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**STAR-SHAPEDNESS AND CO-STAR-SHAPEDNESS OF FINITE UNIONS AND INTERSECTIONS OF CLOSED HALF-SPACES**

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**Abstract.** In the paper we answer the question when in finite dimensions finite unions and intersections of closed half-spaces are shifts of sets stable under shrinkings or dilatations and give explicit descriptions of the kernels of such sets.

## 1 Introduction

Convex closed sets stable under shrinkings and dilatations play an important role in the theory of optimization (see, for example, [1, 2]). Investigation of non-convex sets with these properties and their shifts has been begun by A.M. Rubinov and his co-authors (see [4] and references therein). Our interest in these sets (called further star-shaped and co-star-shaped respectively) and the choice of research methods are due to the fact that in finite dimensions they can be represented as intersections of some collections of finite unions of closed half-spaces. It is of interest to obtain verifiable criteria for a set to be a star-shaped or a co-star-shaped one and give an explicit description of its kernel or co-kernel respectively. In the paper such criteria and descriptions of the kernels are given for finite unions and intersections of closed half-spaces.

The paper consists of three sections. In Theorem 2 of Section 2 we give a necessary condition for a finite union of star-shaped sets to be also star-shaped. Applying Theorem 2 to finite unions of closed half- spaces we obtain in Theorem 3 of Section 3 that these sets are star-shaped if and only if they are strongly star-shaped. From this result it follows (Theorem 4) that a finite union of closed half-spaces is strongly co-star-shaped if and only if it differs from $\mathbb{R}^n$. A necessary condition for a convex set to be co-star-shaped is given in Corollary 3 of Section 2. In Theorem 6 of Section 3 we show that for a polyhedral set it is also sufficient.

## 2 Preliminaries

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space. Throughout the paper we shall denote the inner product of vectors $a$ and $x$ by $\langle a, x\rangle$, the closure, the interior and the boundary of a set $A \subset \mathbb{R}^n$ by $\bar{A}$, $\operatorname{int} A$ and $\operatorname{bd} A$ respectively, the ray $\{\lambda x : \lambda \geq 0\}$ by $\mathscr{L}_x$, the closed ball with center $x_0$ and radius $r$ by $B[x_0, r]$. If $A \subset \mathbb{R}^n$ is convex, then $\operatorname{ri} A$ is its relative interior, $\dim A$ is its dimention and $0^+A$ is its recession cone. The convex cone generated by a nonempty set $C \subset \mathbb{R}^n$ and the affine hull of $C$ will be denoted by $\operatorname{co} C$, $\operatorname{aff} C$ respectively. For a natural number $p$ we shall write $i \in 1 : p$ if $i$ is natural and $i \leq p$.

**Definition 1.** *Let $A$ be a nonempty subset of $\mathbb{R}^n$, $x_0 \in \mathbb{R}^n$. We call the set $A$ star-shaped at the point $x_0$, if*

$$\{tx_0 + (1-t)x : 0 \leq t \leq 1,\ x \in A\} \subset A.$$

*The set of all points at which $A$ is star-shaped is called the kernel of $A$ and denoted by $\operatorname{kern} A$. If $0 \in \operatorname{kern} A$, then $A$ is called radiant.*

It is not difficult to see that the set $\operatorname{kern} A$ is convex. So we can characterize convex sets as star-shaped ones which coincide with their kernels. Since the algebraic operations in $\mathbb{R}^n$ are continuous, the kernel of a closed star-shaped set is also closed. The totality of all star-shaped sets is stable under shifts and

$$\operatorname{kern}(\omega_0 + A) = \omega_0 + \operatorname{kern} A. \tag{1}$$

for a star-shaped set $A$ and a point $\omega_0 \in \mathbb{R}^n$.

**Definition 2** ([6]). *A closed proper set $A \subset \mathbb{R}^n$ is called strongly star-shaped at the point $x_0$ if $x_0 \in \operatorname{int} A$ and for each $x \in \mathbb{R}^n, x \neq 0$ the ray $x_0 + \mathscr{L}_x$ does not intersect the boundary $\operatorname{bd} A$ of the set $A$ more than once.*

*The set of all points at which $A$ is strongly star-shaped is denoted by $\operatorname{kern}_* A$ and called the lower kernel of $A$. If $0 \in \operatorname{kern}_* A$, then $A$ is called strongly radiant.*

A strongly radiant set $A$ can be characterized (see [6]) as a closed radiant set whose gauge

$$\mu_A(x) = \inf\{\lambda > 0 : x \in \lambda A\} \quad (x \in \mathbb{R}^n)$$

is a real-valued continuous function.

A closed convex set $A$ is strongly star shaped if and only if (see [3]) $\operatorname{int} A \neq \emptyset$. Herewith $\operatorname{kern}_* A = \operatorname{int} A$.

