\infty)$ be the set of all α, such that the pair (X,Y) possesses (d,α)-extension property for some $d < \infty$.

We say that the pair (X,Y) possesses convex (d,α)-extension property if it possesses the (d,α)-extension property, and there exists a corresponding extension $\tilde f \in H^\alpha(X,Y)$ of $f \in H^\alpha(F,Y)$ satisfying $\tilde f(X) \subset \overline{\mathrm{co}} f(F)$.

Remark 9.2. *a*) Let also $S_=(X,Y) \subset (0,\infty)$ be the set of all α, such that the pair (X,Y) possesses $(1,\alpha)$-extension property, while $S_{=,c}(X,Y) \subset (0,\infty)$ be the set of all α, such that the pair (X,Y) possesses convex $(1,\alpha)$-extension property. The sets $S_=(X,Y)$ and $S_{=,c}(X,Y)$ for the pairs of spaces under consideration are found in [9].

b) The discrepancy between an arbitrary pair of $\{S_b(X,Y), S_=(X,Y), S_{=,c}(X,Y)\}$ is called the phase transition phenomenon for the pair.

c) For the applications of the results on bounded extension it is very useful to observe that, if a pair (X,Y) has a (d,α)-extension property, and X is C_0-Lipschitz homeomorphic (or C_0-isomorphic if X is Banach) to X_0, while Y is C_1-isomorphic to Y_0, then the pair (X_0,Y_0) has the $(dC_0^\alpha C_1,\alpha)$-extension property, where X_0 can be a quasi-metric or a quasi-Banach space and Y_0 can be a quasi-Banach space.

Markov type and Markov cotype

Let us define the notions of Markov type and Markov cotype.

Definition 9.4. ([34]). *Let (X,d) be a metric space and $p \in (0,\infty]$. The space X is said to possess the* Markov type p *with a constant C_{MT} if, for every $n \in \mathbb{N}$, every stationary reversible Markov chain $\{\xi_k\}_{k\in\mathbb{N}\cup\{0\}}$ with the state set S, and every $f: S \to X$, one has the estimate*

$$\left(\mathrm{E}d\left(f(\xi_n), f(\xi_0)\right)^p\right)^{1/p} \le C_{MT}\left(n\mathrm{E}d\left(f(\xi_1), f(\xi_0)\right)^p\right)^{1/p}.$$

The best constant C_{MT} is designated by $C_{MT}(p,X)$.

Remark 9.3. *a*) Note that we can consider only the chains with strictly positive stationary distributions.

b) K. Ball [15] showed that every metric space (X,d) has the Markov type 1 with the constant 1. Since d^α with $\alpha \in (0,1)$ is still a metric on X, every metric space is also of type α.

c) Markov type properties (type and the constant) are inherited by the subsets of a metric space.

d) In fact, Theorem 1.6 in [15] shows that the definition of Markov type in [15] (Definition 1.6 in [15]) is equivalent to, at least, a formally less restrictive counterpart of Definition 9.4 where only the stationary reversible Markov chains with symmetric transition matrixes are allowed. Thus, if X is of Markov type p with a constant C_{MT} according to Definition 9.4, it is also of Markov type p with not worse constant according to Ball's original definition.

e) There are other notions of type and cotype than Markov or Rademacher ones (see [38]).

To define the Markov cotype, we slightly modify the original definition of Ball (written in the language of matrixes) by substituting the exponent 2 with q.

Definition 9.5. ([15]). *Let X be a normed space and $q \in [1,\infty]$. The space X is said to possess the* Markov cotype p *with a constant C_{MC} if, for every $n \in \mathbb{N}$, $\beta \in (0,1)$,* *symmetric (double) stochastic $n \times n$ matrix A and sequence $\{x_i\}_{i=1}^n \subset X$, one has the estimate*

$$\left(\beta \sum_{i,j=1}^{n} a_{i,j} \left\| \sum_{k=1}^{n} c_{i,k} x_k - \sum_{l=1}^{n} c_{j,l} x_l \right\|_X^q \right)^{1/q} \le C_{MC} \left((1-\beta) \sum_{i,j=1}^{n} c_{i,j} \|x_i - x_j\|_X^q \right)^{1/q},$$

where $C = (1-\alpha)(I-\alpha A)^{-1}$ and $\{a_{i,j}\}$ and $\{c_{i,j}\}$ are the entries of A and C respectively.

For the sake of convenience, we say that X possesses the Markov cotype ∞ with the constant 1 if, instead, one has

$$\max_{i,j=1}^{n} \left\| \sum_{k=1}^{n} c_{i,k} x_k - \sum_{l=1}^{n} c_{j,l} x_l \right\|_X \le \max_{i,j=1}^{n} \|x_i - x_j\|_X.$$

The best constant C_{MC} is designated by $C_{MC}(q, X)$.

Remark 9.4. *a*) Since $\sum_{k=1}^n c_{i,k} x_k - \sum_{l=1}^n c_{j,l} x_l = \sum_{k=1}^n \sum_{l=1}^n c_{i,k} c_{j,l} (x_k - x_l)$, the triangle inequality implies that every Banach space has the Markov cotype ∞ with the constant 1.

b) Let us note that, if a Banach space X is finitely represented in Y possessing the Markov type p with the constant C_{MT} and the Markov cotype q with the constant C_{MC}, then X has the same Markov type and cotype with the same constants.

c) Note that the Markov cotype can also be correctly defined for the convex subsets of a Banach space.

The next theorem relates the (q, h_s)-uniform smoothness to the Markov type q and (p, h_c)-uniform convexity to the Markov cotype p in a quantitative manner.

Theorem 9.4. ([11, 13]). *Assume that $2 \in [q, p] \subset [1, \infty)$.*

a) *If X is a (q, h_s)-uniformly smooth Banach space, then it possesses the Markov type q with the constant*

$$C_{q,h_s} = \left(\inf_{\mu \in (0,1/2]} (1-\mu)^{1-q} h_s(\mu) \right)^{1/q}.$$

b) *If X is a (p, h_c)-uniformly convex Banach space, then it possesses the Markov cotype p with the constant*

$$C_{p,h_c} = 2 \left(\sup_{\mu \in (0,1/2]} (1-\mu) h_c(\mu) \right)^{-1/p}.$$

Adaptation of Ball's scheme

Let us present the adaptation of Ball's approach [15] to the bounded extension problem for Hölder-Lipschitz mappings established in [11, 13]. K. Ball treated the extension of Lipschitz mappings from a Markov type 2 metric space into a Markov cotype 2 Banach space but Naor [33] mentioned that it works also for some Hölder mappings and different values of Markov types and cotypes because $d(x,y)^\alpha$ with $\alpha \in (0,1]$ is a metric if $d(x,y)$ is, and because L_p endowed with the metric $\|x-y\|^{p/2}$ for $p \in (0,2]$ is isometrically embedded into L_2 thanks to Theorem 5.11 in [48].

Theorem 9.5. ([11, 13]). *For $\alpha = p/q \in (0,1]$ with $p, q \in [1,\infty)$, let (X,d) be a metric space possessing the Markov type p with a constant C_{MT}, and let Y be a Banach space possessing the Markov cotype q with a constant C_{MC}. Assume also that $Z \subset X$ and $f \in H^\alpha(Z,Y)$. Then there is an extension $\tilde{f} \in H^\alpha(X,Y^{**})$ of f with*

$$\|\tilde{f}|H^\alpha(X,Y^{**})\| \leq 3^{\frac{p-1}{q}}(C_{MC}^q + 2)^{\frac{1}{q}} C_{MT}^\alpha \|f|H^\alpha(Z,Y)\| \leq (3C_{MT})^\alpha C_{MC}\|f|H^\alpha(Z,Y)\|.$$

Remark 9.5. *a*) As far as the concrete pairs are concerned, K. Ball [15] applied his abstract result on the existence of isomorphic extensions of the Lipschitz mappings from a Markov type 2 space into a Markov cotype 2 space to the pairs of Lebesgue spaces (L_2, L_q) with $q \in [2,\infty)$. More precisely, he showed that the pair possesses the $(1, d_{p,q})$-extension property with the constant $d_{p,q} = 6(q-1)^{-1/2}$. But he also explicitly quantitatively related the uniform convexity and Markov cotype and established the Markov type constant of L_p for $p \in (1,2]$ to be 1. Naor [33] mentioned that Ball's scheme works for the Lipschitz mappings ($\alpha \in (0,1]$) between the pairs (X,Y) of the Markov type p X and Markov cotype p Y spaces and, in addition, found that $S_b(l_p, L_q) = p/\max(q,2)$ for $p \in (1,2]$ and $q \in (1,\infty)$ (interpreting Hölder mappings as Lipschitz; see Remark 9.3, *b*)). Naor, Peres, Schramm and Sheffield [34] showed that L_p has the Markov type 2 for $p \in (2,\infty)$, estimated the Markov type constant using a representation of a Markov chain as a sum of a backward and a forward martingales. Therefore, they completed also the computation of $S_b(L_p, L_q)$ for any pair $p, q \in (1,\infty)$. The pair (L_p, L_q) with $2 \in [q,p]$ was shown to have the $(1, d_{p,q})$-extension property with

$$d_{p,q} \leq 24\sqrt{\frac{p-1}{q-1}}.$$

They conjectured that the constant 24 can be reduced to 1. Theorems 9.4 and 9.5 established in [11] (see also [13]) with the aid of our Markov chain counterpart of Pisier's martingale inequality obtained in [10, 13] provide the estimate $d_{p,q} \leq 6\sqrt{\frac{p-1}{q-1}}$ not only for the pairs of commutative spaces, but also for the pairs of Schatten-von Neumann classes, general noncommutative L_p-spaces or, even, mixed pairs with the same conditions on p and q. The constant 6 above comes from the Ball's scheme meaning that the justification of the conjecture requires to improve Ball's scheme itself.

b) The results in [11, 13] cover the pairs of spaces from the union of the classes of spaces under the consideration in this paper, including the sharpness of the Hölder exponents.

10 Main results: approximation of uniformly continuous mappings

In this section we establish the main results describing the uniform approximation of uniformly continuous mappings from a metric spaces, or a (convex) subset of a space under consideration into another such space by Hölder-Lipschitz mappings.

The best possible smoothness exponents of the approximating Hölder-Lipschitz mappings for the uniformly continuous mappings from the unit ball of L_p l_p into L_q or l_q for various pairs (p,q) were found by I. G. Tsar'kov [42, 43, 44] (see also § 2.1

in [17]). His approach relies on the investigation of the Hölder-Lipschitz regularity of the Mazur maps and M. D. Kirszbraun's extension theorem [29] (see also [28] for generalizations). Earlier [30] he used a different approach utilizing Frechet's extension theorem to approximate the uniformly continuous mappings from a metric space into a superreflexive (uniformly convex) space.

Our approach is based on our counterparts of Tsar'kov's tools developed in [7, 11, 13] and below. We shall deal with the following classes of uniformly continuous mappings.

Definition 10.1. ([30]). *Let X be a metric space with a metric ρ_X and Y a (quasi) Banach space. Assume also that Ω_S is the class of the semiadditive functions $\omega:\ [0,\infty)\to[0,\infty)$ satisfying $\lim_{t\to 0}\omega(t)=\omega(0)=0$.*

Then, by means of $H^\omega(X,Y)$ for $\omega\in\Omega_S$, we designate the class of the continuous mappings $f:\ X\to Y$ satisfying

$$\|f(x)-f(z)\|_Y\le\omega\left(\rho_X(x,z)\right)\ \textit{for every } x,z\in X.$$

Note that, whenever X is metrically convex (for example, X is a convex subset of a normed space with the inherited metric), one has $f\in H^{\omega_f}$ for every uniformly continuous mapping $f:\ X\to Y$, where ω_f is the modulus of continuity of f.

