Eurasian
Mathematical
Journal

EURASIAN MATHEMATICAL JOURNAL

Editorial Board

Editors–in–Chief

V.I. Burenkov, M. Otelbaev, V.A. Sadovnichy

Editors

Managing Editor

A.M. Temirkhanova

Executive Editor

D.T. Matin

SCHUR TEST FOR THE HARDY OPERATOR

E.I. Berezhnoi

Communicated by E.D. Nursultanov

Abstract. For monotonic functions necessary and sufficient conditions are investigated ensuring the equivalence of a function and of an integral containing that function. Factorization theorems (Schur tests) are proved for the classical Hardy operator and its adjoint in Lebesque spaces with monotonic weights.

1 Introduction

The Schur test or, in other words, the Schur extrapolation theorem (see, for example, [6], p. 37, [11], p. 42), is well known in the theory of integral operators with positive kernels. It states that the integral operator K, defined by

$$Kx(t) = \int_{R_+} k(t,s)x(s)ds, \quad t \in R_+,$$

whose kernel satisfies the inequality $k(t,s) \geq 0$ for almost all t, s, is bounded in $L^p \equiv L^p(R_+)$ with $1 < p < \infty$ if and only if there is a positive, almost everywhere finite function u, such that the operator K is bounded both as $K : L_u^\infty \to L_u^\infty$ and $K : L_v^1 \to L_v^1$ where $v = u^{1/p-1}$. In recent years, in connection with various problems of analysis, interest to extrapolation has significantly increased [10, 4, 5, 1, 3]. However, it seems that in all known proofs of the Schur theorem the weight function u is not constructed explicitly which does not allow to investigate its properties.

In this paper, for the classical Hardy operator and its adjoint, necessary conditions and sufficient conditions of such form are given with explicit weight functions.

Let $S(\mu)$ be the space of all functions $x : R_+ \to R$ measurable with respect to the Lebesgue measure μ. Recall that a Banach space $X \subset S(\mu)$, is called ideal [7] if the conditions $y \in X$, $x \in S(\mu)$ and the inequality $|x(t)| \leq |y(t)|$ almost everywhere on R_+ imply that $x \in X$ and $\|x|X\| \leq \|y|X\|$. (The symbol $\|x|X\|$ denotes the norm of x in space X.)

Let $w : R_+ \to R_+$ be a positive function (weight). If X is an ideal space, then the symbol X_w denotes the new ideal space whose norm is given by $\|x|X_w\| = \|wx|X\|$. In particular, if $X = L^p$, then the equality $\|x|L_w^p\| = \|wx|L^p\|$ is true. This definition somewhat differs from the conventional one: usually the weight is included in the measure.

For an ideal space X the symbol $X^{'}$ denotes the space of all integral functionals on X with the standard norm

$$\|y|X^{'}\| = \sup\left\{\int_{R_+} y(t)x(t)dt : \|x|X\| \leq 1\right\}.$$

It is well known that $\{X_w\}^{'} = X^{'}_{1/w}$.

Denote by $K(\downarrow)$ the cone in $S(\mu)$, consisting of all non-negative non-increasing functions, and by $K(\uparrow)$ the cone in $S(\mu)$, consisting of all non-negative non-decreasing functions. Let X be an ideal space in $S(\mu)$. Denote by $K(\downarrow) \cap X$ and $K(\uparrow) \cap X$ the intersection of the space X with the cone $K(\downarrow)$, $K(\uparrow)$ respectively.

It is said that an operator $T : S(\mu) \to S(\mu)$ belongs to the class of sublinear operators $(T \in K)$, if the following conditions

$$|T(x + y)(t)| \leq T|x|(t) + T|y|(t) \quad \text{and} \quad |T(\lambda x)(t)| \leq |\lambda||Tx(t)|, \quad \lambda \in R,$$

are satisfied for all $t \in R_+$.

In particular, the class K contains all integral operators with non-negative kernels.

For $T \in K$ an operator $T^{'}$ is called associative in the L^p-scale if for all $p \in [1, \infty]$ and any weight u the followings inequalities

$$C^{-1}\|T|L^p_u \to L^p_u\| \leq \|T^{'}|(L^p_u)^{'} \to (L^p_u)^{'}\| \leq C\|T|L^p_u \to L^p_u\|$$

are satisfied, where $C > 0$ is independent of p and the weight u.

First of all, let us formulate the Schur extrapolation theorem in the modern form (see, for example, [3]).

