EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 1, Number 3 (2010), 112 – 133

ON CONVERGENCE OF FAMILIES OF LINEAR POLYNOMIAL OPERATORS GENERATED BY MATRICES OF MULTIPLIERS

K. Runovski, H.-J. Schmeisser

Communicated by D.D. Haroske

Key words: trigonometric approximation, convergence, Fourier multipliers, Jackson, Cesaro and Fejér-Korovkin kernels.

AMS Mathematics Subject Classification: 42A10, 42A45, 42B99.

Abstract. The convergence of families of linear polynomial operators with kernels generated by matrices of multipliers is studied in the scale of the L_p-spaces with $0 < p \le +\infty$. An element $a_{n,k}$ of generating matrix is represented as a sum of the value of the generator $\varphi(k/n)$ and a certain "small" remainder $r_{n,k}$. It is shown that under some conditions with respect to the remainder the convergence depends only on the properties of the Fourier transform of the generator φ. The results enable us to find explicit ranges for convergence of approximation methods generated by some classical kernels.

1 Introduction

In this paper we continue the systematic study of methods of trigonometric approximation started in [1] and [5] - [10]. We consider Fourier means, interpolation means and families of linear polynomial operators, which are defined as follows. Let

$$A = \{a_{n,k} \in \mathbb{C} : \ a_{n,-k} = \overline{a_{n,k}}, \, |\, k \,| \le rn, \, n \in \mathbb{N}_0\} \ , \tag{1}$$

where $r \equiv r(A)$ is a real positive number, be *a matrix of multipliers* (We put $a_{n,k} = 0$ if $|k| > rn$.). It generates the kernels $W_n(A)$ given by

$$W_n(A)(h) = \sum_{k \in \mathbb{Z}^d} a_{n,k} \, e^{ikh} \, , \quad n \in \mathbb{N}_0 \, , \ h \in \mathbb{R}^d \ . \tag{2}$$

If $f \in L_p(\mathbb{T}^d) \, , 1 \le p \le \infty$ ($\mathbb{T}^d$ stands for the d-dimensional torus), then the Fourier means are given by

$$\mathcal{F}_n^{(A)}(f;x) = (2\pi)^{-d} \int\limits_{\mathbb{T}^d} f(h) \, W_n(A)(x-h) \, dh \ , \quad n \in \mathbb{N}_0 \, , \ x \in \mathbb{T}^d \, . \tag{3}$$

If f belongs to the space $C(\mathbb{T}^d)$ of continuous 2π - periodic (with respect to each variable) functions then the interpolation means are defined as

$$\mathcal{I}_n^{(A)}(f;x) = (2N+1)^{-d} \cdot \sum_{\nu=0}^{2N} f\left(t_N^{\nu}\right) \cdot W_n(A) \left(x - t_N^{\nu}\right) \ , \ n \in \mathbb{N}_0 \, , \ x \in \mathbb{T}^d \, . \tag{4}$$

Here

$$N = [\rho n]\,,\ \rho \geq r(A);\ t_N^\nu = \frac{2\pi\nu}{2N+1},\quad \nu \in \mathbb{Z}^d\,;\ \sum_{\nu=0}^{2N} \equiv \sum_{\nu_1=0}^{2N} \cdots \sum_{\nu_d=0}^{2N}\ .$$

The functions defined in (1) - (4) are trigonometric polynomials of spherical order not exceeding $r(A)n$. If $f \in L_p(\mathbb{T}^d)\,, 0 < p < \infty$, or if $f \in C(\mathbb{T}^d)$ $(p = +\infty)$ then we consider the functions given by

$$\mathcal{L}_{n;\lambda}^{(A)}(f;x) = (2N+1)^{-d} \cdot \sum_{\nu=0}^{2N} f\left(t_N^\nu + \lambda\right) \cdot W_n(A)\left(x - t_N^\nu - \lambda\right)\ . \tag{5}$$

In the case that $f \in L_p(\mathbb{T}^d)\,, 0 < p < \infty$, formula (5) makes sense for almost all $\lambda \in \mathbb{R}^d$ and $x \in \mathbb{T}^d$. We understand λ as a parameter and call $\{\mathcal{L}_{n;\lambda}^{(A)}\}$ *a family of linear (trigonometric) polynomial operators*. In contrast to classical methods of trigonometric approximation the method of approximation by families is comparatively new (see e.g. [6], [7]). Its systematic study was continued in [1], [5] and other works. For applications of the method, in particular for an algorithm of stochastic approximation (*SA-algorithm*), we refer to [8]. We are interested in the approximation process

$$\mathcal{L}_{n;\lambda}^{(A)}(f;x) \to f(x) \quad (n \to +\infty)$$

The convergence of the family $\{\mathcal{L}_{n;\lambda}^{(A)}\}$ must be understood in the sense of an averaging with respect to the parameter λ. More precisely, we consider the limit process

$$\left\|\mathcal{L}_{n;\lambda}^{(A)}(f;x) - f(x)\right\|_{\overline{p}} = \left(\frac{1}{(2\pi)^d} \int\limits_{\mathbb{T}^d} \left(\int\limits_{\mathbb{T}^d} \left|\mathcal{L}_{n;\lambda}^{(A)}(f;x) - f(x)\right|^p dx\right) d\lambda\right)^{\frac{1}{p}} \to 0$$

if $n \to +\infty$ for admissible p, $0 < p \leq +\infty$(see Section 2 for detailed explanation).