10.1 Abstract Bernstein-Jackson principle

The following common abstract step, reflecting the classical relation between the approximation properties of a mapping and its smoothness, can be extracted from both approaches due to Tsar'kov mentioned above.

Lemma 10.1. *For $d\ge 1$ and $\omega\in\Omega_S$, let (X,Y) be a pair of a metric space X and a Banach space Y possessing the $(d,1)$-extension property, and $f\in H^\omega(X,Y)$. Then, for every $\varepsilon>0$, there exists $f_\varepsilon\in H^1(X,Y)$ satisfying*

$$\|f-f_\varepsilon|C(X,Y)\|\le(1+2d)\omega(\varepsilon)\quad\text{and}\quad\|f_\varepsilon|H^1(X,Y)\|\le 2d\omega(\varepsilon)/\varepsilon.$$

Proof. Thanks to M. Zorn's lemma, there exists a maximal ε-separated subset $M_\varepsilon\subset X$. The restriction $\bar f_\varepsilon$ of f on M_ε is $2\omega(\varepsilon)/\varepsilon$-Lipschitz thanks to the subadditivity of ω. With the aid of the triangle inequality, the proof is finished by choosing f_ε to be a $2d\omega(\varepsilon)/\varepsilon$-Lipschitz extension of $\bar f_\varepsilon$ onto X. □

Sometimes the smoothness of the approximation f_ε is less important than the convex-envelope stability of the images $f(X)$ and $f_\varepsilon(X)$. The next lemma shows how to improve the geometry of the image $f_\varepsilon(X)$ at the expense of its smoothness.

Lemma 10.2. *For $2\in[q,p]\subset(1,\infty)$, let X be a bounded metric space, and let Y be a quasi-Banach space that is isomorphic to a (p,h_c)-uniformly convex and (q,h_s)-uniformly smooth Banach space Z with $d_{BM}(Y,Z)<d$,*

$$c_c=\sup_{\mu\in(0,1/2]}(1-\mu)h_c(\mu)\quad\text{and}\quad c_s=\inf_{\mu\in(0,1/2]}(1-\mu)^{1-q}h_s(\mu).$$

Assume also that $\omega \in \Omega_S$ *and* $g_\omega, h_\omega: (0,\infty) \to (0,\infty)$ *with* $\lim_{t\to 0+} g_\omega(t) = 0$ *are such functions that, for some* $\alpha \in (0,1)$ *and every* $f \in H^\omega(X,Y)$ *and* $\varepsilon > 0$, *there exists* $\tilde{f}_\varepsilon \in H^\alpha(X,Y)$ *satisfying*

$$\|f - \tilde{f}_\varepsilon | C(X,Y)\| \le g_\omega(\varepsilon) \quad \text{and} \quad \|\tilde{f}_\varepsilon | H^\alpha(X,Y)\| \le h_\omega(\varepsilon).$$

Then, for every $f \in H^\omega(X,Y)$ *and* $\varepsilon > 0$, *there exists* $f_\varepsilon \in H^{q\alpha/p}(X,Y)$ *satisfying* $f_\varepsilon(X) \subset \overline{\mathrm{co}}\,(f(X))$,

$$\|f - f_\varepsilon | C(X,Y)\| \le d g_\omega(\varepsilon)^{q/p} \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \left(g_\omega(\varepsilon)^q + c_s c_c^{-q/p} \left(\omega\left(d(X)\right) + 2g_\omega(\varepsilon)\right)^q\right)^{1/q-1/p}$$

and

$$\|f | H^{q\alpha/p}(X,Y)\| \le d h_\omega(\varepsilon)^{q/p} \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \left(g_\omega(\varepsilon)^q + c_s c_c^{-q/p} \left(\omega\left(d(X)\right) + 2g_\omega(\varepsilon)\right)^q\right)^{1/q-1/p}.$$

Moreover, if $p = q = 2$ *and* X *is either bounded, or unbounded, for every* $f \in H^\omega(X,Y)$ *and* $\varepsilon > 0$, *there exists* $f_\varepsilon \in H^\alpha(X,Y)$ *satisfying* $f_\varepsilon(X) \subset \overline{\mathrm{co}}\,(f(X))$,

$$\|f - f_\varepsilon | C(X,Y)\| \le \frac{dc_s^{1/2}}{c_c} g_\omega(\varepsilon)$$

and

$$\|f | H^\alpha(X,Y)\| \le \frac{dc_s^{1/2}}{c_c} h_\omega(\varepsilon).$$

Proof. It is sufficient to choose $f_\varepsilon = \psi_D \circ \tilde{f}_\varepsilon$, where ψ_D is the retraction onto $D = \overline{\mathrm{co}}\,(f(X))$ provided by Corollary 9.1, and take advantage of the conclusion of Corollary 9.1 (with $A = \tilde{f}_\varepsilon(X) \subset D + g_\omega(\varepsilon)B_Y$ if $q < p$) and Corollary 3.1, a). □

Corollary 10.1. *Let* X *be a metric space that is* d_0*-Lipschitz homeomorphic to a metric space* X_0 *possessing the Markov type* 2 *with a constant* C_{MT}, *a bounded* $A \subset X$ *and* $\omega \in \Omega_S$. *Assume also that* Y *is a subspace of a quasi-Banach space* Z *isomorphic to some Banach space* Z_0 *possessing the Markov cotype* 2 *with a constant* C_{MC}, $d_{BM}(Z, Z_0) < d_1$ *and* $d = 3C_{MT}C_{MC}$. *Then, for every* $\varepsilon > 0$ *and* $f \in H^\omega(A,Y)$, *there exists* $f_\varepsilon \in H^\beta(A,Y)$ *satisfying*

$$\|f - f_\varepsilon | C(A,Y)\| \le g_\omega(\varepsilon) \text{ and } \|f_\varepsilon | H^\beta(A,Y)\| \le h_\omega(\varepsilon)$$

for the following combinations β, g_ω *and* h_ω *in the following settings.*

a) One can choose $f_\varepsilon: A \to \overline{\mathrm{co}}\,(f(A))$, $\beta = r_s/r_c$,

$$g_\omega(\varepsilon) = (1+2d)d_1\omega(\varepsilon)\left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{\frac{1}{r_c}} \left(1 + c_s c_c^{-\frac{r_s}{r_c}} \left(2 + \frac{\omega(d(A))}{2d\omega(\varepsilon)}\right)^{r_s}\right)^{\frac{1}{r_s} - \frac{1}{r_c}}$$

and

$$h_\omega(\varepsilon) = 2dd_0^{\frac{r_s}{r_c}} d_1 \omega(\varepsilon)\varepsilon^{-\frac{r_s}{r_c}} \left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}} \right)^{\frac{1}{r_c}} \left(1 + c_s c_c^{-\frac{r_s}{r_c}} \left(2 + \frac{\omega(d(A))}{2d\omega(\varepsilon)} \right)^{r_s} \right)^{\frac{1}{r_s} - \frac{1}{r_c}}$$

if Z_0 *is* (r_c, h_c)*-uniformly convex and* (r_s, h_s)*-uniformly smooth with*

$$c_c = \sup_{\mu \in (0,1/2]} (1-\mu) h_c(\mu) \text{ and } c_s = \inf_{\mu \in (0,1/2]} (1-\mu)^{1-r_s} h_s(\mu).$$

b) One can take $\beta = 1$, $g_\omega(\varepsilon) = (1+2d) d_1 \omega(\varepsilon)$ *and* $h_\omega(\varepsilon) = dd_0 d_1 \omega(\varepsilon)/\varepsilon$. *Moreover, if* Z_0 *is Hilbert, one even has* $f_\varepsilon : A \to \overline{\text{co}}\,(f(A))$.

Proof. Let $T_0 : X \longrightarrow X_0$ and $T_1 : Z \longrightarrow Z_0$ be, correspondingly, a homeomorphism and an isomorphism satisfying $\|T_0|H^1(X, X_0)\| \|T_0^{-1}|H^1(X_0, X)\| \le d_0$ and $\|T_1|\mathcal{L}(Z, Z_0)\| \|T_1^{-1}|\mathcal{L}(Z_0, Z)\| < d_1$, $f \in H^\omega(A, Y)$ and

$$\bar{f} = T_1 \circ f \circ T_0^{-1} : X_0 \longrightarrow T_1 Y \subset Z_0. \tag{1}$$

Thanks to Theorem 9.5 and Part b) of Remark 9.4 the pair (X_0, Y_0) with $Y_0 = T_1 Y$ possesses the $(d, 1)$-extension property. Due to Lemma 10.1, combined with Corollary 3.1, and its proof, for any $\varepsilon = \|T_0^{-1}|H^1(X_0, X)\|\varepsilon' > 0$, there exist $\bar{f}_\varepsilon$ and a maximal ε'-net $\bar{A}_\varepsilon \subset T_0 A = A_0$ satisfying $\bar{f}(x) = \bar{f}_\varepsilon(x)$ for $x \in \bar{A}_\varepsilon$,

$$\|\bar{f} - \bar{f}_\varepsilon|C(X_0, Y_0)\| \le (1 + 2d) \|T_1|\mathcal{L}(Z, Z_0)\| \omega(\varepsilon)$$

and

$$\|\bar{f}_\varepsilon|H^1(X_0, Y_0)\| \le 2d \|T_1|\mathcal{L}(Z, Z_0)\| \omega(\varepsilon)/\varepsilon'.$$

Note that, according to our construction, we have $d\left(\bar{f}(A_0)\right) \le \|T_1|\mathcal{L}(Z, Z_0)\| \omega\left(d(A)\right)$,

$$d_H\left(\bar{f}_\varepsilon(A_0), \bar{f}(A_0)\right) \le d_H\left(\bar{f}_\varepsilon(A_0), \bar{f}(\bar{A}_\varepsilon)\right) \le 2d \|T_1|\mathcal{L}(Z, Z_0)\| \omega(\varepsilon)$$

and

$$æ\left(\bar{f}_\varepsilon(A_0), \bar{f}(A_0)\right) \le æ\left(\bar{f}_\varepsilon(A_0), \bar{f}(\bar{A}_\varepsilon)\right) \le 2d \|T_1|\mathcal{L}(Z, Z_0)\| \omega(\varepsilon). \tag{2}$$

To finish the proof of Part a) and the Hilbert setting in Part b), we construct

$$f_\varepsilon = T_1^{-1} \circ P_D \circ \bar{f}_\varepsilon \circ T_0 : A \to \overline{\text{co}}\,(f(A)), \tag{3}$$

where P_D is the metric projection onto $D = \overline{\text{co}}\,(T_1 f(A))$ (that is 1-Lipschitz if Z_0 is Hilbert). The application of Lemma 9.1, a), Theorem 9.3 and (2) shows that a) and the Hilbert case of b) hold.

To finish the proof of the corollary in the case of the rest of b), we just take

$$f_\varepsilon = T_1^{-1} \circ \bar{f}_\varepsilon \circ T_0 : A \to D = Y.$$

□

10.2 Adaptation of Tsar'kov's scheme I

Unfortunately, for $f \in H^\omega(X,Y)$, the pair (X,Y) may not have the $(c,1)$-extension property. The first approach of Tsar'kov [30] (that covers the case of the uniformly continuous mappings of the unit ball of L_p or l_p into L_q or l_q for $p \in \{1,\infty\}$ and $q=\infty$) is based on Lemma 3.1 suggesting that Y (or $A \subset Y$) is a subset of $l_\infty(Y)$ (or $l_\infty(A)$) and the pair $(X, l_\infty(Y))$ (or $(X, l_\infty(A))$) always has the $(1,1)$-extension property. We use the implementation of this idea quantified in the following way.