It has been proved in [6] that

$$(x_0 \in \operatorname{kern}_* A,\ x \in A,\ \lambda \in [0,1)) \implies ((1-\lambda)x_0 + \lambda x \in \operatorname{int} A), \tag{2}$$

whence it follows that a strongly star-shaped set is star-shaped and regularly closed (see [7]).

The totality of all strongly star-shaped sets is also stable under shifts and (1) is also valid for lower kernels.

**Proposition 1** ([10]). *Let $A \subset \mathbb{R}^n$ be a closed proper set. If $\operatorname{kern}_* A \neq \emptyset$, then it is convex and*

$$\operatorname{kern}_* A \subset \operatorname{kern} A \subset \overline{\operatorname{kern}_* A}. \tag{3}$$

**Corollary 1.** *Let $A$ be strongly star-shaped. Then*

$$\overline{kern_* A} = kern\, A \quad and \quad ri\,(kern_* A) = ri\,(kern\, A).$$

If $\dim(\text{kern}\, A) = n$, then (see [4]) the set $A$ is strongly star-shaped at each point of $\text{int}\,(\text{kern}\, A)$. The set $A = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \leq |x_1|^{-1}\}$ gives an example of a strongly star-shaped set whose kernel has no interior points.

**Definition 3.** *Let $B$ be a nonempty subset of $R^n$, $u \in \mathbb{R}^n$. We call the set $B$ co-star-shaped at the point $u$ if*

$$(x \in B,\ \lambda \geq 1) \implies (u + \lambda(x - u) \in B). \tag{4}$$

*The set of all points at which $B$ is co-star-shaped is called the co-kernel of $B$ and denoted by $kern_\infty B$. If $0 \in kern_\infty B$, then $B$ is called co-radiant.*

In [5] the set $\text{kern}_\infty B$ is defined as the totality of points $u \in \mathbb{R}^n \setminus B$ such that (4) takes place and a set $B$ with nonempty $\text{kern}_\infty B$ is called star-shaped with respect to infinity

We can characterize affine sets as convex ones which coincide with their co-kernels.

**Proposition 2** ([10])**.** *If $B \neq \mathbb{R}^n$ is co-star shaped, then $kern_\infty B \subset (\mathbb{R}^n \setminus \bar{B}) \cup bd\, B$.*

**Proposition 3** ([10])**.** *The co-kernel $kern_\infty B$ of a co-star-shaped set $B$ is convex. It is closed if $B$ is closed.*

There exist non-affine co-star-shaped sets which contain their co-kernels. For example, the set $B = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\} \cup \{(0, x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ enjoys this property ($\text{kern}_\infty B = \{0\}$).

**Proposition 4** ([10])**.** *Let a co-star-shaped set $B \neq \mathbb{R}^n$ be such that*

$$(u_1 \neq u_2,\ u_1, u_2 \in bd\, B) \implies (\exists \lambda \in (0, 1) : \lambda u_1 + (1 - \lambda) u_2 \notin bd\, B),$$

*and $kern_\infty B$ contains more than one element. Then $kern_\infty B \setminus B \neq \emptyset$.*

The totality of all co-star-shaped sets is stable under shifts and

$$\text{kern}_\infty (\omega_0 + B) = \omega_0 + \text{kern}_\infty B. \tag{5}$$

for a co-star-shaped set $B$ and a point $\omega_0 \in \mathbb{R}^n$.

**Proposition 5.** *A nonempty closed convex set $C \subset \mathbb{R}^n$ is co-star-shaped if and only if there exists $\omega \in R^n$ such that $C \subset \omega + 0^+ C$.*

**Proof.** Necessity. Let $C$ be co-star-shaped at the point $\omega$, $x \in C$. We have $\omega + m(x - \omega) = z_m \in C$ for all natural $m$ whence it follows that $x - \omega = \lim\limits_{m \to \infty} \frac{1}{m} z_m \in 0^+ C$ and thereby $C \subset \omega + 0^+ C$.

Sufficiency. Let $C \subset \omega + 0^+ C$, $x \in C$, $\lambda \geq 1$. We have $x - \omega \in 0^+ C$, $\lambda - 1 \geq 0$ whence it follows that $\omega + \lambda(x - \omega) = x + (\lambda - 1)(x - \omega) \in C$. Hence $C$ is co-star-shaped at the point $\omega$. □

**Corollary 2.** *A nonempty closed convex set $C \subset \mathbb{R}^n$ is co-radiant if and only if $C \subset 0^+C$.*

**Corollary 3.** *If a nonempty closed convex set $C \subset \mathbb{R}^n$ is co-star shaped, then* $\dim C = \dim 0^+C$.