In [5] the problem of convergence has been investigated for kernels of type (1) - (2) with

$$A(\varphi) = \{\, a_{n,k}\}:\quad a_{0,0} = 1;\ a_{n,k} = \varphi\left(\frac{k}{\sigma(n)}\right),\ k \in \mathbb{Z}^d,\ n \in \mathbb{N}\ , \tag{6}$$

where $\sigma(n)$ is a certain strongly increasing sequence of positive real numbers of order n and φ is a complex-valued continuous function on $\mathbb{R}^d\,, d \in \mathbb{N}$, with compact support satisfying $\varphi(0) = 1$ and $\varphi(-\xi) = \overline{\varphi(\xi)}$ for each $\xi \in \mathbb{R}^d$. In this case it is $r(A) = r(\varphi) := \sup\{\,|\,\xi\,| : \varphi(\xi) \neq 0\,\}$. We call φ *generator of the kernels* W_n and we use the notations $W_n(\varphi)$ and $\{\,\mathcal{L}_{n;\lambda}^{(\varphi)}\}$ in place of $W_n(A)$ and $\{\,\mathcal{L}_{n;\lambda}^{(A)}\}$, respectively. More precisely, it was shown in [5]

- that the family $\{\mathcal{L}_{\sigma;\lambda}^{(\varphi)}\}$ converges in L_p for all $1 \leq p \leq +\infty$ if the Fourier transform $\widehat{\varphi}$ of the generator φ belongs to $L_1(\mathbb{R}^d)$ and that the convergence in L_p for some $0 < p < 1$ is equivalent to the condition: $\widehat{\varphi} \in L_p(\mathbb{R}^d)$. For this reason the method of approximation by a family of linear trigonometric polynomial operators is relevant both for $p \geq 1$ and $p < 1$ with the range of admissible parameters depending on the properties of its generator φ;

- that the approximation error for families is equivalent to the approximation error of the corresponding Fourier means in the case of $L_p(\mathbb{T}^d)\,, 1 \le p \le +\infty$ and to the approximation error of the corresponding interpolation means in the case of $C(\mathbb{T}^d)$ $(p = +\infty)$;

- that the ranges of convergence (the values of p such that the family converges in L_p) can be determined precisely for appropriate families with classical kernels (Fejér, de la Vallée-Poussin, Rogosinski, Bochner-Riesz) .

In this paper we elaborate an approach to the study of the convergence problem for families which are not generated by some function φ. In particular, we show

- that in many situations the matrix of multipliers can be represented in the form:

$$A = A(\varphi) + R\,,\ \ R = \{\, r_{n,k} \}\,;\ \lim_{n\to+\infty} r_{n,k} = 0,\ \ k \in \mathbb{Z}^d, \tag{7}$$

  where $A(\varphi)$ is of type (6) and the matrix of remainders R satisfies some relevant conditions of "smallness" (see Section 1 for exact definitions) which enable us to reduce the convergence problem for $\{\mathcal{L}^{(A)}_{n;\,\lambda}\}$ to the study of the associated family $\{\mathcal{L}^{(\varphi)}_{\sigma;\,\lambda}\}$;

- that for powers of kernels of type (6), that is, for kernels (q is a natural number)

$$W_n(\varphi, q)(h) = (\gamma_n(\varphi, q))^{-1}(W_n(\varphi))^q(h)\,, \tag{8}$$

  where

$$\gamma_n(\varphi, q) = (2\pi)^{-d} \int\limits_{\mathbb{T}^d} (W_n(\varphi)(h))^q\, dh \tag{9}$$

  is a normalizing factor, the ranges of convergence of corresponding families can be explicitly expressed in terms of the range of convergence of the family generated by the initial kernels $W_n(\varphi)$;

- that for families with classical kernels of types (7) and (8)-(9) (Fejér-Korovkin, Cesaro, (generalized) Jackson) the ranges of convergence can be determined precisely by applying general results.

The paper is organized as follows. Section 2 is devoted to preliminaries. The general results are formulated and proved in Sections 3 and 4. Sharp ranges of convergence of the families generated by (generalized) Jackson, Fejér-Korovkin and Cesaro kernels are determined in Sections 5, 6 and 7-9, respectively. In view of *the comparison principle* stated in [5], Lemma 2.2, the results of this paper formulated for families (5) immediately imply the classical results on convergence of the corresponding Fourier means (3) and the interpolation means (4) in the spaces L_p with $1 \le p \le +\infty$ and C, respectively, in both general and special cases.

2 Definitions, notations and preliminary remarks

$\mathbf{L_p}$-spaces. As usual, $L_p \equiv L_p(\mathbb{T}^d)$, where $0 < p < +\infty$, $\mathbb{T}^d = [0, 2\pi)^d$, is the space of measurable real-valued functions $f(x)$ which are 2π-periodic with respect to each variable such that

$$\| f \|_p \; = \; \Big(\int\limits_{\mathbb{T}^d} |f(x)|^p \, dx \Big)^{1/p} \; < \; +\infty \; .$$

$C \equiv C(\mathbb{T}^d)$ $(p = +\infty)$ is the space of real-valued 2π-periodic continuous functions equipped with the Chebyshev norm

$$\| f \|_\infty \; = \; \max_{x \in \mathbb{T}^d} |f(x)| \; .$$

For L_p-spaces of non-periodic functions defined on a measurable set $\Omega \subseteq \mathbb{R}^d$ we will use the notation $L_p(\Omega)$.