Theorem 10.1. *For $p \in [2,\infty)$, let X be a bounded metric space, and let Y be a quasi-Banach space that is isomorphic to a (p,h_c)-uniformly convex Banach space Z with $d_{BM}(Y,Z) < d$ and $f \in H^\omega(X,Y)$ for some $\omega \in \Omega_S$. Then, for every $\varepsilon > 0$, there exists $f_\varepsilon \in H^{1/p}(X,Y)$ satisfying $f_\varepsilon : \ X \to \overline{\mathrm{co}}\,(f(X))$,*

$$\|f - f_\varepsilon | C(X,Y)\| \le d\,(24p\omega(\varepsilon))^{1/p} \left(\sup_{\mu \in (0,1/2]} (1-\mu) h_c(\mu) \right)^{-1/p} \left(\omega\,(r(X)) + \omega\,(d(X))\right)^{1/p'}$$

and

$$\|f_\varepsilon | H^{1/p}(X,Y)\| \le d\,(16p\omega(\varepsilon)/\varepsilon)^{1/p} \left(\sup_{\mu \in (0,1/2]} (1-\mu) h_c(\mu) \right)^{-1/p} \left(\omega\,(r(X)) + \omega\,(d(X))\right)^{1/p'}.$$

Moreover, if Y is a (p,h_c)-uniformly convex Banach space itself, one should take $d=1$ in these estimates.

Proof. Let $T : Y \longrightarrow Z$ with $\|T|\mathcal{L}(Y,Z)\|\,\|T^{-1}|\mathcal{L}(Z,Y)\| < d$. For $\bar f = T \circ f$ assume that $A = \overline{\mathrm{co}}\left(\bar f(X)\right)$ and, therefore, $\bar f : \ X \to A$. According to Lemma 9.1, a), $l_\infty(A)$ contains an isometric copy $\tilde A$ of A, and we can interpret $\bar f$ as an $H^{\overline{\omega}}(X, l_\infty(A))$-mapping with $\overline{\omega} = \|T|\mathcal{L}(Y,Z)\|\omega$. Due to Lemmas 3.1, b) and 10.1, for every $\varepsilon > 0$, there exists $\tilde f_\varepsilon$ satisfying

$$\|\bar f - \tilde f_\varepsilon | C(X, l_\infty(A))\| \le 3\overline{\omega}(\varepsilon) \quad \text{and} \quad \|\tilde f_\varepsilon | H^1(X, l_\infty(A))\| \le 2\overline{\omega}(\varepsilon)/\varepsilon.$$

To finish the proof with the aid of Corollary 3.1, a), it is left to choose $f_\varepsilon = T^{-1} \circ \phi \circ \tilde f_\varepsilon$, where ϕ is the retraction provided by Theorem 9.1, and note that

$$r\left(\bar f(X), \bar f(X)\right) \le \overline{\omega}\,(r(X)) \ \text{ and } \ d\left(\bar f(X)\right) \le \overline{\omega}\,(d(X)).$$

□

10.3 Adaptations of Tsar'kov's scheme II

To establish the best possible smoothness exponents of the approximating Hölder-Lipschitz mappings for the uniformly continuous mappings of the unit ball of L_p or l_p into L_q or l_q for $p,q \in (1,\infty)$, I.G. Tsar'kov [42, 43, 44] (see also §2.1 in [17]) studied the Hölder-Lipschitz regularity of the Mazur maps and used them to reduce the problem of approximating by Hölder mappings from L_p into L_q to the problem of approximating by Lipschitz mappings. To solve the latter, he followed the Bernstein-Jackson principle (Lemma 10.1) utilizing the $(1,1)$-extension property of the pairs of Hilbert spaces (L_2 or l_2), that is M.D. Kirszbraun's extension theorem [29] (see also [28] for generalizations), instead of Frechet's extension for the pairs (X, l_∞) that he used earlier. A. Naor found a way how to demonstrate the sharpness of the smoothness exponent in the case $q = 1$ in Tsar'kov's result by considering the limit $q \to 1$.

In this section we develop an adaptation of Tsar'kov's approach to the setting of various pairs of function, IG and noncommutative spaces under consideration by means of studying generalised Mazur mappings (simple Mazur ascent and complex Mazur descent), using (the proof of) Corollary 10.1 and Theorems 5.1 and 9.5 instead of Kirszbraun's theorem and Lemma 9.1.

To formulate our three key approximation theorems in this section in a concise manner, we introduce the auxiliary functions

$$\xi(\alpha, W) := (2/\alpha - 1)^{N_{\min}(W)} \text{ and } \eta(\alpha) = 1 + 2^{1-\alpha} \text{ for } \alpha \in (0,1]$$

and $W \in IG$ (see Definition 2.13).

Remark 10.1. *a*) In applications of the following theorems, we can consider proper compositions with translations and limiting arguments to have $f(0) = 0$ and substitute $r(A,0)$ with $r(A,X)$, when X is a quasi-Banach space.

b) We are using Theorem 9.4, *b*) to deduce the Markov cotype of Y from the $(2, h_c)$-uniform convexity of Z_0 in Part *c*), keeping in mind that Z_0/Z_1 inherits the $(2, h_c)$-uniform convexity of Z_0.

Theorem 10.2. *Let X be a quasi-Banach space isomorphic to some $X_0 \in IG_0$ with $d_{BM}(X, X_0) < d_0$, a bounded $A \subset X$, $\omega \in \Omega_S$, $\alpha_0 = \min(p_{\min}(X_0), 2)/2$ and*

$$d = 3\sqrt{p_{\max}(X_0)/\alpha_0 - 1}C_{MC}.$$

Assume also that a quasi-Banach space Z is isomorphic to some Banach space Z_0 possessing the Markov cotype 2 with a constant C_{MC} and $d_{BM}(Z, Z_0) < d_1$. Then, for every $\varepsilon > 0$ and $f \in H^\omega(A, Y)$, there exists $f_\varepsilon \in H^\beta(A, Y)$ satisfying

$$\|f - f_\varepsilon|C(A,Y)\| \le g_\omega(\varepsilon) \text{ and } \|f_\varepsilon|H^\beta(A,Y)\| \le h_\omega(\varepsilon)$$

for the following combinations of β, g_ω and h_ω in the following settings.

a) *If Y is a subspace of Z, one can choose $f_\varepsilon : A \to \overline{\mathrm{co}}\,(f(A))$, $\beta = \alpha_0 r_s/r_c$,*

$$g_\omega(\varepsilon) = (1+2d)d_1\omega(\varepsilon)\left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{\frac{1}{r_c}}\left(1 + c_s c_c^{-\frac{r_s}{r_c}}\left(2 + \frac{\omega(d(A))}{2d\omega(\varepsilon)}\right)^{r_s}\right)^{\frac{1}{r_s}-\frac{1}{r_c}}$$

and

$$h_\omega(\varepsilon) = 2dd_1\omega(\varepsilon)\left(d_0\eta(\alpha_0)\xi(\alpha_0, X_0)r(A,0)^{1-\alpha_0}\varepsilon^{-1}\right)^{\frac{r_s}{r_c}}\left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{\frac{1}{r_c}} \times$$
$$\times\left(1 + c_s c_c^{-\frac{r_s}{r_c}}\left(2 + \frac{\omega(d(A))}{2d\omega(\varepsilon)}\right)^{r_s}\right)^{\frac{1}{r_s}-\frac{1}{r_c}}$$

if Z_0 is (r_c,h_c)-uniformly convex and (r_s,h_s)-uniformly smooth with

$$c_c = \sup_{\mu\in(0,1/2]} (1-\mu)h_c(\mu) \text{ and } c_s = \inf_{\mu\in(0,1/2]} (1-\mu)^{1-r_s} h_s(\mu).$$

b) One can take $\beta=\alpha_0$, $g_\omega(\varepsilon)=C(1+2d)d_1\omega(\varepsilon)$ and

$$h_\omega(\varepsilon) = 2Cd_0dd_1\eta(\alpha_0)\xi(\alpha_0,X_0)r(A,0)^{1-\alpha_0}\omega(\varepsilon)/\varepsilon$$

either if Y is a subspace of Z and $C=1$, or if Y is a factor space Z/Z_1 with $\|I-P_{Z_1}|\mathcal{L}(Z)\|\le C$ for a projector $P_{Z_1}:\ Z\to Z_1$. Moreover, if Z_0 is Hilbert, one has $C=1$ and $f_\varepsilon:\ A\to\overline{\mathrm{co}}\,(f(A))$.

c) If Y is a factor space Z/Z_1, one can choose $f_\varepsilon:\ A\to\overline{\mathrm{co}}\,(f(A))$, $\beta=\alpha_0 r_s/r_c$,

$$g_\omega(\varepsilon) = C_{HI}(1+2d)d_1\omega(\varepsilon)^{\frac{r_s}{r_c}}\omega\left(r(A,0)\right)^{1-\frac{r_s}{r_c}}\left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{\frac{1}{r_c}}\times$$
$$\times\left(1+c_s c_c^{-\frac{r_s}{r_c}}\left(2+\frac{\omega(d(A))^{r_s/r_c}}{2d\omega(\varepsilon)^{r_s/r_c}}\right)^{r_s}\right)^{\frac{1}{r_s}-\frac{1}{r_c}}$$

and

$$h_\omega(\varepsilon) = 2C_{HI}dd_1\omega(\varepsilon)^{\frac{r_s}{r_c}}\omega\left(r(A,0)\right)^{1-\frac{r_s}{r_c}}\left(d_0\eta(\alpha_0)\xi(\alpha_0,X_0)r(A,0)^{1-\alpha_0}\varepsilon^{-1}\right)^{\frac{r_s}{r_c}}\times$$
$$\times\left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{\frac{1}{r_c}}\left(1+c_s c_c^{-\frac{r_s}{r_c}}\left(2+\frac{\omega(d(A))^{r_s/r_c}}{2d\omega(\varepsilon)^{r_s/r_c}}\right)^{r_s}\right)^{\frac{1}{r_s}-\frac{1}{r_c}}$$

if Z_0 is (r_c,h_c)-uniformly convex and (r_s,h_s)-uniformly smooth with c_c and c_s as in Part a) and

$$C_{HI} = \left(2^{1-r_s/r_c}+\left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{1/r_c}\left(1+c_s c_c^{-r_s/r_c}2^{r_s}\right)^{1/r_s-1/r_c}\right).$$

Proof. It is very similar to the proof of Corollary 10.1. In particular, the last step in the proofs of *a*) and *c*) is the application of Theorem 9.3 because the corresponding metric projection is the last mapping in the composition of f_ε. Thus, we describe the differences using notations from that proof. The d_0-Lipschitz homeomorphism becomes linear. We also apply Lemma 10.1 to

$$\bar f = T_1\circ f\circ T_0^{-1}\circ m_{\beta_0}:\ X_1\longrightarrow T_1Y\subset Z_0, \tag{1}$$

where $X_1\in IG_0$ has the same tree $T(X_0)=T(X_1)$, $\min(p_{\min}(X_0),2)p_{X_1}=2p_{X_0}$ and β_0 is the constant function $1/\alpha_0$. The expression for d is provided by Theorems 9.4 and 9.5 with the aid of Theorem 2.2. Eventually, one takes

$$f_\varepsilon = T_1^{-1}\circ P_D\circ\bar f_\varepsilon\circ m_{\alpha_0,a}\circ T_0:\ A\to\overline{\mathrm{co}}\,(f(A)) \text{ for } D=\overline{\mathrm{co}}\,(T_1 f(A)) \tag{2}$$

in the case of Part *a*) and the case of Hilbert $Z_0\supset Y$ in *b*), and

$$f_\varepsilon = P\circ T_1^{-1}\circ\bar f_\varepsilon\circ m_{\alpha_0,a}\circ T_0:\ A\to Y, \tag{3}$$

when Y is a C-complemented subspace in b). In the case of the rest of b), we substitute f in (1) with $B \circ f$, where $B:\ Y = Z/Z_1 \to \operatorname{Ker}(P_{Z_1}) \subset Z$ is the linear right inverse of the quotient map $Q_{Z_1}:\ Z \to Y$ corresponding to P_{Z_1}.