*Proof :* Let $a \in C$, $\omega \in \operatorname{kern}_\infty C$. We have $a + 0^+C \subset C \subset \omega + 0^+C$ whence the equality $\dim C = \dim 0^+C$ follows. □

**Remark 1.** *If the dimension of a nonempty closed convex set $C$ coincides with the dimention of its recession cone, then the set $C$ is not generally speaking co-star-shaped. Indeed, let $C = \{(x,y) \in \mathbb{R}^2 : x > 0,\ y \le \ln x\}$. Then $0^+C = \{(x,y) \in \mathbb{R}^2 : x \ge 0,\ y \le 0\}$, $\dim C = \dim 0^+C = 2$. At the same time there does not exist $\omega \in \mathbb{R}^2$ such that $C \subset \omega + 0^+C$ and according to Proposition 5 the set $C$ is not co-star-shaped.*

**Definition 4** ([5])**.** *A nonempty closed set $B \subset \mathbb{R}^n$ is called strongly co-star-shaped at the point $u$ if*
*1) $u \notin B$;*
*2) $((u + \mathscr{L}_x) \cap B \ne \emptyset) \implies (u + \mathscr{L}_x) \cap B$ contains at least two points and $(u + \mathscr{L}_x) \cap bd\, B$ consists of exactly one point.*

*The set of all points at which $B$ is strongly co-star-shaped is denoted by $kern^* B$ and called the upper kernel of $B$. If $0 \in kern^* B$, then $B$ is called strongly co-radiant.*

A strongly co-radiant set $B$ can be characterized (see [5]) as a closed co-radiant set whose co-gauge

$$\nu_B(x) = \sup\{\lambda > 0 : x \in \lambda B\} \quad (x \in \mathbb{R}^n)$$

is a real-valued nonzero continuous function.

In [5] strongly co-star-shaped sets are called strongly star-shaped with respect to infinity.

If $B$ is strongly co-star-shaped, then (see [5])

$$(u \in \operatorname{kern}^* B,\ x \in B,\ \lambda > 1) \implies (u + \lambda(x - u) \in \operatorname{int} B). \tag{6}$$

Therefore a strongly co-star-shaped set is co-star-shaped and regularly closed.

The totality of all strongly star-shaped sets is also stable under shifts and (5) is valid for upper kernels.

**Proposition 6** ([10])**.** *Let $B \subset \mathbb{R}^n$ be a nonempty closed set. If $kern^* B \ne \emptyset$, then it is convex and*

$$kern^* B \subset kern_\infty B \subset \overline{kern^* B}.$$

**Corollary 4.** *Let $B$ be a strongly co-star-shaped set. Then*

$$\overline{kern^* B} = kern_\infty B \quad \text{and} \quad ri\,(kern^* B) = ri\,(kern_\infty B).$$

**Proposition 7** ([4, 10])**.** *Let $B \ne \mathbb{R}^n$ be a closed co-star-shaped set. If $int\,(kern_\infty B) \ne \emptyset$, then $B$ is a strongly co-star-shaped set and herewith $int\,(kern_\infty B) \subset kern^* B$.*

There exist strongly star-shaped sets whose upper kernels have no interior points. For example, the strongly co-star-shaped set $\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \geq |x_1|^{-1}\}$ enjoys this property.

Let $A \subset \mathbb{R}^n$. We put $\mathscr{C}_r\, A = \overline{\mathbb{R}^n \setminus A}$. If the set $A$ is regularly closed, then $\mathscr{C}_r\, A$ is the complement of the set $A$ in the Boolean algebra of regularly closed subsets of $\mathbb{R}^n$. Therefore $\mathscr{C}_r\, (\mathscr{C}_r\, A) = A$ (see [7]).

**Theorem 1** ([5])**.** *1) If $A \subset \mathbb{R}^n$ is strongly star-shaped, then $\mathscr{C}_r\, A$ is strongly co-star-shaped and kern$_*\, A =$ kern$^*\, \mathscr{C}_r\, A$;*
*2) If $B \subset \mathbb{R}^n$ is strongly co-star-shaped, then $\mathscr{C}_r\, B$ is strongly star-shaped and kern$^*\, B =$ kern$_*\, \mathscr{C}_r\, B$.*

Let $A \subset \mathbb{R}^n$, $A \neq \emptyset$. We put

$$\mathrm{rc}A = \{\lim_{i \to \infty} \lambda_i \omega_i : \lambda_i \downarrow 0,\ \omega_i \in A\}$$

The cone rc$A$ is called the recession cone of the set $A$. The recession cone of the set $A$ coincides with the recession cone of its closure $\bar{A}$. If $A$ is a strongly star-shaped set, then $\mathrm{rc}(\mathscr{C}_r\, A) = \mathscr{C}_r\, (\mathrm{rc}A)$ (see [9]). If $A$ is a convex closed set, then (see [3]) $\mathrm{rc}A = 0^+A$. This equality justifies the use of the term recession cone for the cone rc$A$.