Often we deal with functions in $L_p(\mathbb{T}^{2d})$ which depend on both the main variable $x \in \mathbb{T}^d$ and the parameter $\lambda \in \mathbb{T}^d$. Let us denote by $\| \cdot \|_p$ or $\| \cdot \|_{p;\,x}$ the $L_p(\mathbb{T}^d)$-norm with respect to x. For the $L_p(\mathbb{T}^d)$-norm with respect to the parameter λ we use the symbol $\| \cdot \|_{p;\,\lambda}$. For shortness, $L_{\overline{p}}$ stands for the space $L_p(\mathbb{T}^{2d})$ equipped with the norm

$$\| \cdot \|_{\overline{p}} \; = \; (2\pi)^{-d/p} \, \| \quad \| \cdot \|_{p;\,x} \, \|_{p;\,\lambda} \; . \tag{10}$$

Analogously, we use the symbol $\| \cdot \|_{\overline{\infty}}$ for the norm in the space $C(\mathbb{T}^{2d})$. Clearly, L_p with $0 < p < \infty$ and $C(\mathbb{T}^d)$ can be considered as subspaces of $L_{\overline{p}}$ and $C(\mathbb{T}^{2d})$, respectively. In this case,

$$\| f \|_{\overline{p}} \; = \; \| f \|_p \; , \quad f \in L_p \; (f \in C \text{ if } p = \infty) \; .$$

The functional $\| \cdot \|_{\overline{p}}$ is a norm if and only if $1 \le p \le +\infty$. For $0 < p < 1$ it is a quasi-norm, and the "triangle" inequality is valid for its p-th power. If we put $\widetilde{p} = \min(1, p)$ then the inequality

$$\| f + g \|_{\overline{p}}^{\widetilde{p}} \; \le \; \| f \|_{\overline{p}}^{\widetilde{p}} + \| g \|_{\overline{p}}^{\widetilde{p}} \; , \quad f, \; g \in L_{\overline{p}} \; , \tag{11}$$

will be valid for all $0 < p \le +\infty$. This inequality is very convenient, because both cases can be treated simultaneously. Moreover, for the sake of convenience we shall use the notation "norm" also in the case $0 < p < 1$.

Spaces of trigonometric polynomials. Let σ be a real non-negative number. Let us denote by $\mathcal{T}_\sigma$ the space of all real-valued trigonometric polynomials of (spherical) order σ. It means

$$\mathcal{T}_\sigma \; = \; \Big\{ \; T(x) = \sum_{k \in \mathbb{Z}^d} c_k \, e^{ikx} \; : c_{-k} = \overline{c_k}, \; |\, k \,| \equiv (k_1^2 + \ldots + k_d^2)^{1/2} \le \sigma \; \Big\} \; ,$$

where $kx = k_1 x_1 + \ldots + k_d x_d$ and $\bar{c}$ is a complex conjugate to c. Further, $\mathcal{T}$ stands for the space of all real-valued trigonometric polynomials of arbitrary order. We denote by $\mathcal{T}_{\sigma,p}$, where $0 < p \le +\infty$, the space $\mathcal{T}_\sigma$, if it is equipped with the L_p-norm and we use the symbol $\mathcal{T}_{\sigma,\overline{p}}$ to denote the subspace of $L_{\overline{p}}$ which consists of functions $g(x,\lambda)$ such that $g(x,\lambda)$ as a function of x belongs to $\mathcal{T}_\sigma$ for almost all λ. Clearly, $\mathcal{T}_{\sigma,p}$ can be considered as a subspace of $\mathcal{T}_{\sigma,\overline{p}}$ with identity of the norms. As we can see, in our notation the line over the index p indicates that we are dealing with functions of $2d$ variables.

Types of kernels and generators. Recall that the kernels and conditions with respect to the generator φ of the approximation method have been given in the Introduction (see, in particular, (1), (2), and (6)). Henceforth, the class of all admissible generators will be denoted by $\mathcal{K}$. The different types of kernels under consideration can be classified as follows.

Type (G). These are kernels (2) corresponding to matrices A defined by (6). The main characteristics of the generator φ is the set $\mathcal{P}_\varphi = \{p \in (0,+\infty] : \ \widehat{\varphi} \in L_p(\mathbb{R}^d)\}$, where $\widehat{\varphi}$ is the Fourier transform of φ (see below). The corresponding approximation methods were studied in [5].

Types (GR^α) *and* (GR_α) $(0 < \alpha < 1)$. Let A, $W_n(A)$ be as in (1) and (2). Let $0 < q < +\infty$. We put

$$M_{q;\,A}(n) \;=\; (n+1)^{d(1/q-1)}\,\|\, W_n(A)\,\|_q\,,\ n \in \mathbb{N}_0\,;\quad M_{q;\,A} \;=\; \sup_n M_{q;\,A}(n)\,; \tag{12}$$

$$\mathcal{M}_{q;\,A}(n) = (n+1)^{d(1/q-1)}\,\|\, W_n(A)\,\|_2,\ n \in \mathbb{N}_0\,;\ \mathcal{M}_{q;\,A} = \sup_n \mathcal{M}_{q;\,A}(n)\,. \tag{13}$$

Clearly,

$$\mathcal{M}_{q;\,A}(n) \;=\; (2\pi)^{d/2}(n+1)^{d(1/q-1)}\left(\sum_{|\,k\,|\le rn} |\,a_{n,\,k}|^2\right)^{1/2},\ n \in \mathbb{N}_0\,, \tag{14}$$

by Parseval's equation.

A kernel is said to be of type (GR^α) or (GR_α) if it satisfies (7) with $M_{\alpha;\,R} < +\infty$ or $\mathcal{M}_{\alpha;\,R} < +\infty$ for their matrix R of remainders, respectively. Since

$$M_{\alpha;\,R}(n) \;\le\; (2\pi)^{d(1/\alpha-1/2)}\mathcal{M}_{\alpha;\,R}(n)\,,\ n \in \mathbb{N}_0\,, \tag{15}$$

by Holder's inequality, each kernel of type (GR_α) is also of type (GR^α), that is,

$$GR_\alpha \;\subset\; GR^\alpha\,. \tag{16}$$

Notations (12) will be used also for kernels of type (G). In this case, that is, if $A = A(\varphi)$, we use the symbols $M_{q;\varphi}(n)$ and $M_{q;\varphi}$ instead of $M_{q;\,A}(n)$ and $M_{q;\,A}$, respectively.