As far as Part c) is concerned, the isomorphism T_1 induces the isomorphism $\tilde{T}_1:\ Z/Z_1 \longrightarrow Z_0/\tilde{T}_1 Z_1$ with

$$\|T_1|\mathcal{L}(Z,Z_0)\| = \left\|\tilde{T}_1|\mathcal{L}(Z/Z_1, Z_0/\tilde{T}_1 Z_1)\right\|$$

and

$$\left\|T_1^{-1}|\mathcal{L}(Z_0,Z)\right\| = \left\|\tilde{T}_1^{-1}|\mathcal{L}(Z_0/\tilde{T}_1 Z_1, Z/Z_1)\right\|.$$

Thus, instead of (1), we use

$$\bar{f} = B \circ \tilde{T}_1 \circ f \circ T_0^{-1} \circ m_{\beta_0}:\ X_1 \longrightarrow Z_0, \tag{4}$$

where $B:\ T_1 Y \longrightarrow Z_0$ is the homogeneous inverse of the quotient map $Q_0:\ Z_0 \to T_1 Y$ given in Corollary 3.2. Then we employ Corollary 3.2 to estimate the modulus of continuity of $\bar{f}$ and proceed exactly as above. The counterpart of (2) is

$$f_\varepsilon = \tilde{T}_1^{-1} \circ P_D \circ Q_0 \circ \bar{f}_\varepsilon \circ m_{\alpha_0,a} \circ T_0:\ A \to \overline{\mathrm{co}}\,(f(A)) \ \text{ for } D = \overline{\mathrm{co}}(\tilde{T}_1 A). \tag{5}$$

Note that the quotient space $\tilde{T}_1 Y$ inherits the convexity and smoothness properties of Z_0 and, thus, we use Theorem 9.3 exactly as in a).

In addition to the multiple usage of Lemma 3.1, a), we also employ the regularity estimates for our homogeneous Mazur mappings established in Theorem 5.1. Part c) of Remark 5.1 provides the algebraic identity $I = m_{\beta_0} m_{\alpha_0,a}$. □

Let us recall that functions ω_c and ω_s are defined in Section 2.6.

Theorem 10.3. *Let X be a metric space that is d_0-Lipschitz homeomorphic to a metric space X_0 possessing the Markov type 2 with a constant C_{MT}, a bounded $A \subset X$ and $\omega \in \Omega_S$. Assume also that a quasi-Banach space Z is isomorphic to some $Z_0 \in IG_0$ with $d_{BM}(Z,Z_0) < d_1$, $\alpha_1 = 2/\max(2, p_{\max}(Z_0))$, $d = 6C_{MT}\,(\alpha_1 p_{\min}(Z_0) - 1)^{-1/2}$, $r_s = \min(p_{\min}(Z_0), 2)$ and $r_c = \max(p_{\max}(Z_0), 2) = 2/\alpha_1$. Then, for every $\varepsilon > 0$ and $f \in H^\omega(A,Y)$, there exists $f_\varepsilon \in H^\beta(A,Y)$ satisfying*

$$\|f - f_\varepsilon|C(A,Y)\| \le g_\omega(\varepsilon) \ \text{ and } \ \|f_\varepsilon|H^\beta(A,Y)\| \le h_\omega(\varepsilon)$$

for the following combinations of β, g_ω and h_ω in the following settings.

a) If Y is a subspace of Z, one can choose $f_\varepsilon:\ A \to \overline{\mathrm{co}}\,(f(A))$, $\beta = \alpha_1 r_s/r_c$,

$$g_\omega(\varepsilon) = (1+2d)d_1\eta(\alpha_1)\xi(\alpha_1,Z_0)^2\omega(\varepsilon)\left(1 + \frac{\omega\,(r(A,0))}{2d\omega(\varepsilon)}\right)^{1-\alpha_1}\left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{\frac{1}{r_c}} \times$$
$$\times\left(1 + c_s c_c^{-\frac{r_s}{r_c}}\left(2 + \frac{\omega\,(r(A,0))}{2d\omega(\varepsilon)}\right)^{\alpha_1 r_s}\right)^{\frac{1}{r_s} - \frac{1}{r_c}}$$

and

$$h_\omega(\varepsilon) = 2dd_1\eta(\alpha_1)\xi(\alpha_1, Z_0)^2\omega(\varepsilon)(d_0/\varepsilon)^{\frac{\alpha_1 r_s}{r_c}}\left(1+\frac{\omega\left(r(A,0)\right)}{2d\omega(\varepsilon)}\right)^{1-\alpha_1}\left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{\frac{1}{r_c}} \times\left(1+c_s c_c^{-\frac{r_s}{r_c}}\left(2+\frac{\omega\left(r(A,0)\right)}{2d\omega(\varepsilon)}\right)^{\alpha_1 r_s}\right)^{\frac{1}{r_s}-\frac{1}{r_c}},$$

where $c_s = \omega_s(0, p_{\max}(Z_0), r_s)$, $c_c = \omega_c(0, p_{\min}(Z_0), r_c)$.

b) One can take

$$\beta = \alpha_1, \quad g_\omega(\varepsilon) = C(1+2d)d_1\eta(\alpha_1)\xi(\alpha_1, Z_0)^2\omega(\varepsilon)\left(1+\omega\left(r(A,0)\right)/2d\omega(\varepsilon)\right)^{1-\alpha_1}$$

and

$$h_\omega(\varepsilon) = 2Cdd_0d_1\eta(\alpha_1)\xi(\alpha_1, Z_0)^2\omega(\varepsilon)\varepsilon^{-1}\left(1+\omega\left(r(A,0)\right)/2d\omega(\varepsilon)\right)^{1-\alpha_1}$$

either if Z *is* C*-complemented in* Z*, or if* $Y = Z/Z_1$ *with* $\|I - P_{Z_1}|\mathcal{L}(Z)\| \le C$ *for a projector* $P_{Z_1}:\ Z \to Z_1$*. Moreover,* $C = 1$ *if* $Y = Z$ *or* Z_0 *is Hilbert. In the latter case, one even has* $f_\varepsilon:\ A \to \overline{\mathrm{co}}\,(f(A))$*.*

c) If Y *is a factor space* Z/Z_1*, one can choose* $f_\varepsilon:\ A \to \overline{\mathrm{co}}\,(f(A))$*,* $\beta = \alpha_1 r_s/r_c$*,*

$$g_\omega(\varepsilon) = C_{HI}(1+2d)d_1\eta(\alpha_1)\xi(\alpha_1, Z_0)^2\omega(\varepsilon)^{\frac{r_s}{r_c}} \times\omega\left(r(A,0)\right)^{1-\frac{r_s}{r_c}}\left(1+\frac{\omega\left(r(A,0)\right)^{r_s/r_c}}{2d\omega(\varepsilon)^{r_s/r_c}}\right)^{1-\alpha_1} \times\left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{\frac{1}{r_c}}\left(1+c_s c_c^{-\frac{r_s}{r_c}}\left(2+\frac{\omega\left(r(A,0)\right)^{r_s/r_c}}{2d\omega(\varepsilon)^{r_s/r_c}}\right)^{\alpha_1 r_s}\right)^{\frac{1}{r_s}-\frac{1}{r_c}}$$

and

$$h_\omega(\varepsilon) = 2C_{HI}dd_1\eta(\alpha_1)\xi(\alpha_1, Z_0)^2\omega(\varepsilon)^{\frac{r_s}{r_c}} \times\omega\left(r(A,0)\right)^{1-\frac{r_s}{r_c}}(d_0/\varepsilon)^{\frac{\alpha_1 r_s}{r_c}}\left(1+\frac{\omega\left(r(A,0)\right)^{r_s/r_c}}{2d\omega(\varepsilon)^{r_s/r_c}}\right)^{1-\alpha_1} \times\left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{\frac{1}{r_c}}\left(1+c_s c_c^{-\frac{r_s}{r_c}}\left(2+\frac{\omega\left(r(A,0)\right)^{r_s/r_c}}{2d\omega(\varepsilon)^{r_s/r_c}}\right)^{\alpha_1 r_s}\right)^{\frac{1}{r_s}-\frac{1}{r_c}},$$

where c_s *and* c_c *are as in a) and*

$$C_{HI} = 2^{1-r_s/r_c} + \left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{1/r_c}\left(1+c_s c_c^{-r_s/r_c}2^{r_s}\right)^{1/r_s-1/r_c}.$$

Proof. It is very similar to the proof of Corollary 10.1. Thus, we describe the distinctions using notations from that proof. We apply Lemma 10.1 to

$$\bar{f} = m_{p_{Z_0}/\min\left(p_{Z_0},2\right)} \circ T_1 \circ f \circ T_0^{-1}:\ X_0 \longrightarrow Z_1, \tag{1}$$

where $Z_1 \in IG_0$ has the same tree $T(Z_0) = T(Z_1)$ and $p_{Z_1} = \min(p_{Z_0}, 2)$. The expression for d is provided by Theorems 9.4 and 9.5 with the aid of Theorem 2.2. Eventually, one takes

$$f_\varepsilon = T_1^{-1} \circ P_D \circ m_{\min(p_{Z_0},2)/\alpha_1 p_{Z_0}} \circ m_{\alpha_1,a} \circ \bar{f}_\varepsilon \circ T_0 : \ A \to \overline{\mathrm{co}}\,(f(A)) \ \text{ for } D = \overline{\mathrm{co}}\,(T_1 f(A)) \tag{2}$$

in the case of Part a) and the case of Hilbert Z_0 in b), and

$$f_\varepsilon = P \circ T_1^{-1} \circ m_{\min(p_{Z_0},2)/\alpha_1 p_{Z_0}} \circ m_{\alpha_1,a} \circ \bar{f}_\varepsilon \circ T_0 : \ A \to Y, \tag{3}$$

when Y is a C-complemented subspace in b). In the case of the rest of b), we substitute f in (1) with $B \circ f$, where $B : \ Y = Z/Z_1 \to \mathrm{Ker}(P_{Z_1}) \subset Z$ is the linear right inverse of the quotient map $Q_{Z_1} : \ Z \to Y$ corresponding to P_{Z_1}.

As far as Part c) is concerned, there exists the isomorphism $\tilde{T}_1 : \ Z/Z_1 \longrightarrow Z_0/\tilde{T}_1 Z_1$ as in the preceding proof. Thus, instead of (1), we use

$$\bar{f} = m_{p_{Z_0}/\min(p_{Z_0},2)} \circ B \circ \tilde{T}_1 \circ f \circ T_0^{-1} : \ X_0 \longrightarrow Z_1, \tag{4}$$

where $B : \ T_1 Y \longrightarrow Z_0$ is the homogeneous inverse of the quotient map $Q_0 : \ Z_0 \to T_1 Y$ given in Corollary 3.2. Then we employ Corollary 3.2 to estimate the modulus of continuity of $\bar{f}$ and proceed exactly as above. The counterpart of (2) is

$$f_\varepsilon = \tilde{T}_1^{-1} \circ P_D \circ Q_0 \circ m_{\min(p_{Z_0},2)/\alpha_1 p_{Z_0}} \circ m_{\alpha_1,a} \circ \bar{f}_\varepsilon \circ T_0 : \ A \to \overline{\mathrm{co}}\,(f(A)) \ \text{ for } D = \overline{\mathrm{co}}(\tilde{T}_1 A). \tag{5}$$

Note that the quotient space $\tilde{T}_1 Y$ inherits the convexity and smoothness properties of Z_0 and, thus, we use Theorem 9.3 exactly as in a).