**Theorem 2.** *Let $A_1, \cdots, A_p$ be strongly star-shaped sets,*

$$A = \bigcup_{i=1}^{p} A_i \neq \mathbb{R}^n.$$

*Suppose that the recession cones $rcA_i$ $(i = 1, \cdots, p)$ are regularly closed and the set $A$ is star-shaped. Then*

$$\bigcap_{i=1}^{p} (\mathbb{R}^n \setminus rcA_i) \neq \emptyset.$$

**Proof.** Let $x_0 \in \mathrm{kern}\, A$. Then the nonempty open set $\mathbb{R}^n \setminus A$ is co-star-shaped at the point $x_0$. Let $u_0 \in \mathbb{R}^n \setminus A$ and $r > 0$ be such that $B[u_0, r] \subset R^n \setminus A$. We have, for $i \in 1 : p$,

$$x_0 + \bigcup_{\lambda \geq 1} \lambda(-x_0 + B[u_0, r]) \subset R^n \setminus A \subset R^n \setminus A_i,$$

whence it follows that

$$B[-x_0 + u_0, r] \subset \mathrm{rc}(R^n \setminus A_i) = \mathrm{rc}\overline{R^n \setminus A_i} = \mathrm{rc}(\mathscr{C}_r\, A_i) = \mathscr{C}_r\, (\mathrm{rc}A_i).$$

Hence $-x_0 + u_0 \in \mathrm{int}\, (\mathscr{C}_r\, (\mathrm{rc}A_i)) = \mathbb{R}^n \setminus \mathrm{rc}A_i$. □

## 3 Finite unions and intersections of closed half-spaces

The intersection $A$ of a finite collection of closed half-spaces is convex and thereby it is star-shaped. It is strongly star-shaped if and only if the corresponding open half-spaces have a common point. Herewith kern$_*\, A$ is the intersection of these open half-spaces. While studying finite unions of closed half-spaces we need the explicit description of the interiors of such sets.

**Lemma 1** ([8])**.** *Let*

$$\tilde{A} = \bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \le \mu_i\} \ne \mathbb{R}^n.$$

*Then*

$$int \left( \bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \le \mu_i\} \right) = \bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle < \mu_i\}.$$

**Theorem 3.** *Let*

$$\tilde{A} = \bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \le \mu_i\} \ne \mathbb{R}^n.$$

*The following conditions are equivalent:*
1)$\tilde{A}$ *is star-shaped;*
2)$\bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle > 0\} \ne \emptyset$;
3)$\tilde{A}$ *is strongly star-shaped.*

*If $\tilde{A}$ is star-shaped, then*

$$\emptyset \ne \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \le \mu_i\} \subset kern\, \tilde{A}, \tag{7}$$

$$\emptyset \ne \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle < \mu_i\} \subset kern_*\, \tilde{A}. \tag{8}$$

*If herewith*

$$\bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \le \mu_i\} \ne \bigcup_{\substack{i=1 \\ i \ne \bar{i}}}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \le \mu_i\} \tag{9}$$

*for each $\bar{i} \in 1 : p$, then*

$$kern_*\, \tilde{A} = \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle < \mu_i\}, \tag{10}$$

$$kern\, \tilde{A} = \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \le \mu_i\}. \tag{11}$$

**Proof.** 1) $\longrightarrow$ 2). Let $A_i = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \le \mu_i\}$. The implication follows by Theorem 2.
2) $\longrightarrow$ 3). Let $x_0 \in \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle > 0\}$. It is obvious that $-tx_0 \in \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle < \mu_i\}$ for $t$ large enough and thereby $\bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle < \mu_i\} \ne \emptyset$. Let $\omega \in \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle < \mu_i\}$, $x \in \tilde{A}$, $\lambda \in [0, 1)$. Then $\langle a_{\bar{i}}, x \rangle \le \mu_{\bar{i}}$ for some $\bar{i} \in 1 : p$

whence it follows that $(1-\lambda)\omega + \lambda x \in \{x \in \mathbb{R}^n : \langle a_{\bar{i}}, x\rangle < \mu_{\bar{i}}\} \subset \operatorname{int} \tilde{A}$. Therefore $\tilde{A}$ is strongly star-shaped at the point $\omega$.
3) $\longrightarrow$ 1) A strongly star-shaped set is a star-shaped one.

Let $\tilde{A}$ be star-shaped. (8) follows from the proof of implications 1) $\longrightarrow$ 2) $\longrightarrow$ 3). Applying Corollary 1 and (8) we obtain

$$\emptyset \neq \bigcap_{i=1}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle \le \mu_i\} = \overline{\bigcap_{i=1}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle < \mu_i\}} \subset \overline{\operatorname{kern}_* \tilde{A}} = \operatorname{kern} \tilde{A},$$

that is (7) holds true. Suppose now that (9) holds true. To prove (10) we show that for each $i_0 \in 1:p$ there exists $x_{i_0} \in \mathbb{R}^n$ such that $\langle a_{i_0}, x_{i_0}\rangle = \mu_{i_0}$ and $\langle a_i, x_{i_0}\rangle > \mu_i$ for $i \in 1:p$, $i \neq i_0$. Indeed, let for example $i_0 = 1$. According to (9)

$$\bigcup_{i=1}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle \le \mu_i\} \neq \bigcup_{i=2}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle \le \mu_i\}.$$