Type (G^q) $(q \in \mathbb{N})$. These are powers of kernels of type (G) given by (8)-(9). It will be shown in Section 4 that $\gamma_n(\varphi,q) \neq 0$ if $n \ge n_0$, where $n_0 \equiv n_0(\varphi,q)$ is a certain integer, and, consequently, the functions $W_n(\varphi,q)(h)$ are well-defined for $n \ge n_0$. Clearly, they belong to $\mathcal{T}_{qr(\varphi)n}$.

Fourier transform and convolution. The Fourier transform and its inverse are given by

$$\widehat{g}(\xi) = \int\limits_{\mathbb{R}^d} g(x)e^{-ix\xi}\,dx\ ,\quad g^{\vee}(x) = (2\pi)^{-d}\int\limits_{\mathbb{R}^d} g(\xi)e^{ix\xi}\,d\xi\ ,\quad g \in L_1(\mathbb{R}^d)\,.$$

Let g_1 and g_2 be functions in $L_1(\mathbb{R}^d)$. The function

$$\widehat{g_1 * g_2}(\xi) = \int\limits_{\mathbb{R}^d} g_1(\xi-\eta)g_2(\eta)\,d\eta \tag{17}$$

is called convolution of g_1 and g_2. Its main property reads as

$$\widehat{g_1 \star g_2}(x) = \widehat{g_1}(x)\cdot\widehat{g_2}(x),\ x\in\mathbb{R}^d. \tag{18}$$

For φ in $\mathcal{K}$ we denote by $(\varphi)_*^q$ its qth *convolution power*, that is, $\varphi * \ldots * \varphi$. Applying (17), (18) we get

$$\begin{aligned}(\varphi)_*^q(0) &= \int\limits_{\mathbb{R}^{d(q-1)}} \varphi(\xi_1)\ldots\varphi(\xi_{q-1})\varphi(-(\xi_1+\ldots+\xi_{q-1}))\,d\xi_1\ldots d\xi_{q-1}\\ &= ((\widehat{\varphi}(\cdot))^q)^{\vee}(0) = (2\pi)^{-d}\int\limits_{\mathbb{R}^d}(\widehat{\varphi}(x))^q\,dx\,.\end{aligned} \tag{19}$$

General linear trigonometric polynomial operators. As it was shown in [5] approximation methods (3) - (5) are examples of linear bounded operators of general type

$$\mathcal{L}_\sigma\ :\ L_p \longrightarrow \mathcal{T}_{r\sigma,\overline{p}}\ \subset\ L_{\overline{p}}\ ,\quad \sigma\geq 0\,, \tag{20}$$

where $0<p\leq+\infty$ and $r>0$. The operator norm of $\mathcal{L}_\sigma$ is given by

$$\|\,\mathcal{L}_\sigma\,\|_{(p)}\ =\ \sup_{\|\,f\,\|_p\leq 1}\ \|\,\mathcal{L}_\sigma(f)\,\|_{\overline{p}}\ . \tag{21}$$

In particular, $\|\,\{\,\mathcal{L}_{n;\lambda}^{(A)}\}\,\|_{(p)}$ stands for the operator norm (quasinorm) of the family $\{\,\mathcal{L}_{n;\lambda}^{(A)}\,\}$ in the sense of (21). The sequence $(\mathcal{L}_\sigma)$ is called *uniformly bounded on* L_p if their operator norms are bounded by a constant independent of σ, that is,

$$\sup_{\sigma\geq 0}\|\,\mathcal{L}_\sigma\,\|_{(p)}\ <\ +\infty\ , \tag{22}$$

and it is said to be *convergent in* L_p if for each $f\in L_p$ $(f\in C$ if $p=\infty)$

$$\lim_{\sigma\to+\infty}\ \|\,f-\mathcal{L}_\sigma(f)\,\|_{\overline{p}}\ =\ 0\ . \tag{23}$$

Obviously, for the approximation processes defined in (3) and (4) these concepts coincide with the norm, the boundedness and the convergence in L_p in usual sense.

Taking into account that both the Uniform Boundedness Principle of functional analysis and the Marcinkiewicz Interpolation Theorem can be extended to the case of quasi-normed spaces (see, for example, [3], Theorem 1.3.2 and Appendix G) we obtain the following statements for the operators (20).

Theorem A (Banach-Steinhaus Convergence Principle). *Let $0 < p \le +\infty$. A sequence $(\mathcal{L}_\sigma)$ of linear bounded operators of type* (20) *converges in L_p (in C, if $p = \infty$) if and only if the following conditions are satisfied:*

a) $\lim_{\sigma \to +\infty} \| e^{ik\cdot} - \mathcal{L}_\sigma(e^{ik\cdot}) \|_{\overline{p}} \; = \; 0$ *for each* $k \in \mathbb{Z}^d$*;*

b) $(\mathcal{L}_\sigma)$ *is uniformly bounded on L_p.*

Theorem B (Marcinkiewicz Interpolation Principle). *Let a sequence $(\mathcal{L}_\sigma)$ of linear operators of type* (20) *be bounded on L_{p_0} and on L_{p_1}, where $0 < p_0 < p_1 \le +\infty$. Then it is bounded on L_p for all $p_0 \le p \le p_1$ and there exists a constant $c(p, p_0, p_1)$ such that*

$$\| \mathcal{L}_\sigma \|_{(p)} \; \le \; c(p, p_0, p_1) \| \mathcal{L}_\sigma \|_{(p_0)}^{1-\theta} \cdot \| \mathcal{L}_\sigma \|_{(p_1)}^{\theta}, \; \sigma \ge 0, \; \left(\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \right). \tag{24}$$

Relations up to constants. By $''A \lesssim B''$ we denote the relation $A(f, \sigma) \le cB(f, \sigma)$, where c is a positive constant independent of $f \in L_p$ (or $f \in C$) and $\sigma \ge 0$. The symbol $'' \asymp ''$ indicates equivalence. It means that $A \lesssim B$ and $B \lesssim A$ simultaneously.