Theorem S. *Let $T, T^{'} \in K$, v be a weight function, $1 < p < \infty$, $\theta = 1/p$ (hence $1 - \theta = 1/p^{'}$, where $p^{'}$ is the conjugate of p). Then the following conditions are equivalent:*

1) an operator T is bounded as

$$T : L^p_v \to L^p_v, \tag{1.1}$$

2) there exist two weight functions v_0 and v_1 such that for almost all $t \in R_+$

$$v(t) = v_1(t)^{\theta} \cdot v_0(t)^{1-\theta} \tag{1.2}$$

and an operator T is bounded both as

$$T : L^1_{v_1} \to L^1_{v_1}, \qquad T : L^{\infty}_{v_0} \to L^{\infty}_{v_0}. \tag{1.3}$$

We note that the implication $(1.2) - (1.3) \implies (1.1)$ follows by the interpolation theorem for positive operators (see, for example, [1]), and the implication $(1.1) \implies (1.2) - (1.3)$ is the essence of the Schur theorem.

Suppose that a function x belongs to $L^1([0, n])$ for every $n \in N$. Then the Hardy operator is defined by the formula

$$Hx(t) = \frac{1}{t}\int_0^t x(s)ds, \quad x \in R_+.$$

Let $w(t) = \frac{1}{t}$ and a function x belong to $L_w^1([n^{-1}, \infty))$ for every $n \in N$. Then the equality

$$Qx(t) = \int_t^\infty \frac{x(s)}{s} ds, \quad x \in R_+,$$

defines the adjoint Hardy operator. It is easy to see that for the Hardy operators theorem S holds.

The following theorem is well known (see, for example, [8, 9].

Theorem M. *Let v be a weight function.*

If $1 < p < \infty$, then the Hardy operator is bounded as an operator $H : L_v^p \to L_v^p$ if and only if

$$\sup_{t>0} \left(\int_t^\infty \left(\frac{v(s)}{s} \right)^p ds \right)^{1/p} \cdot \left(\int_0^t \left(\frac{1}{v(s)} \right)^{p'} ds \right)^{1/p'} < \infty. \tag{1.4}$$

The Hardy operator is bounded as an operator $H : L_{v_1}^1 \to L_{v_1}^1$ if and only if for some $C > 0$

$$\int_t^\infty \frac{v_1(s)}{s} ds \le C v_1(t) \tag{1.5}$$

for almost all $t \in R_+$.

The Hardy operator is bounded as an operator $H : L_{v_0}^\infty \to L_{v_0}^\infty$ if and only if for some $C > 0$

$$\int_0^t \frac{ds}{v_0(s)} \le C \frac{t}{v_0(t)} \tag{1.6}$$

for almost all $t \in R_+$.

In [8, 9] one can also find an analogue of Theorem M for the adjoint Hardy operator Q.

2   Auxiliary results

For the sequel we need some properties of special functions. Let for a function $g : R_+ \to R_+$ and $a \in R_+$

$$\alpha_+(g, a) = \overline{\lim_{t \to \infty}} \frac{g(at)}{g(t)}, \quad \alpha_-(g, a) = \overline{\lim_{t \to 0}} \frac{g(at)}{g(t)},$$

$$\beta_+(g, a) = \underline{\lim}_{t \to \infty} \frac{g(at)}{g(t)}, \quad \beta_-(g, a) = \underline{\lim}_{t \to 0} \frac{g(at)}{g(t)}.$$

First of all, we note the equalities

$$\beta_+(g, a) = \underline{\lim}_{t \to \infty} \frac{g(at)}{g(t)} = \frac{1}{\overline{\lim}_{t \to \infty} \frac{g(at)}{g(t)}} = \frac{1}{\alpha_+(g, a^{-1})}; \quad \beta_-(g, a) = \frac{1}{\alpha_-(g, a^{-1})}.$$

Theorem 2.1. *For a function $g \in K(\downarrow)$ there exists $C > 0$ such that*

$$\int_t^\infty g(s) \frac{ds}{s} \le C g(t) \tag{2.1}$$

for any $t \in R_+$ if and only if there exists $a > 1$ such that

$$\alpha_+(g, a) < 1, \qquad \alpha_-(g, a) < 1. \qquad (2.2)$$

Proof. Suppose that inequality (2.1) holds. We shall prove that the function g satisfies (2.2) by contradiction. Since $g \in K(\downarrow)$, for $a \geq 1$, the inequalities $\alpha_+(g, a) \leq 1$, $\alpha_-(g, a) \leq 1$ are valid. If for any $a > 1$ we have the equality $\alpha_+(g, a) = 1$, then straightforwardly by the definition of $\alpha_+(g, a)$ it follows that for every $l \in N$ there exists a number $t_0 \in R_+$ such that $\frac{g(2^l t_0)}{g(t_0)} > \frac{1}{2}$. Therefore, we have

$$\int_{t_0}^{\infty} g(s) \frac{ds}{s} \geq \int_{t_0}^{2^l t_0} g(s) \frac{ds}{s} \geq g(2^l t_0) \int_{t_0}^{2^l t_0} \frac{ds}{s}$$

$$\geq \frac{g(t_0)}{2} \int_{t_0}^{2^l t_0} \frac{ds}{s} = l \frac{g(t_0) \ln 2}{2}. \qquad (2.3)$$

Since l is an arbitrary integer, (2.3) contradicts (2.1). Thus, the first inequality in (2.2) is proved. Similarly we can prove the second inequality in (2.2).