Next

$$\bigcup_{i=1}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle < \mu_i\} \neq \bigcup_{i=2}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle < \mu_i\}$$

whence it follows that

$$\bigcap_{i=1}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle \ge \mu_i\} \neq \bigcap_{i=2}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle \ge \mu_i\}.$$

Therefore there exists $\bar{x} \in \mathbb{R}^n$ such that $\langle a_i, \bar{x}\rangle \ge \mu_i$ for $i \in 2:p$ and $\langle a_1, \bar{x}\rangle < \mu_1$. Let $x_0 \in R^n \setminus \tilde{A}$. Then $\langle a_i, x_0\rangle > \mu_i$ for all $i \in 1:p$. Therefore there exists $t \in (0,1)$ such that $\langle a_1, t\bar{x} + (1-t)x_0\rangle = \mu_1$. Herewith $\langle a_i, t\bar{x} + (1-t)x_0\rangle > \mu_i$ for $i \in 2:p$.

Let now $\omega \in \operatorname{kern}_* \tilde{A}$. According to Lemma 1 and (4) $\lambda\omega + (1-\lambda)x_1 \in \operatorname{int} \tilde{A} = \bigcup_{i=1}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle < \mu_i\}$ for $\lambda \in (0,1]$. If $\lambda$ is small enough, then $\langle a_i, \lambda\omega + (1-\lambda)x_1\rangle > \mu_i$ for $i \in 2:p$. Therefore $\langle a_1, \lambda\omega + (1-\lambda)x_1\rangle < \mu_1$ for this $\lambda$. Hence $\langle a_1, \omega\rangle < \mu_1$. The inequalities $\langle a_i, \omega\rangle < \mu_i$ for $i \in 2:p$ are proved in the same way and we obtain (10). Then (11) follows by Corollary 1. □

**Remark 2.** *The example of the set*

$$\tilde{A} = \{(x_1, x_2) : -x_1 \le 0\} \cup \{(x_1, x_2) : -x_2 \le 0\} \cup \{(x_1, x_2) : -x_1 - x_2 \le -1\}$$

*and* $\omega = (1/4, 1/4) \in \operatorname{kern}_* \tilde{A} \subset \operatorname{kern} \tilde{A}$ *show that in the general case the inclusions in (7) and (8) are strict.*

**Corollary 5.** *Let vectors* $a_1, \cdots, a_p$ *be linearly independent and* $\mu_1, \cdots, \mu_p$ *be arbitrary numbers. Then the set* $\tilde{A} = \bigcup_{i=1}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle \le \mu_i\}$ *is strongly star-shaped. Herewith*

$$\operatorname{kern}_* \tilde{A} = \bigcap_{i=1}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle < \mu_i\}, \quad \operatorname{kern} \tilde{A} = \bigcap_{i=1}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle \le \mu_i\}.$$

**Proof.** There exist vectors $b_1, \cdots, b_p$ such that

$$\langle a_i, b_j \rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Let $b = b_1 + \cdots + b_p$. Then $\langle a_i, b \rangle = 1$ for all $i$. If $\lambda > \max\limits_{i=1}^{p} \mu_i$, then $\lambda b \notin \bigcup\limits_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \le \mu_i\}$. Therefore by Theorem 3 the set $\tilde{A}$ is strongly star-shaped.

Let $\bar{i} \in 1 : p$. Putting $t_{\bar{i}} = \mu_{\bar{i}},\ t_i = \mu_i + 1$ for $i \in 1 : p,\ i \ne \bar{i},\ \tilde{b} = \sum\limits_{i=1}^{p} t_i b_i$ we have

$$\tilde{b} \in \bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \le \mu_i\}, \quad \tilde{b} \notin \bigcup_{\substack{i=1 \\ i \ne \bar{i}}}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \le \mu_i\}.$$

Therefore equalities (10) and (11) hold true. □

**Proposition 8.** *Let*

$$\bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \le \mu_i\} \ne \mathbb{R}^n.$$

*The equality*

$$\bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \le \mu_i\} = \bigcup_{j=1}^{q} \{x \in \mathbb{R}^n : \langle b_j, x \rangle \le \nu_j\} \tag{13}$$

*takes place if and only if*

$$co\,\{(a_i, \mu_i)_{i=1}^{p}\} + \{(0, \mu) : \mu \le 0\} = co\,\{(b_j, \nu_j)_{j=1}^{q}\} + \{(0, \mu) : \mu \le 0\}. \tag{14}$$

**Proof.** According to Lemma 1 (13) takes place if and only if

$$\bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \ge \mu_i\} = \bigcap_{j=1}^{q} \{x \in \mathbb{R}^n : \langle b_j, x \rangle \ge \nu_j\}. \tag{15}$$

