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ON INCREASE AT INFINITY OF ALMOST HYPOELLIPTIC POLYNOMIALS

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Abstract. It is proved that an almost hypoelliptic polynomial $P(\xi) = P(\xi_1, \cdots, \xi_n)$ is increasing at infinity, i. e. $|P(\xi)| \to \infty$ as $|\xi| \to \infty$, if and only if the number n of variables of P is invariant with respect to any linear nondegenerate transformation $T : R^n \longrightarrow R^n \cdot$

1 Introduction

We shal use following standard notation: $N-$ the set of all natural numbers, $N_0 = N \cup \{0\}$, $N_0^n = N_0 \times \cdots \times N_0-$ the set of all n-dimensional multi-indices, R^n- the n-dimensional Euclidian space.

For $\xi \in R^n$ and $\alpha \in N_0^n$ we put $|\xi| = \sqrt{\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $D_j = \partial/\partial\xi_j$ $(j = 1, \cdots, n)\cdot$

A polynomial $P(\xi) = P(\xi_1, \cdots, \xi_n)$ is said to be hypoelliptic (see [6]) if for all $0 \neq \nu \in N_0^n$

$$|P^{(\nu)}(\xi)|/|P(\xi)| \equiv |D^\nu P(\xi)|/|P(\xi)| \to 0$$

as $|\xi| \to \infty$.

A polynomial $P(\xi)$ is called hyperbolic with respect to the real vector $A \in R^n$ (see [3], or [6], Definition 12.3.3), if $P_m(A) \neq 0$ and there exists a real number τ_0 such that $P(\xi + i\tau A) \neq 0$, if $\xi \in R^n$ and $\tau < \tau_0$, where $P_m(\xi)$ is the m-homogeneous principal part of $P(\xi)$.

A polynomial $P(\xi)$ is called hyperbolic by Gårding if P is hyperbolic with respect to some vector A.

Definition 1.1. We say that a polynomial P is more powerful than a polynomial Q and write $Q < P$, if for some constant $C > 0$

$$|Q(\xi)| \leq C(1 + |P(\xi)|) \;\; \forall \xi \in R^n.$$

Definition 1.2. A polynomial P is called **almost hypoelliptic** (see [8]) if $D^\alpha P < P$ for all $\alpha \in N_0^n$.

Existence, uniqueness, smoothness etc. of solutions to many problems for general differential equations depend on the behaviour at infinity of the characteristic polynomials (complete symbols) of corresponding equations (operators).

Elliptic, semielliptic and hypoelliptic polynomials increase at infinity (see [6], Theorem 11.1.1), while hyperbolic by Gårding (consequently hyperbolic by Petrovsky) (see [3] and [12]) or almost hypoelliptic polynomials can remain bounded under infinite argument increase (see [7] or [5]).

Therefore the problem of obtaining conditions under which a polynomial P in several variables increases at infinity naturally arises. We denote by I_n the set of all polynomials $P(\xi) = P(\xi_1, \cdots, \xi_n)$ in n variables such that

$$|P(\xi)| \to \infty \quad \text{as} \quad |\xi| \to \infty.$$

Our purpose in the present paper is finding conditions on a non-hypoelliptic polynomial P under which $P \in I_n$.

In the general case this problem is not solved hitherto, but there are some results in this direction.

In connection with numerous problems of the theory of hypoelliptic differential equations B. Pini [13], L. Cattabriga [1], J. Friberg [2], E. Pehkonen [11] and others obtained conditions for $P \in I_n$ in various special cases. In our opinion the most general result is due to V.P. Mikhailov who in his work [10] studied a class of the so-called non-degenerate (regular) polynomials which belong to I_n. Similar results have been obtained by L.R. Volevich and S.G. Gindikin in [14].

In [4] we obtained necessary and sufficient conditions ensuring that a general two-dimensional polynomial belongs to I_2 in terms of multiplicity of the roots of certain homogeneous subpolynomials.

In [9] V.N. Margaryan and G.G. Tonoyan obtained necessary and sufficient conditions for two-dimensional almost hypoelliptic polynomials to belong to I_2 in terms of linear transformations.

Here we find necessary and sufficient conditions ensuring that an n-dimensional almost hypoelliptic polynomial belongs to I_n for any $n \geq 2$.

First consider following example: let $n = 2$ and

$$P(\xi) = P(\xi_1, \xi_2) = \sum_{j=0}^{m} a_j (\xi_1 + b\,\xi_2)^{2(m-j)},$$

where $m \in N$, $a_j \geq 0$, $(j = 0, 1, \cdots, m)$, $a_0 > 0$, $b \neq 0$.

It is easy to verify that

1) this is an almost hypoelliptic polynomial,

2) $P \notin I_2$,

3) by the linear non-degenerate transformation $\eta_1 = \xi_1 + b\xi_2$, $\eta_2 = \xi_2$ this polynomial goes into the polynomial in one variable

$$Q(\eta) = Q(\eta_1) = \sum_{j=0}^{m} a_j \eta_1^{2j}.$$

This simple example suggests us to investigate the relationship between the fact that $P \in I_n$ and the behaviour of P under linear non-degenerate transformations. This leads to the following definition