Now w show that if g satisfies condition (2.2), then it satisfies (2.1).

Let $a_1 > 1$. Then we can find $t_1(a_1)$ such that for all $t \geq t_1(a_1)$, the inequality

$$\frac{g(a_1 t)}{g(t)} \leq q_1 < 1$$

is satisfied.

We note that the monotonicity of g implies that the latter inequality is valid for all $t \geq t_1(a_1)$ and $a \geq a_1$.

Thus for all $t \geq t_1(a_1)$ the inequalities

$$\frac{g(a_1^2 t)}{g(t)} \leq \frac{g(a_1^2 t)}{g(a_1 t)} \cdot \frac{g(a_1 t)}{g(t)} \leq q_1^2$$

are valid, and similarly we find that for every $k \in N$ and for all $t \geq t_1(a_1)$ the inequality

$$\frac{g(a_1^k t)}{g(t)} \leq q_1^k$$

is satisfied.

In the same manner it can be shown that there exist $q_2 \in (0, 1)$, and sufficiently large $a_2 > 1$ and $t_2(a_2) > 0$ such that

$$\frac{g(t)}{g(a_2^{-k} t)} \leq q_2^k$$

for all $k \in N$ and for all $t \in (0, t_2(a_2)]$.

Let $a = \max\{a_1, a_2\}$, $q = \max\{q_1, q_2\}$, $t_1 = t_1(a)$, $t_2 = t_2(a)$. Without loss of generality, we assume that the inequality $t_2 < t_1$ is true. Suppose $t \in R_+$. There are three possible cases: a)$t < t_2$, b) $t_2 \leq t < t_1$, c) $t \geq t_1$.

First, assume that $t < t_2$.

Then
$$\int_t^\infty g(s)\frac{ds}{s} = \sum_{i=0}^\infty \int_{a^i t}^{a^{i+1}t} g(s)\frac{ds}{s}$$

$$= \sum_{\{i: ta^{i+1} \leq t_2\}} \int_{a^i t}^{a^{i+1}t} g(s)\frac{ds}{s} + \sum_{\{i: t_2 < ta^{i+1} < t_1\}} \int_{a^i t}^{a^{i+1}t} g(s)\frac{ds}{s}$$

$$+ \sum_{\{i: ta^i \geq t_1\}} \int_{a^i t}^{a^{i+1}t} g(s)\frac{ds}{s} = I_1 + I_2 + I_3. \tag{2.4}$$

Let $i_1 = \max\{i : ta^i \leq t_2\}$, $i_2 = \min\{i : ta^i \geq t_1\}$. Then by the monotonicity of g we obtain

$$I_1 = \sum_{\{i: ta^{i+1} \leq t_2\}} \int_{a^i t}^{a^{i+1}t} g(s)\frac{ds}{s} = \sum_0^{i_1} \int_{a^i t}^{a^{i+1}t} g(s)\frac{ds}{s}$$

$$\leq \sum_0^{i_1} g(a^i t) \int_{a^i t}^{a^{i+1}t} \frac{ds}{s} = \ln a \sum_0^{i_1} g(a^i t) \leq \ln a \cdot g(t) \sum_0^{i_1} q^i \leq g(t)\frac{\ln a}{1-q}. \tag{2.5}$$

Similarly, we prove that the inequalities

$$I_3 = \sum_{\{i: ta^i \geq t_2\}} \int_{a^i t}^{a^{i+1}t} g(s)\frac{ds}{s} = \sum_{i_2}^\infty \int_{a^i t}^{a^{i+1}t} g(s)\frac{ds}{s} \leq \sum_{i_2}^\infty g(a^i t) \int_{a^i t}^{a^{i+1}t} \frac{ds}{s}$$

$$= \ln a \sum_{i_2}^\infty g(a^i t) \leq \ln a \cdot g(a^{i_2} t) \sum_0^\infty q^i \leq g(a^{i_2} t)\frac{\ln a}{1-q} \leq g(t_1)\frac{\ln a}{1-q} \tag{2.6}$$

are satisfied.