3 Families generated by kernels of type (GR)

In this section we formulate and prove some results on explicit ranges of convergence of families (5) generated by kernels of types (GR^α) and (GR_α). Our approach is mainly based on the following lemmas which have been proved in [5].

Lemma 1. *Let A be of type* (1) *, $0 < p \le +\infty$, $\widetilde{p} = \min(1, p)$, $\widehat{p} = p$ for $0 < p < +\infty$ and $\widehat{p} = 1$ for $p = +\infty$. Then*

$$M_{\widehat{p};\,A}(n) \; \lesssim \; \| \{ \mathcal{L}_{n;\,\lambda}^{(A)} \} \|_{(p)} \; \lesssim \; M_{\widetilde{p};\,A}(n), \; n \in \mathbb{N}_0 . \tag{25}$$

Lemma 2. *Let $0 < p \le 1$, $\varphi \in \mathcal{K}$. Then $M_{p;\,\varphi} < +\infty$ if and only if $p \in \mathcal{P}_\varphi$, where $\mathcal{P}_\varphi = \{ p \in (0, +\infty] : \widehat{\varphi} \in L_p(\mathbb{R}^d) \}$.*

Lemma 3. *For any $T \in \mathcal{T}_{\rho n}$, $n \in \mathbb{N}_0$, $\rho \ge r(A)$, $\lambda \in \mathbb{R}^d$*

$$\mathcal{L}_{n;\,\lambda}^{(A)}(T; x) \; = \; \sum_{k \in \mathbb{Z}^d} a_{n,\,k}\, c_k e^{ikx} \;, \tag{26}$$

where c_k are the Fourier coefficients of T.

Our result on convergence of families generated by kernels of type (GR^α) can be formulated as follows.

Theorem 1. *Let A be of type (GR^α) for some $0 < \alpha < 1$, that is, $A = A(\varphi) + R$, where $\varphi \in \mathcal{K}$ and $R = \{ r_{n,k} \}$ satisfies $M_{\alpha;\,R} < +\infty$. Assume that $1 \in \mathcal{P}_\varphi$ and $\alpha \notin \mathcal{P}_\varphi$. Then the family $\{ \mathcal{L}_{n,\lambda}^{(A)} \}$ converges in L_p if and only if $p \in \mathcal{P}_\varphi$.*

Proof. Let $0 < p \le +\infty$. Suppose $k \in \mathbb{Z}^d$, $A \in \mathbb{R}^d$ and $n \ge |k|/\rho$. Using (26) we get

$$\| e^{ik\cdot} - \mathcal{L}_{n;\lambda}^{(A)}(e^{ik\cdot}) \|_{\widetilde{p}} = | 1 - a_{n,k} | \cdot \| e^{ik\cdot} \|_p = (2\pi)^{d/p} \left| 1 - \varphi\Big(\frac{k}{\sigma(n)}\Big) - r_{n,k} \right| .$$

Taking into account that φ is continuous, $\sigma(n) \asymp n$, $\lim\limits_{n\to+\infty} r_{n,k} = 0$ by (7) we obtain

$$\lim_{n\to+\infty} \| e^{ik\cdot} - \mathcal{L}_{n;\lambda}^{A}(e^{ik\cdot}) \|_{\widetilde{p}} = 0 , \quad k \in \mathbb{Z}^d . \tag{27}$$

In view of Theorem A we conclude therefrom that the family $\{ \mathcal{L}_{n,\lambda}^{(A)} \}$ converges in L_p if and only if it is uniformly bounded on L_p.

Let first $p \in \mathcal{P}_\varphi$. Since $1 \in \mathcal{P}_\varphi$, $\widetilde{p}$ also belongs to $\mathcal{P}_\varphi$ and $M_{\widetilde{p};\varphi} < +\infty$ by Lemma 2. In view of (12) and $\widetilde{p} > \alpha$ we get by Nikol'skij's inequality for different norms for trigonometric polynomials (see, e.g., [12], Proposition 3.3.2)

$$M_{\widetilde{p};R}(n) \le c\, M_{\alpha;R}(n) , \ n \in \mathbb{N}_0 . \tag{28}$$

This implies $M_{\widetilde{p};R} < +\infty$. Hence, we obtain

$$\sup_n \| \{ \mathcal{L}_{n;\lambda}^{(A)} \} \|_{(p)}^{\widetilde{p}} \lesssim M_{\widetilde{p};A}^{\widetilde{p}} \le M_{\widetilde{p};\varphi}^{\widetilde{p}} + M_{\widetilde{p};R}^{\widetilde{p}} < +\infty .$$

by Lemma 1 and $\{ \mathcal{L}_{n;\lambda}^{(A)} \}$ converges in L_p.

Let now $p \ge \alpha$ and $p \notin \mathcal{P}_\varphi$. In this case $\widehat{p} = \widetilde{p} = p$. By Lemma 1 we get

$$(M_{p;\varphi}(n))^p - (M_{p;R}(n))^p \le (M_{p;A}(n))^p \lesssim \| \{ \mathcal{L}_{n;\lambda}^{(A)} \} \|_{(p)}^{\widetilde{p}} \tag{29}$$

for $n \in \mathbb{N}_0$. Since $M_{p;\varphi} = +\infty$ by Lemma 2 and $M_{p;R} < +\infty$ in view of (28), the supremum at the left-hand side of (29) is equal to $+\infty$. Hence, in this case $\{ \mathcal{L}_{n;\lambda}^{(A)} \}$ is not uniformly bounded and divergent in L_p.