(15) is equivalent to the fact that each inequality $\langle b_j, x \rangle \ge \nu_j$ is a corollary of the system of inequalities $\langle a_i, x \rangle \ge \mu_i$ $(i \in 1 : p)$, and each inequality $\langle a_i, x \rangle \ge \mu_i$ is a corollary of the system of inequalities $\langle b_j, x \rangle \ge \nu_j$ $(j \in 1 : q)$. Applying the Minkowski-Farkas theorem (see [3]) we infer that (15) takes place ifand only if

$$(b_j, \nu_j) \in \text{co}\,\{(a_i, \mu_i)_{i=1}^{p}\} + \{(0, \mu) : \mu \le 0\}\ (j \in 1 : q) \tag{16}$$

and

$$(a_i, \mu_i) \in \text{co}\,\{(b_j, \nu_j)_{j=1}^{q}\} + \{(0, \mu) : \mu \le 0\}\ (i \in 1 : p). \tag{17}$$

(16) and (17) hold true if and only if (14) takes place. □

**Corollary 6.** *(9) takes place if and only if*

$$(a_{\bar{i}}, \mu_{\bar{i}}) \notin co\,\{(a_i, \mu_i)_{i=1, i\neq \bar{i}}^p\} + \{(0, \mu) : \mu \leq 0\}\ (\bar{i} \in 1 : p). \tag{18}$$

**Proof.** (9) takes place if and only if

$$\text{co}\,\{(a_i, \mu_i)_{i=1, i\neq \bar{i}}^p\} + \{(0, \mu) : \mu \leq 0\} \neq \text{co}\,\{(a_i, \mu_i)_{i=1}^p\} + \{(0, \mu) : \mu \leq 0\} \tag{19}$$

for each $\bar{i} \in 1 : p$. Since $(a_{\bar{i}}, \mu_{\bar{i}}) \in \text{co}\,\{(a_i, \mu_i)_{i=1}^p\} + \{(0, \mu) : \mu \leq 0\}$, (18) implies (19).

If $(a_{\bar{i}}, \mu_{\bar{i}}) \in \text{co}\,\{(a_i, \mu_i)_{i=1, i\neq \bar{i}}^p\} + \{(0, \mu) : \mu \leq 0\}$ for some $\bar{i} \in 1 : p$, then $\{(a_i, \mu_i)_{i=1, i\neq \bar{i}}^p\} \subset \text{co}\,\{(a_i, \mu_i)_{i=1}^p\} + \{(0, \mu) : \mu \leq 0\}$, whence it follows that

$$\text{co}\,\{(a_i, \mu_i)_{i=1, i\neq \bar{i}}^p\} + \{(0, \mu) : \mu \leq 0\} = \text{co}\,\{(a_i, \mu_i)_{i=1}^p\} + \{(0, \mu) : \mu \leq 0\}\ (\bar{i} \in 1 : p).$$

Therefore (19) implies (18). □

**Theorem 4.** *A set*

$$\Omega = \bigcup_{i=1}^p \{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq \mu_i\}.$$

*is strongly co-star-shaped if and only if it is a proper subset of* $\mathbb{R}^n$. *Herewith*

$$kern^* \,\Omega = \bigcap_{i=1}^p \{x \in \mathbb{R}^n : \langle a_i, x\rangle > \mu_i\}.$$

**Proof.** Necessity. Let $\Omega$ be strongly co-star-shaped. Then according to Definition 4 of a strongly co-star-shaped set $\Omega$ is a proper subset of $\mathbb{R}^n$ and by Lemma 1 $\text{int}\,\Omega = \bigcup_{i=1}^p \{x \in \mathbb{R}^n : \langle a_i, x\rangle < \mu_i\}$. By Theorem 1 the set $\mathscr{C}_r\,\Omega = \mathbb{R}^n \setminus \text{int}\,\Omega = \bigcap_{i=1}^p \{x \in \mathbb{R}^n : \langle a_i, x\rangle \geq \mu_i\}$ is a strongly star-shaped set and $\text{kern}^*\,\Omega = \text{kern}_*\,(\mathscr{C}_r\,\Omega)$. But

$$\text{kern}_*\,(\mathscr{C}_r\,\Omega) = \bigcap_{i=1}^p \{x \in \mathbb{R}^n : \langle a_i, x\rangle > \mu_i\}.$$

Sufficiency. Let $\Omega$ be a proper subset of $\mathbb{R}^n$. According to Lemma 1

$$\text{int}\,\Omega = \bigcup_{i=1}^p \{x \in \mathbb{R}^n : \langle a_i, x\rangle < \mu_i\}.$$

Hence $\Omega$ is regularly closed. Then

$$\mathscr{C}_r\,\Omega = \bigcap_{i=1}^p \{x \in \mathbb{R}^n : \langle a_i, x\rangle \geq \mu_i\}$$

is a nonempty regularly closed subset of $\mathbb{R}^n$. Therefore

$$\text{int}\,(\mathscr{C}_r\,\Omega) = \mathbb{R}^n \setminus \Omega = \bigcap_{i=1}^p \{x \in \mathbb{R}^n : f_i(x) > \mu_i\}) \neq \emptyset,$$

and thereby $\mathscr{C}_r\,\Omega$ is strongly star-shaped. Since $\Omega = \mathscr{C}_r\,(\mathscr{C}_r\,\Omega)$, the set $\Omega$ is strongly co-star-shaped by Theorem 1. □