Let
$$c_1 = \frac{1}{g(t_1)} \int_{t_2/a}^{t_1} \frac{ds}{s}.$$

Then we have

$$I_2 = \sum_{\{i: t_2 < ta^{i+1} < t_1\}} \int_{a^i t}^{a^{i+1}t} g(s)\frac{ds}{s} \leq \int_{t_2/a}^{t_1} g(s)\frac{ds}{s} \leq c_1 g(t_1). \tag{2.7}$$

Combining estimates (2.5)-(2.7), we finally obtain

$$\int_t^\infty g(s)\frac{ds}{s} = I_1 + I_2 + I_3$$

$$\leq \frac{\ln a}{1-q}g(t) + c_1 g(t_1) + \frac{\ln a}{1-q}g(t_1) \leq \left(c_1 + 2\frac{\ln a}{1-q}\right)g(t).$$

Thus, for $t < t_2$ inequality (2.1) is proved. In the case $t_2 \leq t < t_1$ in (2.4) there is no first term I_1 and if $t_1 \geq t$ then in (2.4) there is only term I_3. The proof of (2.1) in these cases is similar to the case $t < t_2$. $\qquad\square$

A dual fact also holds.

Theorem 2.2. *For a function $g \in K(\uparrow)$ there exists $C > 0$ such that*

$$\int_0^t g(s)\frac{ds}{s} \le Cg(t)$$

for any $t \in R_+$ if and only if there exists $a > 1$ such that

$$\beta_+(g, a) > 1, \quad \beta_-(g, a) > 1.$$

The proof of Theorem 2.2 is similar to that of Theorem 2.1.

For a function $g : R_+ \to R_+$ we put ut $\widehat{g}(t) = \frac{t}{g(t)}$ and apply Theorem 2.2 to the function $\widehat{g}$. We formulate the corresponding result as a corollary.

Corollary 2.1. *Let $g : R_+ \to R_+$ be such that the function $\widehat{g} \in K(\uparrow)$. Then there exists $C > 0$ such that*

$$\int_0^t \widehat{g}(s)\frac{ds}{s} \le C\widehat{g}(t)$$

for any $t \in R_+$ if and only if there exists $a \in (0, 1)$ such that

$$\beta_+(g, a) > a, \quad \beta_-(g, a) > a.$$

The proof of the corollary is obtained from the following chain of equalities:

$$\underline{\lim}_{t\to 0}\frac{\widehat{g}(at)}{\widehat{g}(t)} = \underline{\lim}_{t\to 0}\frac{at}{t}\frac{g(t)}{g(at)} = a\underline{\lim}_{t\to 0}\frac{g(t)}{g(at)} = a\underline{\lim}_{\tau\to 0}\frac{g(a^{-1}\tau)}{g(\tau)} = a\beta_+(g, a^{-1}).$$

3   Main results

Theorem 3.1. *Let for $1 < p < \infty$, $\theta = \frac{1}{p}$ (hence $1 - \theta = \frac{1}{p'}$), $v_0, v_1 \in K(\downarrow)$, and $v(t) = v_1^\theta(t)\, v_0^{1-\theta}(t)$.*

Assume that the Hardy operator H is bounded both as $H : L_{v_1}^1 \to L_{v_1}^1$ and $H : L_{v_0}^\infty \to L_{v_0}^\infty$.

Then the operator H is bounded as $H : L_v^p \to L_v^p$. Moreover, $v \in K(\downarrow)$, there exists $C > 0$ such that for all $t \in R_+$

$$\int_t^\infty \left(\frac{v(s)}{s}\right)^p ds \le C\frac{v(t)^p}{t^{p-1}}, \tag{3.1}$$

$$\int_0^t \left(\frac{1}{v(s)}\right)^{p'} ds \le C\frac{t}{v^{p'}(t)}, \tag{3.2}$$

for $a \in [1, \infty)$

$$\alpha_+(v, a) \le (\alpha_+(v_1, a))^\theta, \quad \alpha_-(v, a) \le (\alpha_-(v_1, a))^\theta, \tag{3.3}$$

and for $a \in (0, 1)$

$$\beta_+(v, a) \ge (\beta_+(v_0, a))^{1-\theta}, \quad \beta_-(v, a) \ge (\beta_-(v_0, a))^{1-\theta}. \tag{3.4}$$

Proof. First we prove (3.1). Under the assumptions of Theorem 3.1 by Theorem M it follows that for v_0 and v_1 conditions (1.5)-(1.6) are satisfied. According to the equalities

$$\int_t^\infty \left(\frac{v(s)}{s}\right)^p ds = \int_t^\infty \left(\frac{v_0^{1-\theta}(s)v_1^\theta(s)}{s}\right)^p ds$$

$$= \int_t^\infty v_1(s)\, s\, \frac{v_0^{1/\theta-1}(s)}{s^{1/\theta}}\frac{ds}{s} = \int_t^\infty v_1(s)\, \frac{v_0^{1/\theta-1}(s)}{s^{1/\theta-1}}\frac{ds}{s},$$

non-increasing of the function $\frac{v_0(s)}{s}$ and condition (5) we obtain

$$\int_t^\infty v_1(s)\, \frac{v_0^{1/\theta-1}(s)}{s^{1/\theta-1}}\frac{ds}{s} \le \frac{v_0^{1/\theta-1}(t)}{t^{1/\theta-1}}\int_t^\infty v_1(s)\frac{ds}{s}$$

$$C\, v_1(t)\, \frac{v_0^{1/\theta-1}(t)}{t^{1/\theta-1}} = C\, \frac{(v_1^\theta(t)\, v_0^{1-\theta}(t))^p}{t^{1/\theta-1}} = C\, \frac{v^p(t)}{t^{1/\theta-1}}.$$