Definition 1.3. We call a polynomial $P(\xi) = P(\xi_1, \cdots, \xi_n)$ stable with respect to a linear non-degenerate transformation $T : R^n \longrightarrow R^n$, $T\xi = \eta$ (with respect to a non-degenerate matrix $T = (t_i^j)_{i,j=1}^n$) if the polynomial $Q_T(\eta) = P(T^{-1}\eta)$ depends on variables $\eta_1, \cdots, \eta_n$. A polynomial $P(\xi) = P(\xi_1, \cdots, \xi_n)$ is said to be unstable with respect to a linear nondegenerate transformation T if the polynomial $Q_T(\eta)$ depends on variables $\eta_{i_1}, \cdots, \eta_{i_k}$ with $k < n$. □

2 Some properties of almost hypoelliptic and general polynomials

Let

$$R(\xi) = R(\xi_1, \cdots, \xi_n) = \sum_{|\alpha|=m} r_\alpha \xi^\alpha \tag{2.1}$$

be a homogeneous polynomial of order m (in the sequel m-homogeneous),

$$\Sigma(R) = \{\xi \in R^n, R(\xi) = 0\}$$

and

$$\Sigma_m(R) = \{\xi \in \Sigma(R),\ ord_R\, \xi = m\} \equiv \{\xi \in \Sigma(R), \sum_{|\alpha|\le m-1} |D^\alpha R(\xi)| = 0\}.$$

It is easy to check that the set $\Sigma_m(R)$ is a linear manifold. Indeed, let $\eta^1, \eta^2 \in \Sigma_m(R)$. Then by Taylor's formula, for any numbers a and b, we obtain that

$$R(a\eta^1 + b\eta^2) = \sum_{|\alpha|\le m} a^{m-|\alpha|} \frac{R^{(\alpha)}(\eta^1)}{\alpha!}\ (b\eta^2)^\alpha = \sum_{|\alpha|=m} \frac{R^{(\alpha)}(\eta^1)}{\alpha!}\ (b\eta^2)^\alpha$$

$$= \sum_{|\alpha|=m} r_\alpha\ \alpha!\ \frac{1}{\alpha!}\ (b\eta^2)^\alpha = b^m \sum_{|\alpha|=m} r_\alpha\ (\eta^2)^\alpha = b^m R(\eta^2) = 0.$$

Lemma 2.1. *Let P be an almost hypoelliptic polynomial of order m, $T = (t_i^j)_{i,j=1}^n$ be a $n \times n$ non-degenerate matrix and $\eta = T\xi$. Then the polynomial $Q(\eta) = Q_T(\eta) = P(T^{-1}\eta)$ is almost hypoelliptic.*

Proof. Since for any $\alpha \in N_0^n$ the polynomial $D_\eta^\alpha Q(\eta) = D_\eta^\alpha P(T^{-1}\eta)$ is a linear combination of $\{(D_\xi^\beta P)(T^{-1}\eta)\}$ for $\beta \in N_0^n$, $|\beta| = |\alpha|$, it follows that using the almost hypoellipticity of P there exist positive constants C_1 and C_2 such that

$$\sum_{|\alpha|\le m} |D_\eta^\alpha Q(\eta)| = \sum_{|\alpha|\le m} |D_\eta^\alpha P(T^{-1}\eta)|$$

$$\le C_1 \sum_{|\beta|\le m} |(D_\xi^\beta P)(T^{-1}\eta)\}| \le C_2[1 + |P(T^{-1}\eta)| = C_2[1 + |Q(\eta)|\,] \quad \forall \eta \in R^n.$$

□

For an m-homogeneous polynomial $R(\xi) = R(\xi_1, ..., \xi_n)$ by $\sigma_{n,m}(R)$ we denote the maximal number of linearly independent (in R^n) elements $\xi \in \Sigma_m(R)$.

Clearly every polynomial $P(\xi) = \sum_{|\alpha|\leq m} \gamma_\alpha \xi^\alpha$ of order m can be represented as the sum of j-homogeneous polynomials $(j = 0, 1, \cdots, m)$:

$$P(\xi) = \sum_{j=0}^{m} P_j(\xi) = \sum_{j=0}^{m} \sum_{|\alpha|=j} \gamma_\alpha \xi^\alpha \cdot \tag{2.2}$$

Lemma 2.2. *If an almost hypoelliptic polynomial $P \notin I_n$, then*

$$1 \leq \sigma_{n,m}(P_m) \leq n-1 \cdot \tag{2.3}$$

Proof. It is required to prove only the left-hand side of (2.3). Since the set $\Sigma_m(P_m)$ is a linear manifold, to prove the left-hand side of (2.3) it siffices to show the existence of a non-zero point $\eta \in \Sigma_m(P_m)$.

Since $P \notin I_n$, there exists a sequence $\{\xi^s\}$ and a constant $a_1 > 0$ such that $|\xi^s| \to \infty$ as $s \to \infty$ and

$$|P(\xi^s)| \leq a_1 \quad (s = 1, 2, \cdots) \cdot \tag{2.4}$$

The vectors $\eta^s = \xi^s / |\,\xi^s|$ $(s = 1, 2, \cdots)$ are of unit length, hence the sequence $\{\eta^s\}$ has an accumulation point $\eta \in R^n$, $|\,\eta| = 1$ and, by passing to a subsequence we may assume that $\eta^s \to \eta$ as $s \to \infty$. It is easily seen that $\eta \in \Sigma(P_m) \cdot$ Let us show that $\eta \in \Sigma_m(P_m)$.