By Theorem 4 and Corollary 4 we obtain

**Corollary 7.** *A set*

$$\Omega = \bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \le \mu_i\}.$$

*is co-star-shaped. If $\Omega \neq \mathbb{R}^n$, then*

$$kern_\infty \, \Omega = \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \ge \mu_i\}.$$

**Theorem 5.** *A nonempty set*

$$V = \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \le \mu_i\} \neq \emptyset,$$

*is co-star-shaped if and only if*

$$\bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \ge c_i\} \neq \emptyset, \quad \text{where} \quad c_i = \sup_{x \in V} \langle a_i, x\rangle \quad (i \in 1:p). \tag{20}$$

*If $V$ is co-star-shaped, then*

$$kern_\infty \, V = \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \ge c_i\}. \tag{21}$$

*If*

$$\bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \le \mu_i\} \neq \bigcap_{\substack{i=1 \\ i \neq \bar{i}}}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \le \mu_i\}, \tag{22}$$

*for each $\bar{i} \in 1:p$, then the set $V$ is co-star-shaped if and if*

$$\bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \ge \mu_i\} \neq \emptyset. \tag{23}$$

*If $V$ is co-star-shaped and (22) takes place, then*

$$kern_\infty \, V = \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \ge \mu_i\}, \tag{24}$$

**Proof.** Necessity. Let $u \in \text{kern}_\infty V$, $x \in V$, $\lambda > 1; i \in 1:p$. Since $u + \lambda(x-u) \in V$ we have $\langle a_i, u\rangle + \lambda\langle a_i, x-u\rangle \le \mu_i$. Dividing by $\lambda$ and letting $\lambda$ tend to $\infty$ we obtain $\langle a_i, x\rangle \le \langle a_i, u\rangle$. Therefore $c_i = \sup\limits_{x \in V} \langle a_i, x\rangle \le \langle a_i, u\rangle$ whence it follows that $\text{kern}_\infty \subset \bigcap\limits_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \ge c_i\}$. Hence (20) takes place.

Sufficiency. Let (20) take place, $u \in \bigcap\limits_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \ge c_i\}$, $x \in V$. We have $\langle a_i, x\rangle \le c_i \le \langle a_i, u\rangle$ and therefore

$$\langle a_i, u + \lambda(x-u)\rangle = \langle a_i, x\rangle + (\lambda - 1)\langle a_i, x-u\rangle \le \mu_i \quad (i \in 1:p, \ \lambda \ge 1).$$

Hence

$$\bigcap_{i=1}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle \geq c_i\} \subset \operatorname{kern}_\infty V,$$

and (21) is proved.

Let now (22) take place. Then $c_i = \mu_i$ for all $i \in 1 : p$. Indeed, let, for example, $i = 1$. Then there exists $\tilde{x} \in \mathbb{R}^n$ such that $\langle a_1, \tilde{x}\rangle > \mu_1$ and $\langle a_i, \tilde{x}\rangle \leq \mu_i$ for $i \in 2 : p$. Let $x_0 \in V$. Then $\langle a_i, x_0\rangle \leq \mu_i$ for $i \in 1 : p$. If $\langle a_1, x_0\rangle = \mu_1$, then $c_1 = \mu_1$. If $\langle a_1, x_0\rangle < \mu_1$ there exists $t \in (0, 1)$ such that $\langle a_1, t\tilde{x} + (1 - t)x_0\rangle = \mu_1$. Herewith $\langle a_i, t\tilde{x} + (1 - t)x_0\rangle \leq \mu_i$ for $i \in 2 : p$. Hence $t\tilde{x} + (1 - t)x_0 \in V$ and we again obtain $c_1 = \mu_1$. Therefore $V$ is co-star-shaped if and only if (23) holds true and (24) follows from (21). □

**Theorem 6.** *A nonempty polyhedral set $V \subset \mathbb{R}^n$ is co-star-shaped if and only of $\dim V = \dim 0^+V$.*

**Prroof.** The necessity follows from Corollary 3. To prove the sufficiency we represent the polyhedral set $V$ in the form

$$\bigcap_{i=1}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq c_i\} \neq \emptyset, \quad \text{where} \quad c_i = \sup_{x\in V}\langle a_i, x\rangle \quad (i \in 1 : p),$$

and consider two cases.

Case 1: $\dim V = n$. Then $\bigcap_{i=1}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle < 0\} = \operatorname{int} 0^+V \neq \emptyset$ and thereby there exists $\omega_0 \in \mathbb{R}^n$ such that $\langle a_i, \omega_0\rangle \geq c_i$ for all $i \in 1 : p$. According to Theorem 5 the set $V$ is co-star-shaped.