In addition to the multiple usage of Lemma 3.1, a), we also employ the regularity estimates for our homogeneous Mazur mappings established in Theorem 5.1. Remark 5.1 provides the algebraic identity

$$I = m_{\min(p_{Z_0},2)/\alpha_1 p_{Z_0}} m_{\alpha_1,a} m_{p_{Z_0}/\min(p_{Z_0},2)}.$$

□

Theorem 10.4. *Let X be a quasi-Banach space with $d_{BM}(X, X_0) < d_0$ for some $X_0 \in IG_0$, a bounded $A \subset X$, $\omega \in \Omega_S$ and $\alpha_0 = \min(p_{\min}(X_0), 2)/2$. Assume also that a quasi-Banach space Z is isomorphic to some $Z_0 \in IG_0$ with $d_{BM}(Z, Z_0) < d_1$, $\alpha_1 = 2/\max(2, p_{\max}(Z_0))$, $r_s = \min(p_{\min}(Z_0), 2)$, $r_c = \max(p_{\max}(Z_0), 2)$ and*

$$d = 6 \left(\frac{p_{\max}(X_0)/\alpha_0 - 1}{\alpha_1 p_{\min}(Z_0) - 1} \right)^{1/2}.$$

Then, for every $\varepsilon > 0$ and $f \in H^\omega(A, Y)$, there exists $f_\varepsilon \in H^\beta(A, Y)$ satisfying

$$\|f - f_\varepsilon | C(A, Y)\| \le g_\omega(\varepsilon) \text{ and } \|f_\varepsilon | H^\beta(A, Y)\| \le h_\omega(\varepsilon)$$

for the following combinations of β, g_ω and h_ω in the following settings.

a) If Y is a subspace of Z, one can choose $f_\varepsilon : A \to \overline{\mathrm{co}}\,(f(A))$, $\beta = \alpha_0\alpha_1 r_s/r_c$,

$$g_\omega(\varepsilon) = (1+2d)d_1\eta(\alpha_1)\xi(\alpha_1, Z_0)^2\omega(\varepsilon)\left(1+\frac{\omega\left(r(A,0)\right)}{2d\omega(\varepsilon)}\right)^{1-\alpha_1}\left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{\frac{1}{r_c}} \times\left(1+c_s c_c^{-\frac{r_s}{r_c}}\left(2+\frac{\omega\left(r(A,0)\right)}{2d\omega(\varepsilon)}\right)^{\alpha_1 r_s}\right)^{\frac{1}{r_s}-\frac{1}{r_c}}$$

and

$$h_\omega(\varepsilon) = 2dd_1\eta(\alpha_1)\xi(\alpha_1, Z_0)^2\omega(\varepsilon)\left(d_0\eta(\alpha_0)\xi(\alpha_0, X_0)/\varepsilon\right)^{\frac{\alpha_1 r_s}{r_c}}\left(1+\frac{\omega\left(r(A,0)\right)}{2d\omega(\varepsilon)}\right)^{1-\alpha_1} \times\left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{\frac{1}{r_c}}\left(1+c_s c_c^{-\frac{r_s}{r_c}}\left(2+\frac{\omega\left(r(A,0)\right)}{2d\omega(\varepsilon)}\right)^{\alpha_1 r_s}\right)^{\frac{1}{r_s}-\frac{1}{r_c}},$$

where $c_s = \omega_s(0, p_{\max}(Z_0), r_s)$, $c_c = \omega_c(0, p_{\min}(Z_0), r_c)$.

b) One can take $\beta = \alpha_0\alpha_1$,

$$g_\omega(\varepsilon) = (1+2d)d_1\eta(\alpha_1)\xi(\alpha_1, Z_0)^2\omega(\varepsilon)\left(1+\frac{\omega\left(r(A,0)\right)}{2d\omega(\varepsilon)}\right)^{1-\alpha_1}$$

and

$$h_\omega(\varepsilon) = 2dd_1\eta(\alpha_1)\xi(\alpha_1, Z_0)^2\omega(\varepsilon)\left(d_0\eta(\alpha_0)\xi(\alpha_0, X_0)/\varepsilon\right)^{\alpha_1}\left(1+\frac{\omega\left(r(A,0)\right)}{2d\omega(\varepsilon)}\right)^{1-\alpha_1}$$

if Z is C-complemented in Z, or if Y is a factor space Z/Z_1 with $\|I - P_{Z_1}|\mathcal{L}(Z)\| \le C$ for a projector $P_{Z_1} : Z \to Z_1$. Moreover, $C = 1$ if $Y = Z$ or Z_0 is Hilbert. In the latter case, one even has $f_\varepsilon : A \to \overline{\mathrm{co}}\,(f(A))$.

If Y is a factor space Z/Z_1,

c) one can choose $f_\varepsilon : A \to \overline{\mathrm{co}}\,(f(A))$, $\beta = \alpha_0\alpha_1 r_s/r_c$,

$$g_\omega(\varepsilon) = C_{HI}(1+2d)d_1\eta(\alpha_1)\xi(\alpha_1, Z_0)^2\omega(\varepsilon)\left(r(A,0)\right)^{1-\frac{r_s}{r_c}} \times \times\left(1+\frac{\omega\left(r(A,0)\right)}{2d\omega(\varepsilon)}\right)^{1-\alpha_1}\left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{\frac{1}{r_c}}\left(1+c_s c_c^{-\frac{r_s}{r_c}}\left(2+\frac{\omega\left(r(A,0)\right)}{2d\omega(\varepsilon)}\right)^{\alpha_1 r_s}\right)^{\frac{1}{r_s}-\frac{1}{r_c}}$$

and

$$h_\omega(\varepsilon) = 2C_{HI}dd_1\eta(\alpha_1)\xi(\alpha_1, Z_0)^2\omega(\varepsilon)\left(d_0\eta(\alpha_0)\xi(\alpha_0, X_0)/\varepsilon\right)^{\frac{\alpha_1 r_s}{r_c}}\left(r(A,0)\right)^{1-\frac{r_s}{r_c}} \times \times\left(1+\frac{\omega\left(r(A,0)\right)}{2d\omega(\varepsilon)}\right)^{1-\alpha_1}\left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{\frac{1}{r_c}}\left(1+c_s c_c^{-\frac{r_s}{r_c}}\left(2+\frac{\omega\left(r(A,0)\right)}{2d\omega(\varepsilon)}\right)^{\alpha_1 r_s}\right)^{\frac{1}{r_s}-\frac{1}{r_c}},$$

where c_s and c_c are as in a) and

$$C_{HI} = 2^{1-r_s/r_c} + \left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{1/r_c}\left(1+c_s c_c^{-r_s/r_c}2^{r_s}\right)^{1/r_s-1/r_c}.$$

Proof. As above, it is very similar to the proof of Theorem 10.2. In fact, it is close to a composition of parts of the proofs of Theorems 10.2 and 10.3. Thus, we describe the differences using notations from that proof. The d_0-Lipschitz homeomorphism becomes linear. We apply Lemma 10.1 to

$$\bar{f} = m_{p_{Z_0}/\min(p_{Z_0},2)} \circ T_1 \circ f \circ T_0^{-1} \circ m_{\beta_0} : \ X_1 \longrightarrow Z_1, \tag{1}$$

where $X_1 \in IG_0$ has the same tree $T(X_0) = T(X_1)$, $\min(p_{\min}(X_0), 2)p_{X_1} = 2p_{X_0}$ and β_0 is the constant function $1/\alpha_0$, and $Z_1 \in IG_0$ has the same tree $T(Z_0) = T(Z_1)$ and $p_{Z_1} = \min(p_{Z_0}, 2)$. The expression for d is provided by Theorems 9.4 and 9.5 with the aid of Theorem 2.2. Eventually, one takes

$$\begin{aligned} &f_\varepsilon = T_1^{-1} \circ P_D \circ m_{\min(p_{Z_0},2)/\alpha_1 p_{Z_0}} \circ m_{\alpha_1,a} \circ \bar{f}_\varepsilon \circ m_{\alpha_0,a} \circ T_0 : \ A \to \overline{\text{co}}\,(f(A)) \\ &\text{for } D = \overline{\text{co}}\,(T_1 f(A)) \end{aligned} \tag{2}$$

in the case of Part *a*) and the case of Hilbert Z_0 in *b*), and

$$f_\varepsilon = P \circ T_1^{-1} \circ m_{\min(p_{Z_0},2)/\alpha_1 p_{Z_0}} \circ m_{\alpha_1,a} \circ \bar{f}_\varepsilon \circ m_{\alpha_0,a} \circ T_0 : \ A \to Y, \tag{3}$$

when Y is a C-complemented subspace in *b*). In the case of the rest of *b*), we substitute f in (1) with $B \circ f$, where $B : \ Y = Z/Z_1 \to \text{Ker}(P_{Z_1}) \subset Z$ is the linear right inverse of the quotient map $Q_{Z_1} : \ Z \to Y$ corresponding to P_{Z_1}.

As far as Part *c*) is concerned, there exists the isomorphism $\tilde{T}_1 : \ Z/Z_1 \longrightarrow Z_0/\tilde{T}_1 Z_1$ with the same norms of itself and the inverse. Thus, instead of (1), we use

$$\bar{f} = m_{p_{Z_0}/\min(p_{Z_0},2)} \circ B \circ \tilde{T}_1 \circ f \circ T_0^{-1} \circ m_{\beta_0} : \ X_1 \longrightarrow Z_1, \tag{4}$$

where $B : \ T_1 Y \longrightarrow Z_0$ is the homogeneous inverse of the quotient map $Q_0 : \ Z_0 \to T_1 Y$ given in Corollary 3.2. Then we employ Corollary 3.2 to estimate the modulus of continuity of $\bar{f}$ and proceed exactly as above. The counterpart of (2) is

$$\begin{aligned} &f_\varepsilon = \tilde{T}_1^{-1} \circ P_D \circ Q_0 \circ m_{\min(p_{Z_0},2)/\alpha_1 p_{Z_0}} \circ m_{\alpha_1,a} \circ \bar{f}_\varepsilon \circ m_{\alpha_0,a} \circ T_0 : \ A \to \overline{\text{co}}\,(f(A)) \\ &\text{for } D = \overline{\text{co}}(\tilde{T}_1 A). \end{aligned} \tag{5}$$

As above, we use Theorem 9.3 exactly as in *a*).

In addition to the multiple usage of Lemma 3.1, *a*), we also employ the regularity estimates for our homogeneous Mazur mappings established in Theorem 5.1 and rely on the algebraic identities obtained in the proofs of Theorems 10.2 and 10.3 (following from Remark 5.1). □

The following lemma shows how to handle the situation when the domain space is isomorphic to a quotient of a "good" space. It follows immediately from Corollaries 3.2 and 3.1, *a*).