The proof of inequality (3.2) is similar to the above proof. One needs to use non-decreasing of the function $\frac{1}{v_1(s)}$ and condition (1.6):

$$\int_0^t \left(\frac{1}{v(s)}\right)^{p'} ds = \int_0^t \frac{1}{(v_0^{1-\theta}(s)\, v_1^\theta(s))^{p'}} ds$$

$$= \int_0^t \frac{1}{v_0(s)\, v_1^{1/(1-\theta)-1}(s)} ds \le \frac{1}{v_1^{1/(1-\theta)-1}(t)}\int_0^t \frac{1}{v_0(s)} ds$$

$$\le C\, \frac{t}{v_0(t)\, v_1^{1/(1-\theta)-1}(t)} = C\, \frac{t}{(v_0^{1-\theta}(t)\, v_1^\theta(t))^{p'}} = C\frac{t}{v(t)^{p'}}.$$

The first of inequalities (3.3) can be verified in the following way. Since for $a \ge 1$ the inequality $v_0(at) \le v_0(t)$ holds, we obtain

$$\alpha_+(v,a) = \overline{\lim_{t\to\infty}}\frac{v(at)}{v(t)} = \overline{\lim_{t\to\infty}}\left(\frac{v_1(at)}{v_1(t)}\right)^\theta \left(\frac{v_0(at)}{v_0(t)}\right)^{1-\theta}$$

$$\le \overline{\lim_{t\to\infty}}\left(\frac{v_1(at)}{v_1(t)}\right)^\theta = \left(\overline{\lim_{t\to\infty}}\frac{v_1(at)}{v_1(t)}\right)^\theta = (\alpha_+(v_1,a))^\theta.$$

Similarly one can prove the second inequality in (3.3).

The proof of the first inequality in (3.4) follows since for $0 < a \le 1$ the inequality $v_1(at) \ge v_1(t)$ holds, which implies that

$$\beta_+(v,a) = \underline{\lim_{t\to\infty}}\frac{v(at)}{v(t)} = \underline{\lim_{t\to\infty}}\left(\frac{v_1(at)}{v_1(t)}\right)^\theta \left(\frac{v_0(at)}{v_0(t)}\right)^{1-\theta}$$

$$\ge \underline{\lim_{t\to\infty}}\left(\frac{v_0(at)}{v_0(t)}\right)^{1-\theta} = \left(\underline{\lim_{t\to\infty}}\frac{v_0(at)}{v_0(t)}\right)^{1-\theta} = (\beta_+(v_0,a))^{1-\theta}.$$

Similarly one can prove the second inequality in (3.4). □

The following theorem is one of the main results.

Theorem 3.2. *Let for $1 < p < \infty$, $\theta = 1/p$, $v \in K(\downarrow)$ and let the Hardy operator H be bounded as*

$$H : L_v^p \to L_v^p.$$

Then there exist two weight functions $v_0, v_1 \in K(\downarrow)$, satisfying (1.2), such that the operator H is bounded both as

$$H : L_{v_0}^\infty \to L_{v_0}^\infty, \qquad H : L_{v_1}^1 \to L_{v_1}^1 ,$$

if and only if the following two conditions are satisfied:
    a) there exists $a > 1$ such that

$$\alpha_+(v, a) < 1, \quad \alpha_-(v, a) < 1 , \tag{3.5}$$

and
    b) there exists $a \in (0, 1)$ such that

$$\beta_+(v, a) > a, \quad \beta_-(v, a) > a . \tag{3.6}$$

Proof. If one can find two weight functions $v_0, v_1 \in K(\downarrow)$, satisfying (1.2) and such that the Hardy operator is bounded as an operator in two spaces, then (3.5) follows by (1.5), Theorem 2.1 and inequality (3.3) of Theorem 3.1. Inequalities (3.6) follow by (1.6), Corollary 2.1 and inequalities (3.4) of Theorem 3.1.

If a function v satisfies conditions (3.5)-(3.6), so we can put $v_0(t) \equiv v_1(t) \equiv v(t)$. Then $v_0, v_1 \in K(\downarrow)$, and (1.2) obviously holds. Condition (1.5) follows by (3.5) and Theorem 2.1, and the validity of (1.6) is obtained by (3.6) and Corollary 2.1. $\square$

A dual statement is formulated in the next theorem.