Case 2: $\dim V < n$. Let $I(V)$ and $I(0^+V)$ be sets of indices of constraints that are binding for all points of $V$ or $0^+V$ respectively:

$$I(V) = \{i \in 1 : p : \langle a_i, x\rangle = c_i\,, \forall x \in V\},$$

$$I(0^+V) = \{i \in 1 : p : \langle a_i, x\rangle = 0\,, \forall x \in 0^+V\}.$$

Since $\dim V < n$ we have $I(V) \neq \emptyset$. If $I(V) = \{1, \cdots, p\}$, then $\emptyset \neq V \subset \bigcap_{i=1}^{p}\{x \in \mathbb{R}^n : \langle a_i, x\rangle \geq c_i\}$ and by Theorem 5 the set $V$ is co-star-shaped. Let now $I(V) \neq \{1, \cdots, p\}$. We have $I(V) = I(0^+V)$. Indeed, let $i_0 \in I(V)$, $x \in V$ $x' \in 0^+V$. Then $x + x' \in V$. Hence $c_{i_0} = \langle a_{i_0}, x + x'\rangle = c_{i_0} + \langle a_{i_0}, x'\rangle$ whence it follows that $\langle a_{i_0}, x'\rangle = 0$. Thus $I(V) \subset I(0^+V)$. To prove the inverse inclusion we consider a maximal linearly independent subset $\{a_{i_1}, \cdots, a_{i_k}\}$ of the set $\{a_i : i \in I(V)\}$. Since $I(V) \subset I(0^+V)$ and $\dim V = \dim 0^+V$ we have $a_i = \lambda_1^{(i)} a_{i_1} + \cdots + \lambda_k^{(i)} a_{i_k}$ for each $i \in I(0^+V)$. Then for $x \in V$ and $i \in I(0^+V)$ we obtain $\langle a_i, x\rangle = \lambda_1^{(i)} c_{i_1} + \cdots + \lambda_k^{(i)} c_{i_k}$ that is $I(0^+V) \subset I(V)$.

Let $z \in \operatorname{aff} V$. Then $\langle a_i, z\rangle = c_i$ for all $i \in I(V)$. There exists $\omega_0 \in 0^+V$ such that $\langle a_i, \omega_0\rangle < 0$ for all $i \in J = \{1, \cdots, p\} \setminus I(V)$. Consider $\omega = z - \lambda\omega_0$ with $\lambda > \max_{i\in J} \frac{\langle a_i, z\rangle - c_i}{\langle a_i, \omega_0\rangle}$. Then $\langle a_i, z - \lambda\omega_0\rangle = c_i$ for $i \in I(V)$ and $\langle a_i, z - \lambda\omega_0\rangle > c_i$ for $i \in J$. Hence by Theorem 5 the set $V$ is co-star-shaped. □

**Theorem 7.** *A nonempty polyhedral set*

$$V = \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \le \mu_i\}$$

*is strongly co-star-shaped iff* $\bigcap\limits_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle < 0\} \neq \emptyset$. *Herewith*

$$\mathit{kern}^* V \supset \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle > \mu_i\}.$$

*If (22) takes place, then*

$$\mathit{kern}^* V = \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle > \mu_i\}.$$

**Proof.** Necessity. Let $V$ be strongly co-star-shaped. According to Theorem 1 the set

$$\mathscr{C}_r V = \mathbb{R}^n \setminus \operatorname{int} V = \bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \ge \mu_i\}$$

is a proper strongly star-shaped subset of $\mathbb{R}^n$ and $\operatorname{kern}^* V = \operatorname{kern}_* \mathscr{C}_r V$. By Theorem 3 we have

$$\bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle < 0\} \neq \emptyset, \tag{25}$$

$$\bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle > \mu_i\} \subset \operatorname{kern}^* V,$$

and $\bigcap\limits_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle > \mu_i\} = \operatorname{kern}^* V$, if for each $\bar{i} \in 1 : p$

$$\bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \ge \mu_i\} \neq \bigcup_{\substack{i=1 \\ i\neq \bar{i}}}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \ge \mu_i\}. \tag{26}$$

The latter occurs iff for each $\bar{i} \in 1 : p$

$$\bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle < \mu_i\} \neq \bigcap_{\substack{i=1 \\ i\neq \bar{i}}}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle < \mu_i\} \text{ for all } \bar{i} \in 1 : p. \tag{27}$$

Under condition (25) we obtain (27) from (22).

Sufficiency. Let $\bigcap\limits_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle < 0\} \neq \emptyset$. According to Theorem 3 a proper subset $U = \bigcup\limits_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle \ge \mu_i\}$ of $\mathbb{R}^n$ is strongly star-shaped. By Lemma 1 $\operatorname{int} U = \bigcup\limits_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x\rangle > \mu_i\}$, therefore by Theorem 1 the set $V = \mathbb{R}^n \setminus \operatorname{int} U = \mathscr{C}_r U$ is strongly co-star-shaped. □

**Corollary 8.** *A nonempty polyhedral set $V$ is strongly co-star-shaped if and only if* $\dim(0^+V) = n$.

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