Lemma 10.3. *Let X be a quasi-Banach space isomorphic to a Banach space X_0 that is (r_c, h_c)-uniformly convex and (r_s, h_s)-uniformly smooth with $d_{BM}(X, X_0) < d_0$,*

$$c_c = \sup_{\mu \in (0,1/2]} (1-\mu) h_c(\mu), \ c_s = \inf_{\mu \in (0,1/2]} (1-\mu)^{1-r_s} h_s(\mu),$$

and let $Z = X/X_1$ be a factor space of X, and a bounded $A \subset Z$. For a quasi-Banach space Y, assume that, for some $\beta \in (0, 1]$ and an arbitrary bounded $B \subset X$ and $\omega \in \Omega_S$, one has the approximation property: for every $\varepsilon > 0$ and $f \in H^\omega(B, Y)$, there exists $f_\varepsilon \in H^\beta(B, Y)$ satisfying

$$\|f - f_\varepsilon|C(B, Y)\| \le g_\omega(\varepsilon) \text{ and } \|f_\varepsilon|H^\beta(B, Y)\| \le h_\omega(\varepsilon).$$

Then, given $\omega \in \Omega_S$, for every $\varepsilon > 0$ and $f \in H^\omega(A, Y)$, there exists $f_\varepsilon \in H^{\beta r_s/r_c}(A, Y)$ satisfying

$$\|f - f_\varepsilon|C(A, Y)\| \le g_\omega(\varepsilon) \text{ and } \|f_\varepsilon|H^{\beta r_s/r_c}(A, Y)\| \le C^\beta_{HI} r(A, 0)^{\beta(1-r_s/r_c)} h_\omega(\varepsilon),$$

where

$$C_{HI} = 2^{1-r_s/r_c} + \left(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\right)^{1/r_c} \left(1 + c_s c_c^{-r_s/r_c} 2^{r_s}\right)^{1/r_s - 1/r_c}.$$

Remark 10.2. Let us note that, depending on the parameters of the spaces, Theorem 10.1 can provide better Hölder-Lipschitz regularity exponents than Theorems 10.2 – 10.4.

In the next corollary we see that Theorem 5.2 permits to substitute the IG_0 spaces with the IG_{0+} spaces in Theorems 10.2 – 10.4.

Corollary 10.2. *a) With occasionally different ξ and η, the conclusions of Theorems* 10.2 – 10.4 *remain true if $X_0 \in IG_+$ in Theorems* 10.2 *and* 10.4 *with $\alpha_0 = \min(p_{\min}(X_0), 2)/2$, and $Z_0 \in IG_+$ in Theorems* 10.3 *and* 10.4 *with $1/\alpha_1 = \max(p_{\max}(Z_0), 2)/2$, $r_s = \min(p_{\min}(Z_0), 2)$ and $r_c = \max(p_{\max}(Z_0), 2)$.*

b) With occasionally different ξ and η, the conclusions of Theorems 10.2 *and* 10.3 *remain true if $X_0 = L_{p_0}(\mathcal{M}_0) \in \Gamma_5$ in Theorems* 10.2 *with $\alpha_0 = \min(p_0, 2)/2$ and/or $Z_0 = L_{p_1}(\mathcal{M}_1) \in \Gamma_5$ in Theorems* 10.3 *with $1/\alpha_1 = \max(p_1, 2)/2$, $r_s = \min(p_1, 2)$ and $r_c = \max(p_1, 2)$.*

Proof. *a)* Lemmas 8.2 and 8.1 allow us to reduce the proof of *a)* to the setting $X_0 \in IG_{0+}$ and $Z_0 \in IG_{0+}$. The details of this reduction are the same as in the proof of Part *b)* below. According to Theorems 2.2 and 2.4 the exponents p, q of the best (p, h_c)-uniform convexity and (q, h_s)-uniform smoothness are computed in a similar way leading to the same outcomes from the applications of the results on metric projections and homogeneous inverses from Sections 3.2 and 9.2. The usage of the abstract Mazur ascent from Definition 5.3 in tandem with Theorem 5.2 completely substitutes the usage of the simple Mazur ascent from Definition 5.2 and Theorem 5.1, *b)*, while Part *a)* of Theorem 5.1 remains applicable. Indeed, the reason for employing the simple Mazur ascent in Theorems 10.2 – 10.4 is the reduction of the approximation problem for an arbitrary pair to the pair of a Markov type 2 space and a Markov type 2 space that require to change all the parameters $p_{X_0}(i) < 2$ and all the parameters $p_{Z_0} > 2$ to the value 2 that is achieved by using the Mazur ascents with the parameters $\alpha_0 = \min(p_{\min}(X_0), 2)/2$ and $1/\alpha_1 = \max(p_{\max}(Z_0), 2)/2$ correspondingly. Theorem 5.2 with $\beta \le \alpha$ permits to perform the same tasks with the same exponents.

In details, we only need to find the substitutions for the pair of mutually inverse operators

$$(m_{\alpha_0,a}, m_{\beta_0}) \text{ and } \left(m_{\min(p_{Z_0},2)/\alpha_1 p_{Z_0}} \circ m_{\alpha_1,a}, m_{p_{Z_0}/\min(p_{Z_0},2)}\right)$$

used in the proofs of Theorems 10.2 – 10.4, while we would like to maximize only the smoothness of the substitutes for $m_{\alpha_0,a} : X_0 \to X_1$ and $m_{\min(p_{Z_0},2)/\alpha_1 p_{Z_0}} \circ m_{\alpha_1,a} : Z_1 \to Z_0$, where $p_{\min}(X_1) \geq 2$ and $p_{\max}(Z_1) \leq 2$.

In the case of $m_{\alpha_0,a}$, we choose $P_I = \{i \in P_{nc}(X_0) : p_{X_0}(i) \geq 2\}$ and use the abstract Mazur ascent mapping provided by Theorem 5.2 with $\beta = \alpha_0$. Both this mapping and its inverse are α_0-Hölder according to the same theorem.

In the second case, we choose $P_I = \{i \in P_{nc}(Z_0) : p_{Z_0}(i) \leq 2\}$. Note that $p_{Z_1}(i) = 2$ for $i \in P_n c(Z_0) \setminus P_I$, and one needs to transform them to the values p_{Z_0} and, possibly, also to increase the value $2 = p_{Z_1}(i_0)$ for (possibly commutative spaces) at the vertex(es) i_0 with $p_{Z_0}(i_0) = p_{\max}(Z_0)$. This first step is achieved by the abstract Mazur ascent mapping from Theorem 7.5 with $\beta = \alpha_1$. Afterwards we use the same complex Mazur descent (note that it deals only with the “commutative” vertexes while the noncommutative are already “in place") to return the parameters that became too large back to their values for Z_0. In the opposite direction we increase all the noncommutative parameters between $p_{\max}(Z_0)$ and 2 to the values not less than 2 utilizing the degenerated ascent described in Remark 5.3. Then we reduce them all to the value $P_{\max}(Z_0)$ by the appropriate complex descent and, eventually, return back to Z_1 with the aid of the inverse of the first abstract Mazur ascent above provided by Theorem 5.2.

The proof of Part *b*) requires even less changes. If $X_0 = L_{p_0}(\mathcal{M}_0)$ and/or $0 = L_{p_1}(\mathcal{M}_1)$, one works, correspondingly, with the spaces $\overline{X}_0$ and/or $\overline{Z}_0$ provided by Theorem 8.6 instead of X_0 and/or Z_0. Substituting p_0 and p_1 with 2, we also invoke Theorem 8.6 in the same manner to obtain the Hilbert spaces $\overline{X}_2$ and $\overline{Z}_2$ and extend by continuity the union of the mappings $m_{r,2}$ and $m_{2,r}$ for $r \in \{r, 2\}$ provided by Theorem 4.5, as in the proof of Theorem 6.8, construct the homogeneous Hölder homeomorphisms between $\overline{X}_0$ and $\overline{X}_2$, and between $\overline{Z}_0$ and $\overline{Z}_2$. □

10.4 Pairs of abstract Banach lattices

To reveal the intimate relation between the approximation of uniformly continuous mappings and the Hölder classification of spheres, let us note that Theorem 6.2 implies the following theorem.

Theorem 10.5. *Let X and Y be Banach lattices with a weak unit that do not contain $l_\infty(I_n)$ uniformly (by n). Assume also that $f : D \to Y$ is uniformly continuous on a bounded $D \subset X$ with the modulus of continuity dominated by a nondecreasing subadditive $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to 0} \omega(t) = 0$. Then there exists $\gamma \in (0, 1]$, such that, for every $\varepsilon > 0$, there is $f_\varepsilon \in H^\gamma(X, Y)$ satisfying*

$$\|f(x) - f_\varepsilon(x)\|_Y \leq \varepsilon \text{ for } x \in D.$$

Moreover, one also has $\gamma = 1/4$ and $f_\varepsilon(X) \subset \overline{\text{co}}(f(D))$ if Y is a Hilbert space.

Proof. We essentially follow Tsar'kov's scheme II of the reduction of the case of a pair of Lebesgue spaces to the case of a pair of Hilbert spaces followed by the application of Kirszbraun's extension theorem [29]. The difference is in the usage of the Hölder homeomorphisms provided by Theorem 6.2 instead of our counterparts of the Mazur mapping and the composition with a metric projection in the special case of Hilbert Y.

Indeed, let $D_\delta \subset D$ be a maximal δ-separated subset of D, and let $\psi_0 : X \overset{(\alpha_0,\beta_0)}{\longleftrightarrow} H_0$ and $\psi_1 : Y \overset{(\alpha_1,\beta_1)}{\longleftrightarrow} H_1$ be the homogeneous Hölder homeomorphisms supplied by Theorem 6.2 (H_0 and H_1 are Hilbert). Assume also that $\phi = \psi_1 \circ f \circ \psi_0^{-1} : D_0 \to H_1$ and $D_{0\delta} = \psi_0(D_\delta)$, where $D_0 \subset H_0$. Let $\left\| \psi_0^{-1} \right| H^{\beta_0}(\psi_0(D), X)\| = C_0$ and $\|\psi_1 | H^{\alpha_1}(f(D), H_1)\| = C_1$. Therefore, the modulus $\omega(t,\phi)$ of continuity of ϕ on $\psi_0(D)$ satisfies

$$\omega(t,\phi) \le C_1 \left(\omega\left(C_0 t^\alpha\right)\right)^\beta = w(t).$$

Since w is subadditive, the restriction of ϕ to $D_{0\delta}$ is Lipschitz with $\|\phi | H^1(D_{0\delta}, H_1)\| \le 2w(\delta')/\delta'$, where $\delta' = (\delta/C_0)^{1/\beta_0}$, and, thus, is extended to $\phi_\delta : H_0 \to H_1$ satisfying $\|\phi_\delta | H^1(H_0, H_1)\| \le 2w(\delta')/\delta'$ with the aid of Kirszbraun's extension theorem. Now we choose $f_\delta = \psi_1^{-1} \circ \phi_\delta \circ \psi_0$ and use Corollary 3.1 to conclude that $f_\delta \in H^{\alpha_0\beta_1}(D, Y)$. Clearly, for every $x \in D$, there is $y \in D_\delta$ with $\|x - y\|_X < \delta$, and, therefore, one has

$$\begin{aligned} \|f(x) - f_\delta(x)\|_Y &\le \|f(x) - f(y)\|_Y + \|f_\delta(x) - f_\delta(y)\|_Y \\ &\le \omega(\delta) + \|f_\delta | H^{\alpha_0\beta_1}(D, Y)\| \delta^{\alpha_0\beta_1}. \end{aligned} \tag{1}$$

Since $\lim_{t\to 0} \omega(t) = 0$, the right-hand side of (1) is less than a given $\varepsilon > 0$ for sufficiently small $\delta > 0$. The proof of the general case is finished because $x \in D$ was arbitrary.