Since $D^\alpha(P - P_m)(\xi) = const \equiv C_\alpha$ for any $\alpha \in N_0^n, |\alpha| = m-1$, by the almost hypoellipticity of P we obtain

$$\sum_{|\alpha|=m-1} |D^\alpha P_m(\xi)| = \sum_{|\alpha|=m-1} |\,[D^\alpha P_m(\xi) + C_\alpha] - C_\alpha|$$

$$= \sum_{|\alpha|=m-1} |D^\alpha P(\xi) - C_\alpha| \leq \sum_{|\alpha|=m-1} |D^\alpha P(\xi)| + a_2 \leq a_3[1 + |P(\xi)|\,] \quad \forall \xi \in R^n,$$

where

$$a_2 = \sum_{|\alpha|=m-1} |C_\alpha|, \quad a_3 = a_3(P, a_2) > 0.$$

By this and (2.4) it follows that

$$\sum_{|\,\alpha|=m-1} |D^\alpha P_m(\xi^s)\,| \leq a_3(a_1 + 1) \quad (s = 1, 2, \cdots) \cdot \tag{2.5}$$

Since $|\,\xi^s| \to \infty$, $|\,\eta^s| \to \eta$, as $s \to \infty$ and for $|\,\alpha| = m-1$ the polynomial $D^\alpha P_m$ is a linear homogeneous function, it follows by (2.5) that $D^\alpha P_m(\eta) = 0$ for all $\alpha \in N_0^n, |\,\alpha\,| = m-1 \cdot$ Therefore by the generalized Euler's formula for homogeneous functions we obtain that for any $\beta \in N_0^n, |\beta| \leq m-1$

$$D^{\beta}P_m(\eta) = \frac{(|\beta|+1)!}{(m-|\beta|)!} \sum_{|\gamma|=m-|\beta|-1} \frac{1}{\gamma!} D^{\beta+\gamma}P_m(\eta)\,\eta^{\gamma} = 0,$$

i.e. $\eta \in \Sigma_m(P_m)$, which proves the left-hand side of (2.3). □

Remark 2.1. The assumption of almost hypoellipticity in the above lemma is essential. Indeed, for the 2-homogeneous not almost hypoelliptic polynomial $P(\xi) = \xi_1^2 - \xi_2^2 \notin I_2$ the set $\Sigma_2(P) \setminus \{0\}$ is empty, i.e. $\sigma_{2,2} = 0$.

The following three statements are true for arbitrary polynomials.

Lemma 2.3. *Let P be a polynomial of order m, T be a non-degenerate $n \times n$ matrix, $\eta = T\xi$ and $Q(\eta) = P(T^{-1}\eta)$. Then*

a) $ord\, Q = m$, and if P and Q are represented in the form (2.2), then $Q_j(\eta) = P_j(T^{-1}(\eta))$, $j = 1, \cdots, m$

b) if P is an m-homogeneous polynomial then

b1) Q is also m-homogeneous,

b2) $T : \Sigma_m(P) \longrightarrow \Sigma_m(Q)$.

Proof. First we prove statement b1). For any $t > 0$ and $\eta \in R^n$ we have $Q(t\,\eta) = P(T^{-1}(t\,\eta)) = P(t\,T^{-1}\eta) = t^m P(T^{-1}\eta) = t^m Q(\eta)$, which proves statement b1).

To prove statement a) let us represent P and Q as the sum of homogeneous polynomials (see (2.2), where $m(Q)$ is unknown). By the linearity of the transformation T and by the proved part of the lemma $Q_j(\eta) = P_j\,(T^{-1}\eta) = P_j(\xi)$ $(j = 0, 1, \cdots, m)$ and $ord\, Q = \max_{0 \le j \le m}\{ord\, Q_j\} = ord\, P_m = m$, which proves statement a).

Since it is obvious that $T\tau \in \Sigma(Q)$ for $\tau \in \Sigma(P)$ and

$$\Sigma_m(P_m) = \bigcap_{|\,\alpha| \le m-1} \Sigma(D^{\alpha}P_m),$$

we get $T(\,\Sigma_m\,(P_m)) = \Sigma_m\,(Q_m)$, which proves statement b2). □

Lemma 2.4. *Let a polynomial P be unstable with respect to a linear non-degenerate transformation $T : R^n \longrightarrow R^n$ (see Definition 1.3) and $Q < P$ (see Definition 1.1). Then Q is unstable with respect to T.*

Proof. By the unstability of P there exists a number $k \in N$, $k \le n-1$ such that (by renumbering of variables) $p\,(\eta) \equiv P(T^{-1}\eta) = p\,(\eta_1, \cdots, \eta_k)$. Since $Q < P$, there exists a constant $C > 0$ such that for all $\xi, \eta \in R^n$

$$|\,q\,(\eta)| \equiv |\,Q(T^{-1}\eta)| = |\,Q(\xi)| \le C[1 + |\,P(\xi)\,|]$$

$$= C[1 + |\,P\,(T^{-1}\eta)|] = C[1 + |\,p\,(\eta_1, \cdots, \eta_k)|\,],$$

which means that the polynomial q depends only on the variables $\eta_1, \cdots, \eta_k$, i.e. the polynomial Q is unstable with respect to T. □

Lemma 2.5. *Let $P \in I_n$. Then P is stable with respect to any linear non-degenerate transformation.*

Proof. Assume to the contrary that P is unstable with respect to some non-degenerate $n \times n$ matrix $T = (t_i^j)_{i,j=1}^n$ and $\eta = T\xi$. Then $p(\eta) = P(T^{-1}\eta)$ is a polynomial in k variables $\eta_1, \cdots, \eta_k$, with $k \leq n-1$.