Finally we consider the case $0 < p < \alpha$. Assume that the family $\{ \mathcal{L}_{n;\lambda}^{(A)} \}$ converges in L_p. Then $\{ \mathcal{L}_{n;\lambda}^{(A)} \}$ being uniformly bounded in L_p and L_1 should be uniformly bounded in L_α by the interpolation argument (Theorem B). This contradicts the above result. □

We give a few remarks. Theorem 1 contains the General Convergence Theorem for kernels of type (G), which we have proved in [5], Theorem 4.1, as a special case corresponding to $R = 0$. To find an estimate for the value $M_{\alpha;R}$ can be a rather sophisticated problem. The replacement of the requirement $M_{\alpha;R} < +\infty$ by the stronger condition $\mathcal{M}_{\alpha;R} < +\infty$ is one possible approache to its solution. Another method is based on the concept of *majorant*.

Lemma 4. *Suppose $\psi \in \mathcal{K}$ and $\widehat{\psi}(x) = O(|x|^{-\delta})$, $(x \to +\infty)$ for some $\delta > d$. Then*

$$M_{p;\psi}(n) \lesssim \begin{cases} 1 & , \ d/\delta < p \le +\infty \\ (\ln(n+1))^{1/p} & , \ p = d/\delta \\ n^{d/p-\delta} & , \ 0 < p < d/\delta \end{cases} . \tag{30}$$

Proof. By Poisson's summation formula and properties of the Fourier transform we get

$$W_n(\psi)(x) = \sum_{\nu\in\mathbb{Z}^d} \widehat{\psi(\cdot/n)}(2\pi\nu - x) = n^d \sum_{\nu\in\mathbb{Z}^d} \widehat{\psi}(n(2\pi\nu - x))$$

for $x \in [-\pi,\pi]^d$. It follows therefrom the estimate

$$\begin{aligned} |\, W_n(\psi)(x)\,| \ &\leq \ n^d\Big(\,|\,\widehat{\psi}(-nx)\,| \,+\, c\sum_{\nu\neq 0} |\, n\nu\,|^{-\delta}\Big) \\ & n^d\,|\,\widehat{\psi}(-nx)\,| \,+\, c_1 n^{-(\delta-d)}\ , \end{aligned} \tag{31}$$

where c and c_1 do not depend on x and n. Inequality (31) immediately implies (30) for $p = +\infty$. Taking into account that $n^{-(\delta-d)}$ is bounded and integrating (31) over $[-\pi,\pi]^d$ we conclude

$$\begin{aligned} \|\, W_n(\psi)\,\|_p^p \ &\lesssim \ n^{dp} \int\limits_{[-\pi,\pi]^d} |\,\widehat{\psi}(nx)\,|^p\, dx \lesssim n^{d(p-1)} \int\limits_{1\leq|x|\leq 2\pi n} |\,x\,|^{-\delta p}\, dx \lesssim \\ & n^{d(p-1)} \int\limits_1^n r^{-\delta p + d - 1}\, dr \end{aligned} \tag{32}$$

for $0 < p < +\infty$. Now (30) follows from (2.8) and (1.3). □

In view of (1.6) and (2.6) we obtain by Theorem 1 the following statement. In particular, it contains the result on convergence of families generated by kernels of type (GR_α) as a special case.

Theorem 2. *Let $A = A(\varphi) + R' + R''$, where $\varphi \in \mathcal{K}$, $R' = \{\, r_{n,k}\}$ satisfies $\mathcal{M}_{\alpha;\,R'} < +\infty$ for some $0 < \alpha < 1$ and $R'' = \{\, r''_{n,k}\}$ has the representation*

$$r''_{n,\,k} \ = \ \lambda_n \psi\left(\frac{k}{n}\right)\ ,\ n \in \mathbb{N};\ \psi \in \mathcal{K}\,,$$

where

$$\psi \in \mathcal{K};\ \widehat{\psi}(x) = O(|x|^{-\delta}),\ (x \to +\infty)\,;\ \lambda_n = O(n^{\delta - d/\alpha})\,;\ d < \delta < d/\alpha\,.$$

Assume that $1 \in \mathcal{P}_\varphi$ and $\alpha \notin \mathcal{P}_\varphi$. Then the family $\{\, \mathcal{L}^{(A)}_{n,\lambda}\,\}$ converges in L_p if and only if $p \in \mathcal{P}_\varphi$.

Proof. By (15) and Lemma 4 we obtain

$$\begin{aligned} M^\alpha_{\alpha;\,R'+R''} \ &\leq \ M^\alpha_{\alpha;\,R'} + M^\alpha_{\alpha;\,R''} = M^\alpha_{\alpha;\,R'} + \left(\sup_n\, |\lambda_n|\, M_{\alpha;\,\psi}(n)\right)^\alpha \leq \\ &\leq \ (2\pi)^{d(1-\alpha/2)} \mathcal{M}^\alpha_{\alpha;\,R'} + c\,\left(\sup_n |\lambda_n|\, n^{d/\alpha - \delta}\right)^\alpha < +\infty. \end{aligned}$$

Now the statement follows from Theorem 1. □

4 Families generated by powers of kernels of type (G)

In this Section we consider kernels of type (G^q). Let $\varphi \in \mathcal{K}$ and $q \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^d} (\,\widehat{\varphi}(x))^q \, dx \neq 0\,. \tag{33}$$

For the kernels

$$W_0(h) \equiv 1\,;\; W_n(h) = \sum_{k\in\mathbb{Z}^d} \varphi\left(\frac{k}{n}\right) e^{ikh}\,,\; n \in \mathbb{N}\,, \tag{34}$$

of type (G) we introduce their *qth power* by (8)-(9). It is easy to see that

$$\gamma_n \equiv \gamma_n(\varphi, q) = (2\pi)^{-d} \int_{\mathbb{T}^d} (W_n(\varphi))^q \, dh = \sum_{k_1+\ldots+k_q=0} \prod_{j=1}^{q} \varphi\left(\frac{k_j}{n}\right)\,. \tag{35}$$

Up to the factor $n^{d(1-q)}$ these quantities are the Riemannian sums of the integral at the left-hand side of (19). In view of (33) they are not equal to 0 for $n \geq n_0$, where $n_0 \equiv n_0(\varphi, q)$ is a certain integer. By this reason the kernels $W_n(\varphi, q)$ are well defined by (8)-(9) at least for $n \geq n_0$.