Theorem 3.3. *Let for $1 < p < \infty$, $\theta = 1/p$, $v \in K(\uparrow)$ and let the Hardy operator Q be bounded as*

$$Q : L_v^p \to L_v^p. \tag{3.7}$$

Then there exist two weight functions $v_0, v_1 \in K(\uparrow)$, satisfying (1.2), such that the operator Q is bounded both as

$$Q : L_{v_0}^\infty \to L_{v_0}^\infty, \qquad Q : L_{v_1}^1 \to L_{v_1}^1 ,$$

if and only if the following two conditions are satisfied:
    a) there exists $a \in (0, 1)$ such that

$$\alpha_+(v, a) < 1, \quad \alpha_-(v, a) < 1 , \tag{3.8}$$

and
    b) there exists $a > 1$ such that

$$\beta_+(v, a) > a^{-1}; \quad \beta_-(v, a) > a^{-1} . \tag{3.9}$$

Proof. By duality condition (3.7) holds if and only if the operator H is bounded as

$$H : (L_v^p)' \to (L_v^p)'. \tag{3.10}$$

By the equality $(X_w)' = X'_{1/w}$ (3.10) can be rewritten as

$$H : L_{1/v}^{p'} \to L_{1/v}^{p'}.$$

We set $w = 1/v$. Since $v \in K(\uparrow)$, it follows that $w \in K(\downarrow)$. In addition, we have $\frac{1}{p'} = 1 - \theta$. According to the previous theorem there exist two weight functions $w_0, w_1 \in K(\downarrow)$, satisfying the identity $w = w_0^\theta \cdot w_1^{1-\theta}$, and such that the Hardy operator bounded as

$$H : L_{w_0}^\infty \to L_{w_0}^\infty, \quad H : L_{w_1}^1 \to L_{w_1}^1 \tag{3.11}$$

if and only if
   a) there exists $a > 1$ such that

$$\alpha_+(w, a) < 1; \quad \alpha_-(w, a) < 1 \tag{3.12}$$

and
   b) there exist $a \in (0, 1)$ such that

$$\beta_+(w, a) > a; \quad \beta_-(w, a) > a. \tag{3.13}$$

Next, we have

$$\alpha_+(w, a) = \varlimsup_{t \to \infty} \frac{w(at)}{w(t)} = \varlimsup_{t \to \infty} \frac{v(t)}{v(at)} = \varlimsup_{\tau \to \infty} \frac{v(a^{-1}\tau)}{v(\tau)} = \alpha_+(v, a^{-1}),$$

$$\alpha_-(w, a) = \varlimsup_{t \to 0} \frac{w(at)}{w(t)} = \varlimsup_{t \to 0} \frac{v(t)}{v(at)} = \varlimsup_{\tau \to 0} \frac{v(a^{-1}\tau)}{v(\tau)} = \alpha_-(v, a^{-1}),$$

$$\beta_+(w, a) = \varliminf_{t \to \infty} \frac{w(at)}{w(t)} = \varliminf_{t \to \infty} \frac{v(t)}{v(at)} = \varliminf_{\tau \to \infty} \frac{v(a^{-1}\tau)}{v(\tau)} = \beta_+(v, a^{-1}),$$

$$\beta_+(w, a) = \varliminf_{t \to 0} \frac{w(at)}{w(t)} = \varliminf_{t \to 0} \frac{v(t)}{v(at)} = \varliminf_{\tau \to 0} \frac{v(a^{-1}\tau)}{v(\tau)} = \beta_+(v, a^{-1}).$$

Therefore conditions (3.12)-(3.13) are equivalent to conditions (3.8)-(3.9).

It remains to note that $\frac{1}{w} = (\frac{1}{w_0})^\theta (\frac{1}{w_1})^{1-\theta}$, and that (3.11) is equivalent to the boundedness of the operator Q is bounded as

$$Q : L_{\frac{1}{w_1}}^\infty \to L_{\frac{1}{w_1}}^\infty, \quad Q : L_{\frac{1}{w_0}}^1 \to L_{\frac{1}{w_0}}^1.$$

$\square$

Conditions (1.5)-(1.6) ensuring the boundedness of H in spaces $L_{v_1}^1$ and $L_{v_0}^\infty$ imply that natural assumptions on the weight functions are as follows:

$$v_1(t) \in K(\downarrow), \quad \frac{t}{v_0(t)} \in K(\uparrow). \tag{3.14}$$

Let $1 < p < \infty$, $\theta = \frac{1}{p}$ and the function v be defined by $v(t) = v_1^\theta(t) \, v_0^{1-\theta}(t)$. Then by (3.14) it immediately follows that

$$\frac{v(t)}{t^{1-\theta}} \in K(\downarrow). \tag{3.15}$$

Therefore it is natural to look for an analogue of Theorem 3.2 with weights v satisfying condition (3.15).