Since $k < n$ the system of linear algebraic equations

$$\sum_{j=1}^{n} t_i^j \xi_j = 0, \quad i = 1, \cdots, k \tag{2.6}$$

has a nonzero solution $\tau \in R^n$ and by the homogeneity of system (2.6) $\xi^s = s\tau$ will be a solution to (2.6) for any $s \in N\cdot$ Then $\eta^s = T\xi^s = 0$ for all $s \in N$ and $P(\xi^s) = p(\eta^s) = const$ $(s = 1, 2, \cdots)\cdot$ Since $|\xi^s| = s|\tau| \to \infty$ as $s \to \infty$, this contradicts the condition $P \in I_n\cdot$ □

Lemma 2.6. *Let R be a m-homogeneous polynomial $(m \geq 1)$ with $\Sigma_m(R) = \{0\}$. Then there exists a constant $C > 0$ such that*

$$|\xi| \leq C[1 + \sum_{|\alpha|=m-1} |D^\alpha R(\xi)|] \quad \forall \xi \in R^n\cdot \tag{2.7}$$

Proof. If the set of all linear homogeneous polynomials $\{D^\alpha R;\ |\alpha| = m-1\}$ has no common real non-zero root then

$$r(\xi) = \sum_{|\alpha|=m-1} |D^\alpha R(\xi)|^2$$

is an elliptic homogeneous polynomial of order two for which the inequality

$$|\xi|^2 \leq C_1[1 + r(\xi)] \quad \forall \xi \in R^n$$

with a constant $C_1 > 0$ is well known. This inequality implies inequality (2.7).

Let $0 \neq \tau \in R^n$ be a common real root of the polynomials $\{D^\alpha R;\ |\alpha| = m-1\}$. We will show that $\tau \in \Sigma_m(R)$. For this it sufficies to prove that τ is a common real root of the homogeneous polynomials $\{D^\alpha R;\ 0 \leq |\alpha| \leq m-2\}$.

By the generalized Euler's formula for homogeneous functions we have that for any homogeneous polynomial $D^\alpha R,\ 0 \leq |\alpha| \leq m-2$

$$D^\alpha R(\tau) = \frac{1}{(m-|\alpha|)!} \sum_{|\beta|=m-|\alpha|-1} \frac{(D^\beta D^\alpha R)(\tau)}{\alpha!} \tau^\beta$$

$$= \frac{1}{(m-|\alpha|)!} \sum_{|\gamma|=m-1, \gamma \geq \alpha} \frac{D^\gamma R(\tau)}{(\gamma-\alpha)!} \tau^{\gamma-\alpha} = 0.$$

This means that $0 \neq \tau \in \Sigma_m(R)$, which contradicts the assumtion $\Sigma_m(R) = \{0\}$. □

Futher assume that if the polynomial $P(\xi) = P(\xi_1, \cdots, \xi_n)$ is represented in form (2.2) and the principal part P_m depends on variables $\xi_1, \cdots, \xi_k$: $P_m(\xi) = P_m(\xi_1, \cdots, \xi_k)$ for some $k < n$, then by $\Sigma_{m,k}(P_m)$ we denote the set of points $\tau \in R^k$ such that $D^\alpha P_m(\tau) = 0$ for all $\alpha \in N_0^k$, $|\alpha| = m-1$.

Corollary 2.1. *Let the $m-$homogeneous principal part P_m of the polynomial $P(\xi) = P(\xi_1, \cdots, \xi_n)$ depend on the variables $\xi_1, \cdots, \xi_k$ for some $k < n$ and $\Sigma_{m,k}(P_m) = \{0\}$. Then there exists a constant $C > 0$ such that*

$$\sqrt{\xi_1^2 + \cdots + \xi_k^2} \leq C[1 + \sum_{|\alpha|=m-1} | D^\alpha P(\xi)|] \quad \forall \xi \in R^n. \tag{2.8}$$

Proof. Using representation (2.2) we have that for all $\xi \in R^n$

$$\sum_{\alpha \in N_0^k, |\alpha|=m-1} |D^\alpha P_m(\xi)| = \sum_{\beta \in N_0^n, |\beta|=m-1} |D^\beta P_m(\xi)|$$

$$= \sum_{\beta \in N_0^n,\, |\beta|=m-1} |D^\beta P(\xi) - \sum_{j=0}^{m-1} D^\beta P_j(\xi)| \leq \sum_{|\beta|=m-1} |D^\beta P(\xi)| + const,$$

therefore (2.8) follows from (2.7). □

Lemma 2.7. *Let an almost hypoelliptic polynomial $P(\xi) = P(\xi_1, \cdots, \xi_n)$ be represented in form (2.2), where the principal part P_m depends on the variables $\xi_1, \cdots, \xi_k$ for some $k < n$ and $\Sigma_{m,\,k}(P_m) = \{0\}$. We put $\xi' = (\xi_1, \cdots, \xi_k)$, $\xi'' = (\xi_{k+1}, \cdots, \xi_n)$ and represent the polynomial P in the form*

$$P(\xi) = P(\xi', \xi'') = P_m(\xi') + \sum_{\alpha' \in N_0^k} (\xi')^{\alpha'} r_{\alpha'}(\xi'')$$

$$= P_m(\xi') + r_{0'}(\xi'') + \sum_{0 \neq \alpha' \in N_0^k} (\xi')^{\alpha'} r_{\alpha'}(\xi''), \tag{2.9}$$

where $\alpha' = (\alpha_1, \cdots, \alpha_k)$, $|\alpha'| + ord\, r_{\alpha'} \leq m-1$.

Then $P \in I_n$ if and only if $r_{0'} \in I_{n-k}$.