If $Y = H_1$, then $f_\delta \in H^{1/4}(H, Y)$ due to Theorem 6.2 and the composition rule (Corollary 3.1, a)). Considering $\tilde{f}_\delta = P \circ f_\delta$, where P is the metric projection onto $\overline{\mathrm{co}}(f(D))$, we obtain the identity $f - \tilde{f}_\delta = P \circ (f - f_\delta)$. Since P is 1-Lipschitz according to the Phelps characterisation of Hilbert spaces, we achieve the additional properties $\tilde{f}_\delta \in H^{1/4}(X, Y)$ and $\tilde{f}_\delta(D) \subset \overline{\mathrm{co}}(f(D))$ finishing the proof of the theorem. □

10.5 Pairs of concrete spaces

Corollary 10.1 and Theorems 10.1 − 10.4 suggest the following definition.

Definition 10.2. *Let X be a metric space and its bounded subset, and let Y be a quasi-Banach space. Assume also that $\beta, \gamma, \delta \in (0, 1]$. We say that the pair (X, Y) possesses the* (β, γ, δ)-uniform approximation property *if there exist constants $C_g, C_h > 0$ and exponents $\beta_0, \beta_1 \in [0, 1)$ depending on the (parameters of) the spaces X and Y, such that, for every $\omega \in \Omega_S$, bounded $A \subset X$, $f \in H^\omega(A, Y)$ and $\varepsilon > 0$, there exists $f_\varepsilon \in H^\beta(A, Y)$ satisfying*

$$\|f - f_\varepsilon | C(A, Y)\| \le C_0 \omega\left(r(A)\right)^{\beta_1} \omega(\varepsilon)^\gamma$$

and

$$\|f_\varepsilon | H^\beta(A, Y)\| \le C_1 r(A)^{\beta_0} \omega\left(r(A)\right)^{\beta_1} \omega(\varepsilon)^\gamma \varepsilon^{-\delta}.$$

If, in addition, one also has $f_\varepsilon :\ A \longrightarrow \overline{\mathrm{co}}\left(f(A)\right)$, we say that the pair (X, Y) possesses the convex (β, γ, δ)-uniform approximation property.

One uses the term β-uniform approximation property *when* γ *and* δ *are not important. If, in addition, the pair* (X, Y) *does not possess the* β'*-uniform approximation property for any* $\beta' \in (0, \beta)$*, we say that the pair* (X, Y) *possesses the* sharp β-uniform approximation property.

Theorem 10.6. *For* $i \in I_5 \cup \{0\}$*, let* X *be a metric space and* $X_i, Y_i \subset \Gamma_i$*. Assume also that* $Z \in \{X_i, Y_i\}$ *has the form:*
(i) $Z \in \left\{ Z^{s_Z}_{p_Z, q_Z, a_Z}(G_Z),\ \tilde{Z}^{s_Z, A_Z}_{p_Z, q_Z, a_Z}(G_Z),\ Z^{s_Z}_{p'_Z, q'_Z, a'_Z}(G_Z)^*,\ \tilde{Z}^{s_Z, A_Z}_{p'_Z, q'_Z, a'_Z}(G_Z)^* \right\}$ *with admissible* a_Z *if* $Z \in \{X_1, Y_1\}$*,*
(ii) $Z \in \left\{ Z^{s_Z}_{p_Z}(G_Z),\ Z^{s_Z}_{p'_Z}(G_Z)^* \right\}$ *if* $Z \in \{X_2, Y_2\}$*,*
(iii) $Z \in \left\{ Z^{s_Z}_{p_Z, q_Z}(\mathbb{R}^n)_w,\ Z^{s_Z}_{p'_Z, q'_Z}(\mathbb{R}^n)^*_w \right\}$ *if* $Z \in \{X_3, Y_3\}$*,*
(iv) $Z \in \left\{ Z^{s_Z}_{p_Z, q_Z, \mathcal{F}}(G),\ Z^{s_Z}_{p'_Z, q'_Z, \mathcal{F}}(G)^* \right\}$ *if* $Z \in \{X_4, Y_4\}$*,*
and
(v) $Z = Z_{p_Z}$ *if* $Z \in \{X_5, Y_5\}$*.*
Let also:
1) $2\alpha_0 = \min(p_{\min}(X_0), 2)$, $2\alpha_i = \min\left(p_{X_i \min}, q_{X_i}, 2\right)$, $2/\beta_0 = \max(p_{\max}(X_0), 2)$ *and* $2/\beta_i = \max\left(p_{X_i \max}, q_{X_i}, 2\right)$ *for* $i \in \{1, 3, 4\}$*,*
2) $2\alpha_2 = \min(p_{X_2 \min}, 2)$, $2/\beta_2 = \max(p_{X_2 \max}, 2)$, $2\alpha_5 = \min(p_{X_5}, 2)$ *and* $2/\beta_5 = \max(p_{X_5}, 2)$,
and
3) α_j *and* β_j *be defined by substituting* X_i *with* Y_j *in the above expressions for* α_i *and* β_i *respectively.*
Then the following holds.

a) The pair (X, Y_j) *possesses the convex* $(\beta_j/2, \beta_j/2, \beta_j/2)$*-uniform approximation property for* $i \in I_5 \cup \{0\}$*.*

b) Assume that G_{Y_j} *satisfies the* C*-flexible* λ*-horn condition for* $j \in \{1, 2\}$*, and, if* X_i *is defined as a dual for* $i \in \{1, 2\}$*,* G_{X_i} *satisfies the* C*-flexible* λ*-horn condition too. Let also* X_4 *be not defined as a dual. Then the pair* (X_i, Y_j) *possesses the* $(\alpha_i\beta_j, \beta_j, \beta_j)$*-uniform approximation property for* $i \in I_5 \cup \{0\}$ *and* $j \in \{0, 1, 2, 3, 5\}$*.*

c) Assume that X_i *is as in b) for all* i*, and, if* Y_j *is defined as a dual,* G_{Y_j} *satisfies the* C*-flexible* λ*-horn condition for* $j \in \{1, 2\}$*. Let also* Y_4 *be not defined as a dual. Then the pair* (X_i, Y_j) *possesses the convex* $(\alpha_i\alpha_j\beta_j^2, \alpha_j\beta_j^2, \alpha_j\beta_j^2)$*-uniform approximation property for* $i, j \in I_5 \cup \{0\}$*.*

d) Assume that X_i *is as in b) for all* i*. Then the pair* (X_i, Y_j) *possesses the convex* $(\alpha_i\alpha_j\beta_j^2, \alpha_j^2\beta_j^3, \alpha_j\beta_j^2)$*-uniform approximation property for* $i, j \in I_5 \cup \{0\}$*.*

e) Assume that Y_j *is as in b) for all* j*. Then the pair* (X_i, Y_j) *possesses the* $(\alpha_i^2\beta_i\beta_j, \beta_j, \beta_j)$*-uniform approximation property for* $i, j \in I_5 \cup \{0\}$*.*

f) Assume that Y_j *is as in c) for all* j*. Then the pair* (X_i, Y_j) *possesses the convex* $(\alpha_i^2\beta_i\alpha_j\beta_j^2, \alpha_j\beta_j^2, \alpha_j\beta_j^2)$*-uniform approximation property for* $i, j \in I_5 \cup \{0\}$*.*

g) Assume that Y_j *is as in d) for all* j*. Then the pair* (X_i, Y_j) *possesses the convex* $(\alpha_i^2\beta_i\alpha_j\beta_j^2, \alpha_j^2\beta_j^3, \alpha_j\beta_j^2)$*-uniform approximation property for* $i, j \in I_5 \cup \{0\}$*.*

Moreover, the parameter $\beta = \alpha_i\beta_j$ *is sharp (maximal possible) in Part b) if* $i \neq 4$*,* $\dim(X_5) = \dim(Y_5) = \infty$*, and* X_0 *and* Y_0 *contain copies of* $l_{2\alpha_0}$ *and* l_{2/β_0} *respectively.*

Remark 10.3. *a*) Since the (convex) (β, γ, δ)-uniform approximation property of a pair (X, Y) is stable with respect to the substitution of one or both elements with isomorphic spaces (or the usage of equivalent norms), the applicability of the theorem is noticeably extended by the results on the equivalent characterisations of the spaces under consideration. For example, some spacial cases of X_4 and Y_4, such as the anisotropic Nikol'skii-Besov spaces (on $\mathbb{R}^n$) defined in terms of Fourier multipliers (approximation by entire functions of exponential type) [35] and the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ [41], that are their Lizorkin-Triebel counterparts defined in terms of Fourier multipliers (smooth Littlewood-Paley decompositions) are isomorphic to both (X_3, Y_3) (wavelet characterisations) and $(X_1(\mathbb{R}^n), Y_1(\mathbb{R}^n))$ with the same parameters and an admissible a. Furthermore, the usage of Theorem 10.4 in the proof shows that the substitution of $F^s_{p,q}(\mathbb{R}^n)$ with $F^s_{p,q}(G)$ with an open $G \subset \mathbb{R}^n$ in a pair does not change its (convex) (β, γ, δ)-uniform approximation property as well because such $F^s_{p,q}(G)$ is isomorphic to a complemented subspace in $F^s_{p,q}(\mathbb{R}^n)$.

b) Let us note that, according to Theorem 11.2 with $q = 1$, the exponent $\beta_j/2$ in Part *a*) of Theorem 10.6 is sharp not only when the metric space X is, for example, (a ball in) one of the spaces l_p and l_q $(\mathbb{N}, \{l_p\}_{n\in\mathbb{N}})$ with $p = 1$ or $p = \infty$, but also when $X \in \Gamma_i$ for $i \in \{1, 2\}$ and one of the parameters of X (different from a component of a compatible a_X if $i = 1$) is equal to 1 or ∞ or $X \in \{S_1, S_\infty\}$ thanks to Corollary 3.1 from [13] (see also Remark 8.4 above) complementing the results in the next section by identifying the existence of isomorphic copies of $l_1(I_n)$ and $l_\infty(I_n)$ in these spaces.

c) While X_i may have some parameters equal to 1 thanks to Part *a*), Parts *b*) and *e*) remain true also in the case $p_{\min}(Y_0) = 1$. The sharpness is shown by means of factorizing via a sequence of spaces with converging parameters (see Theorem 2.3 in [17] for an analogous argument due to Naor).

Proof. Theorem 10.1, combined with Theorems 2.2−2.8 and Remark 2.11, *b*) describing the (p, h_c)-uniform convexity of Y_j, implies Part *a*).

According to Section 2.2, X_i and Y_i for $i \in I_4$ are subspaces of the corresponding $IG(l_p, L_p)$ space with the same range of the parameters, and Y_i $(i \neq 4)$ is even complemented thanks to Theorem 8.1 and Remark 8.2. When G_{X_i} possesses the flexible λ-horn condition, X_i, defined as a dual for $i = 1, 2$, is isomorphic to a subspace in the corresponding IG-space with the same parameters due to Section 2.2, Theorem 8.1 and Lemma 8.1. Thus, we obtain Part *b*) with the aid of Theorems 10.2, *b*) and 10.3, *b*) and Corollary 10.2 if $i \in \{0, 5\}$ and/or $j \in \{0, 5\}$. In the rest of *b*), we use Theorem 5.6, *b*).

The sharpness in *b*) is inferred from Theorem 11.2 with the help of Theorems 8.2 − 8.4, Lemmas 8.1 and 6.3 and Remarks 8.2 and 8.3 insuring the existence of the complemented copies of sequence spaces. To check the presence of the corresponding extension properties for various pairs under consideration that are also required in Theorem 11.2, we use Remark 5.1, *b*) or 5.1, *c*) in tandem with Theorems 2.2 − 2.8 or 4.2 − 4.5 respectively.

Parts *c*) and *d*) are deduced, correspondingly, from Parts *a*) and *c*) of Theorems 10.2 − 10.4 assisted by Corollary 10.2 whenever the spaces from $\Gamma_0 \cup \Gamma_1$ are involved. As above, all the exponents (and even parameters) are traced with the aid of Theorems 2.2 − 2.8.