Theorem 3. *Suppose that $\varphi \in \mathcal{K}$ satisfies (33) and let $1 \in \mathcal{P}_\varphi$. Then the family generated by the kernels $W_n(\varphi, q)$ converges in L_p if and only if $p \in q^{-1}\mathcal{P}_\varphi$.*

Proof. By (8) we get

$$W_n(\varphi, q)(h) = \sum_{k\,\in\,\mathbb{Z}^d} a_{n,k}\, e^{ikh}\,,\; n \geq n_0\,, \tag{36}$$

where

$$a_{n,k} \equiv a_{n,k}(\varphi, q) = \gamma_n^{-1} \sum_{k_1+\ldots+k_q=k} \prod_{j=1}^{q} \varphi\left(\frac{k_j}{n}\right)\,,\; k \in \mathbb{Z}^d\,. \tag{37}$$

First we prove that

$$\lim_{n\to+\infty} a_{n,k} = 1 \tag{38}$$

for each $k \in \mathbb{Z}^d$. If $k \in \mathbb{Z}^d$ and $n \geq n_0$ then

$$\begin{aligned} a_{n,k} - 1 &= \gamma_n^{-1} \left(\sum_{k_1+\ldots+k_q=k} \prod_{j=1}^{q} \varphi\left(\frac{k_j}{n}\right) - \sum_{k_1+\ldots+k_q=0} \prod_{j=1}^{q} \varphi\left(\frac{k_j}{n}\right) \right) = \\ &= \gamma_n^{-1} \left(\sum_{k_1,\ldots,k_{q-1}} \prod_{j=1}^{q-1} \varphi\left(\frac{k_1}{n}\right) \ldots \varphi\left(\frac{k_{q-1}}{n}\right) \cdot \right. \\ &\quad \cdot \left. \left(\varphi\left(\frac{k}{n} - \frac{k_1+\ldots+k_{q-1}}{n}\right) - \varphi\left(-\frac{k_1+\ldots+k_{q-1}}{n}\right) \right) \right) \end{aligned} \tag{39}$$

by (35) and (37).

In order to estimate the right-hand side of (39) we observe first that

$$\lim_{n \to +\infty} n^{d(1-q)} \gamma_n \; = \; (2\pi)^{-d} \int_{\mathbb{R}^d} (\widehat{\varphi}(x))^q \, dx \; \neq \; 0 \tag{40}$$

by (33) and (35). Moreover, by uniform continuity of φ for each $\varepsilon > 0$ there exists an integer n_1 such that

$$\sup_{\xi \in \mathbb{R}^d} \left| \varphi\left(\frac{k}{n} - \xi\right) - \varphi(-\xi) \right| \; < \; \varepsilon \, , \; n \geq n_1 \, . \tag{41}$$

Taking into account that the number of nonvanishing items in the sum over $k_1, \ldots, k_{q-1}$ in (39) does not exceed $cn^{d(q-1)}$, where c is a positive constant independent of n and using (40) and (41) we get from (39)

$$\begin{aligned} |\, a_{n,k} - 1 \,| \; &\leq \; c' n^{d(1-q)} \left| \sum_{k_1, \ldots, k_{q-1}} \sup_{\xi \in \mathbb{R}^d} \left| \varphi\left(\frac{k}{n} - \xi\right) - \varphi(-\xi) \right| \times \right. \\ &\quad \left. \times \; \left(\sup_{\xi \in \mathbb{R}^d} |\, \varphi(\xi) \,| \right)^{q-1} \right| \; \leq \; c'' \varepsilon \end{aligned}$$

for sufficiently large n (k is fixed). This completes the proof of (38).

Let $0 < s < +\infty$. Combining (8), (9) and (40) we see that

$$\begin{aligned} n^{d(1/s-1)} \, \| \, W_n(\varphi, q) \, \|_s \; &= \; n^{d(1/s-1)} \, |\gamma_n^{-1}| \, \| \, W_n(\varphi) \, \|_{qs}^q \; \asymp \\ &\asymp \; \left(n^{d(1/qs-1)} \, \| \, W_n(\varphi) \, \|_{qs} \right)^q \end{aligned}$$

It follows the equivalence

$$M_{s;\,A}(n) \; \asymp \; M_{qs;\,\varphi}(n) \, , \; n \geq n_0 \, , \tag{42}$$

using the notations in (12), where $A = \{ \, a_{n,k} \}$ is given by (37).

Let $p \in q^{-1}\mathcal{P}_\varphi$, that is, $qp \in \mathcal{P}_\varphi$. Then $q\widetilde{p} \in \mathcal{P}_\varphi$ as well. Indeed, for $0 < p \leq 1$ it is obvious. If $1 \leq p \leq +\infty$, then $q\widetilde{p} = q \in \mathcal{P}_\varphi$ because $q \geq 1$ and $1 \in \mathcal{P}_\varphi$. By the General Convergence Theorem [5], Theorem 4.1, (see also Theorem 1 with $R = 0$)

$$\sup_n \| \, \{ \, \mathcal{L}_{n;\,\lambda}^{(\varphi)} \} \, \|_{(q\widetilde{p})} \; < \; +\infty \, . \tag{43}$$

Applying Lemma 1 and (42) with $s = \widetilde{p}$ we find

$$\| \, \{ \, \mathcal{L}_{n;\,\lambda}^{(A)} \} \, \|_{(p)} \; \lesssim \; M_{\widetilde{p},\,A}(n) \; \asymp \; M_{q\widetilde{p};\,\varphi}(n) \; \lesssim \; \| \, \{ \, \mathcal{L}_{n;\,\lambda}^{(\varphi)} \} \, \|_{(q\widetilde{p})} \tag{44}$$

for $n \geq n_0$. By (43), (44) the family $\{ \, \mathcal{L}_{n;\,\lambda}^{(A)} \}$ is uniformly bounded on L_p. Thus, its convergence follows from (38) and Theorem A.