Proof. Since $P(0', \xi'') = r_{0'}(\xi'')$ the necessity is obvious.

To prove the sufficiency we argue by contradiction. Suppose that $r_{0'} \in I_{n-k}$ and $P \notin I_n$, i.e. there exist a sequence $\xi^s = \{(\xi_1^s, \cdots, \xi_n^s)\}_{s=1}^\infty$ and a number $C_1 > 0$ such that $|\xi^s| \to \infty$ as $s \to \infty$ and

$$|P(\xi^s)| \leq C_1 \quad (s = 1, 2, \cdots)\cdot \tag{2.10}$$

This, together with Corollary 2.1, implies

$$| (\xi')^s| \equiv \sqrt{(\xi_1^s)^2 + ... + (\xi_k^s)^2} \leq C_2 \quad (s = 1, 2, \cdots) \tag{2.11}$$

with a number $C_2 > 0$.

In addition let us show that (under the assumptions of the lemma) (2.11) implies that there exists a constant $C_3 > 0$ such that for all $\alpha^{'} \in N_0^k,\ 0 < |\alpha^{'}| \leq m-1$

$$|\, r_{\alpha'}((\xi^{''})^s)| \leq C_3 \quad (s = 1, 2, \cdots). \tag{2.12}$$

If $r_{\alpha'}((\xi^{''})^s) = 0$ for all $s \in N$, the estimate (2.12) is obvious. Let $r_{\alpha'}((\xi^{''})^s) \neq 0$ for infinitely many $s \in N\cdot$ This estimate we prove by the reverse induction on $\alpha^{'} \in N_0^k$; $|\alpha^{'}| \geq 1$.

For $\alpha^{'} \in N_0^k$; $|\alpha^{'}| = m-1$ (2.12) immediately follows by the condition $|\alpha^{'}| + ord\, r_{\alpha'} \leq m-1$.

Assume that estimate (2.12) hold for $\alpha^{'} \in N_0^k$; $2 \leq r \leq |\alpha^{'}| \leq m-1$ and prove it for $|\alpha^{'}| = r-1$.

By the almost hypoellipticity of P we have for any $\alpha^{'} \in N_0^k,\ |\alpha^{'}| = r-1$ (see representation (2.9))

$$|D^{\alpha^{'}} P(\xi)| = |D^{\alpha^{'}} P_m(\xi^{'}) + \sum_{\beta^{'} \geq \alpha^{'}, |\beta^{'}| > |\alpha^{'}} \frac{(\beta^{'})!}{(\beta^{'} - \alpha^{'})!} (\xi^{'})^{\beta^{'} - \alpha^{'}} r_{\beta^{'}}(\xi^{''})$$

$$+ (\alpha^{'}!)\, r_{\alpha'}(\xi^{''})| \leq C_4 [\, 1 + |P(\xi)|\,] \quad \forall \xi \in R^n$$

with a constant $C_4 > 0\cdot$

From here, by the inductive hypothesis and applying estimates (2.10), (2.11) we get with positive constants C_5 and C_6

$$|r_{\alpha'}[\,(\xi^{''})^s]\,| \leq C_5 \{\, [\, 1 + |P(\xi^s)|\,] + |D^{\alpha^{'}} P_m((\xi^{'})^s)| +$$

$$+ \sum_{\beta^{'} \geq \alpha^{'}, |\beta^{'}| > |\alpha^{'}} \frac{(\beta^{'})!}{(\beta^{'} - \alpha^{'})!} |\,[\,(\xi^{'})^s\,]^{\beta^{'} - \alpha^{'}}|\, |r_{\beta'}[\,(\xi^{''})^s\,]\,|\} \leq C_6 \quad (s = 1, 2, \cdots),$$

which proves inequality (2.12) for all $\alpha^{'} \in N_0^k$; $0 < |\alpha| \leq m-1$.

Applying estimates (2.10) - (2.12) we get with a constant $C_7 > 0$

$$|r_{0'}[\,(\xi^{''})^s\,]\,| = |P(\xi^s) - P_m[\,(\xi^{'})^s\,] - \sum_{|\alpha^{'}| \geq 1} [\,(\xi^{'})^s\,]^{\alpha^{'}} r_{\alpha'}[\,(\xi^{''})^s\,]\,| \leq C_7 \ \ (s = 1, 2, \cdots),$$

which contradicts the assumption $r_{0'} \in I_{n-k}\cdot$ □

Corollary 2.2. *Assume that the hypothesis of Lemma 2.7 are fulfilled except condition* $\Sigma_{m,k}(P_m) = \{0\}$ *and let* $P \notin I_n$. *Then* $r_{0'} \notin I_{n-k}$.

3 Main result

Here we prove the converse of Lemma 2.5 for almost hypoelliptic polynomials.

Lemma 3.1. *Let an almost hypoelliptic polynomial* $P(\xi) = P(\xi_1, \cdots, \xi_n)$ *be stable with respect to any linear nondegenerating transformation. Then* $P \in I_n$.

Proof. Suppose that there is an almost hypoelliptic polynomial $P \notin I_n$, i.e. there exist a sequence $\{\xi^s\}$ and a constant $C > 0$ such that $|\xi^s| \to \infty$ as $s \to \infty$ and

$$|P(\xi^s)| \leq C \quad (s = 1, 2, \cdots)\cdot \tag{3.1}$$

Let us show that P is unstable with respect to a sertain linear non-degenerate transformation. We prove this by induction on n.

For $n = 2$ this statement is proved in [9], however we give here the proof in the case $n = 2$ in terminology convenient for us.