To establish $e)$, $f)$ and $g)$, we add the usage of Lemma 10.3 to the above considerations leading, respectively, to $b)$, $c)$ and $d)$ and, thus, finishing the proof of the theorem. □

11 Tsar'kov's phenomenon and sharpness of Hölder exponents

The approximation of uniformly continuous mappings between Banach spaces by Hölder-Lipschitz mappings is closely related to the problem of the existence of homogeneous Hölder-Lipschitz homeomorphisms of Banach spaces. Tsar'kov's phenomenon for a pair of Banach spaces X and Y is the existence of an exponent $\alpha \in (0,1)$ such that not every uniformly continuous mapping from a unit ball of X into Y can be arbitrary well-approximated by an α-Hölder mapping. In this section, we relate the best Hölder smoothness of the homeomorphisms between the spheres of the spaces under consideration with the limiting exponents describing Tsar'kov's phenomenon.

The best possible smoothness exponents of the Hölder-Lipschitz mappings approximating the uniformly continuous mappings of the unit ball of L_p or l_p into L_q or l_q for various pairs (p,q) were found by I. G. Tsar'kov [42, 43, 44] (see also §2.1 in [17]). I. G. Tsar'kov studied the Hölder-Lipschitz regularity of the classical Mazur mappings and used them to reduce the problem of approximating by Hölder mappings from L_p into L_q to the problem of approximating by Lipschitz mappings. To solve the latter, he followed the Bernstein-Jackson principle utilizing the $(1,1)$-extension property of the pairs of Hilbert spaces (L_2 or l_2), that is M. D. Kirszbraun's extension theorem [29] (see also [28] for generalizations), instead of Frechet's extension for the pairs (X, l_∞) that he used earlier [30]. A. Naor found a way how to demonstrate the sharpness of the smoothness exponent in the case $q=1$ in Tsar'kov's result by considering the limit $q \to 1$.

In 1993, I.G. Tsar'kov [45] had solved the problem of the uniform approximation of a set-valued uniformly continuous mapping f from a uniformly smooth X into the set of closed convex subsets of a uniformly convex Y by means of a single-valued mapping f_ε satisfying

$$r_X\left(f(x), f_\varepsilon(x)\right) \le \varepsilon \text{ for } x \in X$$

and

$$\|f_\varepsilon(x_1) - f_\varepsilon(x_2)\|_Y \le C\left(\omega_Y^{-1}\left(\Omega_X\left(C\|x_1 - x_2\|_X\right)\right) + \|x_1 - x_2\|_X\right) \text{ for } x_1, x_2 \in D,$$

where D is an arbitrary bounded subset of X, $C = C(D)$, and ω_Y and Ω_X are the classical moduli of uniform convexity and smoothness of Y and X correspondingly. He had also established the sharpness of the Hölder regularity exponent $\frac{\min(p,2)}{\max(q,2)}$ of f_ε for the reflexive $X = L_p$ and $Y = L_q$.

The essence of our approach is in the next lemma relating Hölder homeomorphisms and the approximation problem.

Lemma 11.1. *Let X_0, X_1, Y_0, Y_1 be Banach spaces and $\gamma_0, \gamma_1, \delta_0, \delta_1 \in (0,1]$. Assume also that the pairs (X_0, Y_0) and (X_1, Y_1) possess the γ_0 and γ_1-uniform approximation properties respectively, while the pairs (X_0, X_1) and (Y_0, Y_1) possess the sharp δ_0 and*

δ_1-uniform approximation properties correspondingly. Then

$$\alpha(Y_0, X_1) \le \min\left(\delta_1/\gamma_1, \delta_0/\gamma_0\right).$$

Proof. Assume that $\phi: \ Y_0 \overset{(\alpha,\beta)}{\longleftrightarrow} X_1$ for some $\alpha, \beta \in (0,1]$. If $f_0: \ B_{X_0} \to X_1$ and $f_1: \ B_{Y_0} \to Y_1$ are some uniformly continuous mappings, then so are the compositions $\phi^{-1} \circ f_0$ and $f_1 \circ \phi^{-1}$ that can be uniformly approximated by some $g_0 \in H^{\gamma_0}(B_{X_0}, Y_0)$ and $g_0 \in H^{\gamma_1}(B_{X_1}, Y_1)$. Therefore, the compositions $\phi \circ g_0$ and $g_1 \circ \phi$ approximate, respectively, the original mappings f_0 and f_1. The application of Lemma 3.1, a) finishes the proof by implying

$$\alpha \le \min\left(\delta_1/\gamma_1, \delta_0/\gamma_0\right).$$

□

Theorem 11.1. *Let $X, Y \in IG_+([1,\infty))$ have the same tree $T(X) = T(Y)$ (and, thus, common P). Assume also that, for every $\varepsilon > 0$ and $m \in \mathbb{N}$, there exist*

$$p \in I(X) \cap (\{p_{\min}(X), p_{\max}(X)\} + (-\varepsilon, \varepsilon)) \ \text{ and } \ q \in I(Y) \cap (\{p_{\min}(Y), p_{\max}(Y)\} + (-\varepsilon, \varepsilon)),$$

such that X and Y contain C_0-isomorphic and C_1-complemented copies of $l_p(I_m)$ an $l_q(m)$ correspondingly for some $C_0, C_1 \in [1,\infty)$. Then

$$\alpha(X, Y) \le \min\left(\frac{\min(p_{\min}(X), 2)}{\min(p_{\min}(Y), 2)}, \frac{\max(p_{\max}(X), 2)}{\max(p_{\max}(Y), 2)}\right).$$

In particular, $\alpha(X, H) \le \min(p_{\min}(X), 2)/2$ and $\alpha(H, X) \le 2/\max(p_{\max}(X), 2)$ if $Y = H = X_{\bar{2}}$.

Proof.]We use Lemma 11.1 with $Y_0 = X$, $X_1 = Y$ and $X_0 = Y_1 = X_{\bar{2}}$. Its conditions are verified with the aid of the proof of Parts a) and b) of Theorem 10.6 and Remark 10.3, b), while the conditions of Theorem 11.2 follow from the conditions of the assumed existence of the uniformly complemented and isomorphic subspaces and Remark 8.3.

□

11.1 Sharpness: key approximation theorem

This section is dedicated to Tsar'kov's approximation lemma that is extended and extensively used in [7, 11, 13]. Let us recall that the space of continuous mappings from X into Y is correctly defined.

Definition 11.1. *Let X and Y be metric and Banach spaces correspondingly. Then $C(X, Y)$ is the Banach space of all continuous mappings from X into Y with the norm*

$$\|f|C(X, Y)\| = \sup_{x \in X} \|f(x)\|_Y.$$

Lemma 11.2. ([42, 43, 44, 17]) *For some $1 \le p < q < \infty$ and $\alpha > p/q$ and every $n \in 2\mathbb{N}$, let ψ_n be an element of $H^\alpha(B(l_p(I_n), l_q(I_n))$ satisfying*

$$C = \sup_{n \in 2\mathbb{N}} \|\psi_n|H^\alpha(B(l_p(I_n), l_q(I_n))\| < \infty.$$

Then one has

$$\|M_{p/q} - \psi_n | C(B(l_p(I_n), l_q(I_n))\| \geq (1 - C(n/2)^{1/q-\alpha/p})/2,$$

where $M_{p/q} : l_p(I_n) \longrightarrow l_q(I_n)$, $x \longmapsto \|x|l_p\|^{p/q-1}x$ *is the Mazur mapping.*

The following theorem shows the sharpness of the majority of our approximation results in Section 10 and underpins the applicability of Lemma 11.1 to the study of the sharpness of the Hölder exponents of the homogeneous Hölder homeomorphisms of the pairs of Banach spaces under consideration.

Theorem 11.2. *For* $2 \in [q,p] \subset [1,\infty)$ *and* $d, C_0, C_1, C_2 \geq 1$, *let* X *be a bounded metric space containing* C_0*-Lipschitz homeomorphic copy of the unit ball* $B(l_q(I_n))$ *of* $l_q(I_n)$ *for every* $n \in \mathbb{N}$, *and let* Y *be a quasi-Banach space containing a* C_2*-complemented and* C_1*-isomorphic copy of* $l_p(I_n)$ *for every* $n \in \mathbb{N}$. *Let also the pair* (X,Y) *possess the* $(d,q/p)$*-extension property. Assume also that, for every* $\omega \in \Omega_S$, *there is* $g_\omega : (0,\infty) \to (0,\infty)$, *such that, for every* $f \in H^\omega(X,Y)$ *and* $\varepsilon > 0$, *there exists* $f_\varepsilon \in H^\alpha(X,Y)$ *satisfying*

$$\|f - f_\varepsilon | C(X,Y)\| \leq \varepsilon \quad \text{and} \quad \|f_\varepsilon | H^\alpha(X,Y)\| \leq g_\omega(\varepsilon).$$

Then we have $\alpha \leq q/p$.

Remark 11.1. *Theorem* 11.2 *works equally well even if* X *is* L_1 *or an appropriate metric space.*

Proof. Assume that $\alpha > q/p$. For $n \in \mathbb{N}$, let $\phi_n : B(l_q(I_n)) \to X$, $T_n : l_p(I_n) \to Y$ and $P_n : Y \to \operatorname{Im} T_n$ be a corresponding C_0-homeomorphism, a C_1-isomorphism and a projector satisfying $\|P_n|\mathcal{L}(Y)\| \leq C_2$. Choosing $\bar{f}_n = T_n \circ M_{q/p} \circ \phi_n^{-1} \in H^{q/p}(\operatorname{Im}\phi_n, Y)$, where $M_{q/p} \in H^{q/p}(B(l_q), l_p)$ is the Mazur mapping (see Lemma 11.2), we utilize the $(d,q/p)$-extension property of the pair (X,Y) to extend it to $f_n \in H^{q/p}(X,Y)$ satisfying

$$\|f_n|H^{q/p}(X,Y)\| \leq$$
$$\leq d\|\phi_n^{-1}|H^1(\operatorname{Im}\phi_n, B(l_q(I_n)))\|^{q/p}\|T_n|\mathcal{L}(l_p(I_n),Y)\| \cdot \|M_{q/p}|H^{q/p}(B(l_q),l_p)\|. \quad (1)$$

This means, in particular, that $\{f_n\}_{n\in\mathbb{N}} \subset H^\omega(X,Y)$ for some $\omega(t) = c_3 t^{q/p}$. Therefore, for every $\varepsilon' > 0$, there exists $g_n \in H^\alpha(X,Y)$ satisfying

$$\|f_n - g_n|C(X,Y)\| \leq \varepsilon' \text{ and } \|g_n|H^\alpha(X,Y)\| \leq g_\omega(\varepsilon'). \quad (2)$$

Observing that $T_n^{-1} \circ P_n \circ f_n \circ \phi_n = M_{q/p}$, we define $\psi_n = T_n^{-1} \circ P_n \circ g_n \circ \phi_n$ and use (1) and (2) to infer the estimates

$$\|M_{q/p} - \psi_n|C(B(l_q(I_n)), l_p(I_n))\| \leq \varepsilon'\|T_n^{-1}|\mathcal{L}(\operatorname{Im}T_n, l_q(I_n))\|C_2 = \varepsilon$$

and

$$\|\psi_n|H^\alpha(B(l_q(I_n)), l_p(I_n))\| \leq C_4 g_\omega(C_5\varepsilon). \quad (3)$$

Now we choose $\varepsilon = 1/3$ in (3) to achieve the contradiction with Lemma 11.2, providing, for sufficiently large n, the estimate

$$\|M_{q/p} - \psi_n|C(B(l_q(I_n)), l_p(I_n))\| > 1/3.$$

□

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