Theorem 3.4. *Let $1 < p < \infty$, $\theta = 1/p$, v be is a weight function satisfying (3.15), $w(t) = \frac{v(t)}{t^{1-\theta}}$, and let the Hardy operator H be bounded as*

$$H : L_v^p \to L_v^p.$$

Then there exist two weight functions v_0, v_1 satisfying (1.2) and (3.14), such that the operator H is bounded both as

$$H : L_{v_0}^\infty \to L_{v_0}^\infty, \qquad H : L_{v_1}^1 \to L_{v_1}^1 ,$$

if and only if the following two conditions are satisfied:
  a) there exists $a > 1$ such that

$$\alpha_+(w, a) < 1, \quad \alpha_-(w, a) < 1, \tag{3.16}$$

and
  b) there exists $a \in (0,1)$ such that

$$\beta_+(w, a^{-1}) > a; \quad \beta_-(w, a^{-1}) > a. \tag{3.17}$$

Proof. Necessity. Inequality (1.5) and Theorem 2.1 imply that there exists $a_1 > 1$, for which the inequalities
$$\alpha_+(v_1, a_1) < 1, \quad \alpha_-(v_1, a_1) < 1$$
are satisfied.

It follows by (1.6) and Theorem 2.2 that there exists $a_0 > 1$, for which the inequalities
$$\beta_+(\widehat{v_0}, a_0) > 1, \quad \beta_-(\widehat{v_0}, a_0) > 1$$
are satisfied.

Therefore, taking into account conditions (3.14) with $a > 1$ we have

$$\alpha_+(w, a) = \varlimsup_{t \to \infty} \frac{(at)^{\theta-1} \, v_0(at)^{1-\theta} v_1(at)^\theta)}{t^{\theta-1} \, v_0(t)^{1-\theta} v_1(t)^\theta} \leq \varlimsup_{t \to \infty} \frac{v_1(at)^\theta)}{v_1(t)^\theta} = (\alpha_+(v_1, a))^\theta$$

and

$$\alpha_-(w, a) \leq (\alpha_-(v_1, a))^\theta .$$

This implies the inequality (3.16).
Analogously, we have
$$\beta_+(w, a^{-1}) \geq \beta_+(\widehat{v_0}, a)^{1-\theta}$$

and
$$\beta_-(w, a^{-1}) \geq \beta_-(\widehat{v_0}, a)^{1-\theta}.$$

This implies the inequality (3.17).

Sufficiency. We define, for $\gamma \in (0, 1)$, the functions v_0 and v_1 by the following equalities

$$v_0(t) = t\left(\frac{v(t)}{t^{1-\theta}}\right)^{(1-\gamma)/(1-\theta)} = tw(t)^{(1-\gamma)/(1-\theta)}, \quad v_1(t) = \left(\frac{v(t)}{t^{1-\theta}}\right)^{\gamma/\theta} = w(t)^{\gamma/\theta}.$$

Then direct calculations show that

$$v(t) \equiv v_1^\theta(t)\, v_0^{1-\theta}(t), \quad v_1(t) \in K(\downarrow), \quad \frac{v_0(t)}{t} \in K(\downarrow),$$

$$\alpha_+(v_1, a) = \alpha_+(w, a)^{\gamma/\theta}, \quad \alpha_-(v_1, a) = \alpha_-(w, a)^{\gamma/\theta}, \tag{3.18}$$

and

$$\beta_+(\widehat{v_0}, a) = \beta_+(w, a^{-1})^{(1-\gamma)/(1-\theta)}, \quad \beta_-(\widehat{v_0}, a) = \beta_-(w, a^{-1})^{(1-\gamma)/(1-\theta)}. \tag{3.19}$$

It follows by (3.18), (3.16), Theorems 2.1, and Theorem M, that H is bounded as an operator $H : L^1_{v_1} \to L^1_{v_1}$.

Similarly, it follows by (3.19), (3.17), Theorems 2.2, and Theorem M that operator H is bounded as an operator $H : L^\infty_{v_0} \to L^\infty_{v_0}$. $\qquad\square$

Passing to the adjoint operator and dual space, it is easy to obtain the dual statement.