Let the polynomial $P(\xi) = P(\xi_1, \xi_2)$ be represented in form (2.2). Since $P \notin I_n$, by Lemma 2.2 $\sigma_{2,\,m}(P_m) = 1$, i.e. the set $\Sigma_{\,m}(P_m)$ contains a non-zero element. Let $\tau \in \Sigma_{\,m}(P_m)$, $|\tau| = 1$. Then either $\tau = (0, 1)$, or $\tau = (1, 0)$, or $\tau_1\tau_2 \neq 0$. Since the first two cases can be treated analogously, we consider only two possibilites 1) $\tau = (1, 0)$, 2) $\tau_1\tau_2 \neq 0$.

In the first case $P_m(\xi_1, \xi_2) = \gamma_m\, \xi_2^m$ with $\gamma_m \neq 0$. Without loss of generality it can be assumed that $\gamma_m = 1$. Then (see (2.9))

$$P(\xi) = \xi_2^m + \sum_{j=0}^{m-1} \xi_2^j\, q_j(\xi_1).$$

Since $P \notin I_2$, by Corollary 2.2 $q_0 \notin I_1$. On the other hand, since q_0 is a polynomial in one variable, $ord\, q_0 = 0$, i.e. $q_0(\xi_1) \equiv const \equiv C_0$.

By this and the almost hypoellipticity of P we obtain that, for any $j:\ 1 \leq j \leq m-1$, with a constant $\kappa_1 > 0$

$$|\, q_j(\xi_1)\,| = \frac{1}{j!}\,|\, D_2^j P(\xi_1, 0)| \leq \kappa_1\,[\,1 + |\,P(\xi_1, 0)\,|\,]$$

$$= \kappa_1[1 + |\,q_0(\xi_1)|\,] = \kappa_1(C_0 + 1) \quad \forall \xi_1 \in R^1.$$

This means that $q_j(\xi_1) \equiv const \equiv C_j$ for all $j = 1, \cdots, m-1$, i.e. the polynomial $P(\xi) = \xi_{\,2}^{\,m} + \Sigma_{j=0}^{m-1}\, C_j\, \xi_{\,2}^{\,j}$ depends only on one variable, consequently P is unstable with respect to the identity transformation.

It is easy to verify that in the case 2) $\tau_1\tau_2 \neq 0$ the polynomial P_m can be represented in the form $P_m(\xi) = \gamma(\tau_2\xi_1 - \tau_1\xi_2)^m$, where $\gamma \neq 0$ (see also [7], Lemma 1.1) and, without loss of generality, we assume that $\gamma = 1$.

Let us perform the following change of variables: $\eta_1 = \tau_2\xi_1 - \tau_1\xi_2$, $\eta_2 = \tau_1\xi_1 + \tau_2\xi_2$. This is a non-degenerate linear transformation with the matrix T for which $det\, T = det\, T^{-1} = \tau_1^2 + \tau_2^2 = 1$. Then

$$Q(\eta) = P(\tau_2\eta_1 + \tau_1\eta_2,\ -\tau_1\eta_1 + \tau_2\eta_2) = \eta_1^m + \sum_{j=0}^{m-1} Q_j(\eta_1, \eta_2),$$

where Q_j is a $j-$homogeneous polynomial $(j = 0, 1, \cdots, m-1)$.

Since the matrix T is non-degenerate, by Lemma 2.1 the polynomial Q is almost hypoelliptic. On the other hand since by our assumption $P \notin I_2$ and the matrix T is invertible, we have $Q \notin I_2$.

Now arguing as in case 1) we get

$$Q(\eta_1,\eta_2)=\eta_1^m+\sum_{j=0}^{m-1}C_j\,\eta_1^j\equiv Q(\eta_1),$$

i.e. P is unstable with respect to the non-degenerate linear transformation $\eta=T\xi$, which proves our statment for $n=2$.

Assume that our statement is proved for $2\le n\le r-1$ and prove it for $n=r$.

Since $P\notin I_r$, by Lemma 2.2 $1\le\sigma_{r,m}\equiv l\le r-1\cdot$ Let $\{\tau^1,\cdots,\tau^l\}$ form a basis in $\Sigma_m(P_m)$ and $\{\tau^1,\cdots,\tau^l,\tau^{l+1},\cdots,\tau^r\}$ form a basis in R^r. Denote by U the $r\times r$ matrix $U=(\tau_i^j)$ and let $\eta=U^{-1}\xi\cdot$ By Lemma 2.1 the polynomial $Q(\eta)=P(U\eta)$ is almost hypoelliptic.

Represent Q as the sum of j-homogeneous polynomials $(j=0,1,\cdots,m(Q))$ (see formula (2.2)), where by Lemma 2.3 $m(Q)=m$

$$Q(\eta)=\sum_{j=0}^{m}Q_j(\eta)\cdot$$

Consider the vectors $e^j=U^{-1}\tau^j$ $(j=1,\cdots,r)$, which form a basis in $R^r\cdot$ Here $e_i^j=\delta_i^j$ $(i=1,\cdots,r;j=1,\cdots,l)$, $e_i^{l+k}=0$ $(k=1,\cdots,r-l;i=1,\cdots,l)\cdot$

Since $Q_m(\eta)=P_m(U\eta)$ for all $\eta\in R^r$, we see that $P_m(Ue^j)=P_m(\tau^j)=0$ for all $(j=1,\cdots,l)$. On the other hand by Lemma 2.3 $e^j\in\Sigma_m(Q_m)$ $(j=1,\cdots,l)$.

Let us show that the polynomial Q_m in fact depends only on the variables $\eta_{l+1},\cdots,\eta_r$. For this purpose we prove that for all $\eta\in R^r$ $Q_m(\eta)=Q_m(0,\cdots,0,\eta_{l+1},\cdots,\eta_r)$.

First we show that $Q_m(\eta)=Q_m(0,\eta_2,\cdots,\eta_r)$ for all $\eta\in R^r$. We represent Q_m in the form

$$Q_m(\eta)=\sum_{j=0}^{m}\eta_1^{m-j}q_m^j(\eta_2,\cdots,\eta_r)$$

and prove that $q_m^j(\eta_2,\cdots,\eta_r)\equiv 0$ for all $j=0,1,\cdots,m-1$.

By the j-homogenity of the polynomials $\{q_m^j\}$, $q_m^j(0,\cdots,0)=0$ for $j=1,\cdots,m-1$ and $q_m^0(\eta_2,\cdots,\eta_n)=const\equiv C_0$. On the other hand

$$0=Q_m(e^1)=\sum_{j=0}^{m}1^{m-j}q_m^j(0,\cdots,0)=q_m^0(0,\cdots,0)=C_0=q_m^0(\eta_2,\cdots,\eta_r),$$

therefore $q_m^0(\eta_2,\cdots,\eta_r)\equiv 0$.

Now we proceed by induction in $j=0,1,\cdots,m-1$. Assume that the identity $q_m^j(\eta_2,\cdots,\eta_r)=0$ for all $\eta\in R^r$ holds for $j=1,\cdots,k-1$ $(1\le k\le m-1)$ we will prove it for $j=k$.

By the inductive hypothesis the polynomial Q_m can be represented in the form

$$Q_m(\eta)=\sum_{j=k}^{m}\eta_1^{m-j}\,q_m^j(\eta_2,\cdots,\eta_r)\cdot \tag{3.2}$$

Let

$$q^{\,k}_{\,m}(\eta_2,\cdots,\eta_r)=\sum_{\alpha_2+\cdots+\alpha_r=k}\gamma_{\,(\alpha_2,\,\cdots,\,\alpha_r)}\,\eta_2^{\;\alpha_{\,2}}\cdots\eta_r^{\alpha_{\,r}}\cdot \tag{3.3}$$

Since $e^1\in\Sigma_m(Q_m)$, by (3.2) and (3.3) we have, for any multi-index $\beta=(\beta_2,\cdots,\beta_r):\ |\beta|=k<n$,

$$0=D^\beta Q_m(e^1)=\sum_{j=k}^{m}1^{m-j}D^\beta q^j_m(0,\cdots,0)=D^\beta[\eta_1^{m-k}q^k_m(\eta_2,\cdots,\eta_r)\,]\,|_{\,\eta=e^1}$$

$$+D^\beta[\sum_{j=k+1}^{m}\eta_1^{m-j}q^j_m(\eta_2,\cdots,\eta_r)]|_{\,\eta=e^1}=D^\beta[\gamma_\beta\eta^\beta\,]\,|_{\,\eta=e^1}$$

$$+D^\beta[\sum_{|\alpha|=k,\alpha\neq\beta}\gamma_\alpha\eta^\alpha\,]\,|_{\,\eta=e^1}+D^\beta[\sum_{j=k+1}^{m}\eta_1^{m-j}q^j_m(\eta_2,\cdots,\eta_r)\,]\,|_{\,\eta=e^1},$$

where

$$D^\beta[\,\gamma_\beta\eta^\beta\,]\,|_{\,\eta=e^1}=(\beta!)\gamma_\beta,\ \ D^\beta[\sum_{|\alpha|=k,\;\alpha\neq\beta}\gamma_\alpha\eta^\alpha\,]\,|_{\,\eta=e^1}=0\cdot$$

Therefore

$$(\beta!)\gamma_\beta+\sum_{j=k+1}^{m}[\,D^\beta q^j_m\,](0,\cdots,0)=0\cdot \tag{3.4}$$

Since for $j=k+1,\cdots,m$ $D^\beta q^j_m$ is a $(j-|\beta|)=(j-k)$-homogeneous polynomial and $j-k\geq 1$, we have $[D^\beta q^j_m](0,\cdots,0)=0$ for $j=k+1,\cdots,m$ and by (3.4) it follows that $\gamma_\beta=0$ for any multiindex $\beta=(\beta_2,\cdots,\beta_r):\ |\beta|=k$. By (3.3) this means that $q^k_m(\eta_2,\cdots,\eta_r)=0$ for all $\eta\in R^r$.

By the inductive hypothesis this means that $Q_m(\eta_1,\eta_2,\cdots,\eta_r)=Q_m(0,\eta_2,\cdots,\eta_r)$ for all $\eta\in R^r$, i.e. the polynomial Q_m does not depend on the variable η_1.

In the same way we can see that $Q_m(\eta)=Q_m(0,\cdots,0,\eta_{l+1},\cdots,\eta_r)$ for all $\eta\in R^r$, i.e. that the polynomial Q_m in fact depends on variables $\eta_{l+1},\cdots,\eta_r$ only.