Theorem 3.5. *Let for $1 < p < \infty$, $\theta = 1/p$, v be a weight function such that*

$$v(t)\, t^\theta \in K(\uparrow), \tag{3.20}$$

and let the Hardy operator Q be bounded as

$$Q : L^p_v \to L^p_v. \tag{3.21}$$

Then there exist two weight functions v_0, v_1 satisfying (1.2), such that $v_0(t) \in K(\uparrow)$, $t\, v_1(t) \in K(\uparrow)$ and such that the operator Q is bounded both as

$$Q : L^\infty_{v_0} \to L^\infty_{v_0}, \quad Q : L^1_{v_1} \to L^1_{v_1}, \tag{3.22}$$

if and only if the following two conditions are satisfied:
a) there exist $a > 1$ such that

$$\alpha_+(v, a^{-1}) < a^\theta, \quad \alpha_-(v, a^{-1}) < a^\theta \tag{3.23}$$

and
b) there exist $a \in (0, 1)$ such that

$$\beta_+(v, a) > a^{1-\theta}, \quad \beta_-(v, a) > a^{1-\theta}. \tag{3.24}$$

Proof. Condition (3.21) is equivalent to

$$H : L^{p'}_{1/v} \to L^{p'}_{1/v}.$$

Put $w(t) = 1/v(t)$. Then $\frac{w(t)}{t^\theta} \in K(\downarrow) \iff v(t)t^\theta \in K(\uparrow)$. Let $u(t) = \frac{w(t)}{t^\theta}$. According to the previous theorem there exist two weight functions w_0, w_1, satisfying (3.14), such that $w(t) \equiv w_1^{1-\theta}(t) \cdot w_0^\theta(t)$ and such that the Hardy operator H is bounded both as

$$H : L^\infty_{w_0} \to L^\infty_{w_0}, \qquad H : L^1_{w_1} \to L^1_{w_1}, \tag{3.25}$$

if and only if the following conditions are satified:

a) there exist $a > 1$ such that

$$\alpha_+(u, a) < 1; \quad \alpha_-(u, a) < 1 \tag{3.26}$$

and

b) there exist $a \in (0, 1)$ such that

$$\beta_+(u, a^{-1}) > a; \quad \beta_-(u, a^{-1}) > a. \tag{3.27}$$

Direct calculations show that the following equalities

$$\alpha_+(u, a) = \overline{\lim_{t\to\infty}} \frac{t^\theta v(t)}{(at)^\theta \ v(at)} = a^{-\theta} \cdot \alpha_+(v, a^{-1}); \qquad \alpha_-(u, a) = a^{-\theta} \cdot \alpha_-(v, a^{-1});$$

$$\beta_+(u, a^{-1}) = \underline{\lim_{t\to\infty}} \frac{(at)^\theta v(at)}{t^\theta v(t)} = a^\theta \cdot \beta_+(v, a); \qquad \beta_-(u, a^{-1}) = a^\theta \cdot \beta_-(v, a)$$

hold.

Therefore, conditions (3.26)-(3.27) are equivalent to conditions (3.23)-(3.24).

It remains to note that the equality $\frac{1}{w} \equiv (\frac{1}{w_0})^\theta (\frac{1}{w_1})^{1-\theta}$ holds, and that condition(3.25) is equivalent to condition (3.22). □

Acknowledgments

References

[1] E.I. Berezhnoi, *Interpolation of positive operators in space $\varphi(X_0, X_1)$*, Qualitative and approximate methods in studies of operator equations. Yaroslavl, Yaroslavl State University, 1981, 3 – 12 (in Russian).

[2] E.I. Berezhnoi, *Theorems on the representation of spaces and Schur's lemma,* Doklady of Russian Academy of Sciences. Mathematics 344 (1995), no. 6, 727 – 730 (in Russian).

[3] E.I. Berezhnoi, L. Maligranda, *On extrapolation operators and the limits of applicability of the test Schur,* Doklady of Russian Academy of Sciences. Mathematics, 393 (2003), no. 5, 1 – 4 (in Russian).

[4] M. Christ *Weighted norm inequalities and Schur's lemma,* Studia Math., 78 (1984), no. 3, 309 – 319.

[5] J. Garcia-Cuerva, J. Rubio de Francia *Weighted norm inequalities and related topics,* Amsterdam, North Holland, 1985.

[6] V.B. Korotkov, *Integral operators,* Nauka, Novosibirsk, 1983 (in Russian).

[7] S.G. Krein, Yu.I. Petunin, E.M. Semenov, *Interpolation of linear operators,* Moscow, Nauka, 1978 (in Russian).

[8] A. Kufner, L.-E. Persson *Weighted inequalities of Hardy type,* World Scientific. 2003.

[9] Maz'ya V.G. *Sobolev spaces,* Leningrad. Leningrad State University, 1985 (in Russian).

[10] J.L. Rubio de Francia, *A new technigue in the theory of A_p-weights,* Roma, Topics in modern harmonic analysis, 1983, 571 – 579.

[11] P. Szeptycki, *Notes on integral transformations,* Dissertationes Math. (Rozprawy Mat.), 231 (1984), 1 – 48.

Evgenii Ivanovich Berezhnoi
Department of differential equations
Mathematical faculty
Yaroslavl State University
14 Sovetskaya St.
150000 Yaroslavl, Russia
E-mail: ber@uniyar.ac.ru