Next we put $\eta^{'}=(\eta_1,\cdots,\eta_l)$ and $\eta^{''}=(\eta_{l+1},\cdots,\eta_r\,)$ and represent (see (2.9)) the polynomial Q in form

$$Q(\eta)=Q(\eta^{'},\eta^{''})=Q_m(\eta^{''})+\sum_{\alpha^{''}\in N_0^{r-l}}(\eta^{''})^{\alpha^{''}}q_{\alpha^{''}}(\eta^{'}), \tag{3.5}$$

where $|\alpha^{''}|+ord q_{\alpha^{''}}\leq m-1$ and $\sigma_{k,\,m}(Q_m)=0$, i.e. $\Sigma_{m,\,r-l}(Q_m)=\{0\}$.

Since $\Sigma_{m,r-l}(Q_m)=\{0\}$, and $Q\notin I_r$ it follows by Corollary 2.2 that $q_{0^{''}}\notin I_l$.

Since any nonzero polynomial q in one variable of $ord\,q\geq 1$ increases at infinity and the polynomial $q_{0^{''}}$ depends on l variables it follows that either

1) $l=1$ and then $q_{0^{''}}(\eta^{'})=const\equiv C_{0^{''}}$ for all $\eta^{'}\in R^l$

or

2) $2\leq l\leq r-1$.

By the almost hypoellipticity of Q, in case 1) we obtain, with a constant $\kappa_2 > 0$, for all $\alpha^{''} \in N_0^{r-l}$

$$|q_{\alpha''}(\eta^{'})| = \frac{1}{\alpha''!}|(D^{\alpha''}Q)(\eta^{'}, 0^{''})| \leq \kappa_2(1 + |Q(\eta^{'}, 0^{''})|)$$

$$= \kappa_2(1 + |q_{0''}(\eta^{'})| = \kappa_2(1 + |C_{0''}|) \quad \forall \eta^{'} \in R^l.$$

It follows immediately that $q_{\alpha''}(\eta^{'}) = const \equiv C_{\alpha''}$ for any $\alpha^{''} \in N_0^{r-l}$ and for all $\eta^{'} \in R^l$, i.e.

$$Q(\eta) = Q_m(\eta^{''}) + \sum_{\alpha'' \in N_0^{r-l}} C_{\alpha''}(\eta^{''})^{\alpha^{''}} = Q(0^{'}, \eta^{''}),$$

which in turn means that the polynomial P is unstable with respect to a linear non-degenerate transformation $U : R^r \longrightarrow R^r$.

Consider the case 2) $l \geq 2$. Since Q is almost hypoelliptic, it is easily seen that the polynomial $q_{\alpha''}(\eta^{'}) = Q(\eta^{'}, 0^{''})$ is almost hypoelliptic too. On the other hand since $\Sigma_{m,r-l}(Q_m) = \{0\}$, $l < r$ and $q_{0''} \notin I_l$, by the inductive hypothesis $q_{0''}$ is unstable. This means that there exist a number $l_1 \in N$, $l_1 \leq l-1$ and a non-degenerate matrix $V = (v_i^j)_{i,j=1}^l : R^l \longrightarrow R^l$ such that $\xi' = V\eta'$ and

$$q_{0''}(\eta^{'}) = q_{0''}(V^{-1}\xi') = q^V_{0''}(\xi_1, \cdots, \xi_{l_1}) \quad \forall \eta^{'} \in R^l.$$

Applying again the almost hypoellipticity of Q, we obtain, with a constant $\kappa_3 > 0$, for any $\alpha^{''} \in N_0^{r-l}$ and for all $\eta^{'} \in R^l$

$$|q_{\alpha''}(\eta^{'})| = \frac{1}{\alpha''!}|D^{\alpha''}Q(\eta^{'}, 0^{''}| \leq \kappa_3|Q(\eta^{'}, 0^{''}| \leq \kappa_3[1 + |q_{0''}(\eta^{'})|].$$

By Lemma 2.4 it follows that

$$q_{\alpha''}(\eta^{'}) = q_{\alpha''}(V^{-1}\xi) = q^V_{\alpha''}(\xi_1, \cdots, \xi_{l_1}) \quad \forall \eta^{'} \in R^l.$$

Denote by $t_i^j = v_i^j$ for $i,j = 1, \cdots, l$; $t_i^j = \delta_i^j$ for $i,j = l+1, \cdots, r$; $t_i^j = 0$ in all other cases and put $H = (t_i^j)_{i,j=1}^r$. It is obvious that H is a non-degenerate matrix. Let $T = U\,H$, $\eta = U^{-1}\xi$, $z = H^{-1}\eta$ then $z = T^{-1}\xi$ and

$$P(\xi) = P(Tz) = Q(z) = Q_m(z^{''}) + \sum_{\alpha''}(z^{''})^{\alpha^{''}} q_{\alpha''}(V^{-1}z^{'}) =$$

$$= Q_m(z^{''}) + \sum_{\alpha''}(z^{''})^{\alpha^{''}} q^V_{\alpha''}(z^{'}) = Q_m(z^{''}) + \sum_{\alpha''}(z^{''})^{\alpha^{''}} q^V_{\alpha''}(z_1, \cdots, z_{l_1}).$$

This means that the polynomial $Q(z)$ does not depend on the variables $z_{l_1+1}, \cdots, z_l$, i.e. the polynomial P is unstable with respect to the linear non-degenerate transformation $T : R^r \longrightarrow R^r$. By the inductive hypothesis this completes the proof of the lemma. □

Lemmas 2.5 and 3.1 combined give the main result of this paper:

Theorem 3.1. *An almost hypoelliptic polynomial* $P(\xi) = P(\xi_1, \cdots, \xi_n)$ *belongs to* I_n *if and only if* P *is stable with respect to any linear nondegenerate transformation* $T : R^n \longrightarrow R^n$.

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