Let now $p \notin q^{-1}\mathcal{P}_\varphi$, that is, $qp \notin \mathcal{P}_\varphi$. Then $M_{qp;\,\varphi} = +\infty$ by Lemma 2. In view of (42) with $s = p$ it implies $M_{p,\,A} = +\infty$. Hence, $\{ \, \mathcal{L}_{n;\,\lambda}^{(A)} \}$ is not uniformly bounded on L_p by Lemma 1. The proof is complete. □

5 Families generated by (generalized) Jackson kernels

The **(generalized) Jackson kernels** are defined by

$$J_{n;\,q}(h) \;=\; \gamma_{n,\,q}^{-1}\,(F_n(h))^q\,,\; n \in \mathbb{N}_0\,,\; (2\pi)^{-1}\int_0^{2\pi} (F_n(h))^q)\,dh \;=\; \gamma_{n,\,q}\,, \tag{45}$$

where $q \in \mathbb{N}$ and $F_n(h)$ are the Fejér kernels generated by $\varphi(\xi) = (\,1 - |\,\xi\,|\,)_+$ ($a_+ = \max(a, 0)$). The classical Fejér and Jackson kernels correspond to $q = 1$ and $q = 2$, respectively. In the general case $J_{n;\,q}(h)$ is of type G^q. They have been introduced by S. Stechkin [13] in order to prove the Jackson type estimate (direct theorem of approximation theory) for moduli of smoothness of higher orders. Taking into account that $\mathcal{P}_\varphi = (1/2, +\infty]$ (see, e.g. [5], Section 5) and applying Theorem 3 we immediately obtain the following statement.

Theorem 4. *The family* $\{J_{n;\,\lambda}^{(q)}\}$*,* $q \in \mathbb{N}$*, generated by the generalized Jackson kernels* (45) *converges in* L_p *if and only if* $1/(2q) < p \le +\infty$*.*

Combining this result with the comparison principle [5], Lemma 2.2, we obtain the classical statements on the convergence of the Fourier means and the interpolation means generated by $J_{n;\,q}(h)$ in the spaces L_p for $1 \le p \le +\infty$ and C, respectively (see, e.g. [2], Chapter 7, §2, [13]).

6 Families generated by Fejér-Korovkin kernels

The **Fejér-Korovkin kernels** $K_n(h)$ are defined by (1)-(2), where $A = \{a_{n,\,k} : \,|\,k\,| \le n-2,\, n \ge 2\}$ and

$$a_{n,\,k} \;=\; \frac{(\,n - |\,k\,| + 1\,)\sin\dfrac{|\,k\,|+1}{n}\pi \;-\; (\,n - |\,k\,| + 1\,)\sin\dfrac{|\,k\,|-1}{n}\pi}{2n\sin(\pi/n)}\,. \tag{46}$$

They were introduced by P. Korovkin [4]. The corresponding Fourier means of type (3) turn out to solve the problem of the existence of positive methods of summation which would be relevant for the proof of the Jackson type estimate. In this connection we mention that the Fejér kernels do not have this property. By elementary calculations we get

$$a_{n,\,k} \;=\; \left(1 - \frac{|\,k\,|}{n}\right)\cos\frac{\pi|\,k\,|}{n} \;+\; \frac{1}{\pi}\sin\frac{\pi|\,k\,|}{n} \;+\; \lambda_n\,\sin\frac{\pi|\,k\,|}{n}\,,\; |\,k\,| \;\le\; n-2\,, \tag{47}$$

where

$$\lambda_n \;=\; \frac{1 - (n/\pi)\sin(\pi/n) - 2(\sin(\pi/(2n)))^2}{n\sin(\pi/n)} \ge 0\,,\; n \ge 2\,, \tag{48}$$

Clearly,

$$\lambda_n \;\lesssim\; n^{-2}\,. \tag{49}$$

By (47) $A = A(\varphi) + R' + R''$, where

$$\varphi(\xi) \;=\; \begin{cases} (1-|\,\xi\,|)\cos\pi\xi \;+\; (1/\pi)\sin\pi|\,\xi\,| & ,\ |\,\xi\,| \le 1 \\ 0 & ,\ |\,\xi\,| > 1 \end{cases} \tag{50}$$

$$R' = \{r'_{n,k}\} : \, r'_{n,k} = \begin{cases} -\lambda_{n-1}\sin\dfrac{\pi(n-1)}{n} - 2\,\varphi\left(\dfrac{n-1}{n}\right), & |k| = n-1 \\ 0 & ,\quad \text{otherwise} \end{cases} \tag{51}$$

$$R'' \;=\; \{r'_{n,k}\} : \; r''_{n,k} \;=\; \lambda_n \sin\frac{\pi|\,k\,|}{n}\,, \quad |\,k\,| \le n\,. \tag{52}$$

By straightforward calculation one has

$$\widehat{\varphi}(x) \;=\; 4\pi^2\,\frac{(\cos(x/2))^2}{(x-\pi)^2(x+\pi)^2}\;, \quad \mathcal{P}_\varphi \;=\; (1/4,\, +\infty\,]\;. \tag{53}$$

Using (49) - (51) we obtain

$$\sup_{|k|=n-1} r'_{n,k} \;\lesssim\; n^{-3}.$$

In view of (14) this leads to

$$\mathcal{M}_{1/4;\,R'} \;<\; +\infty\;. \tag{54}$$

To estimate the contribution of R'' we rewrite formula (5.7) as

$$r''_{n,k} \;=\; \lambda_n\psi\left(\frac{k}{n}\right)\;, \quad |\,k\,| \le n\,, \tag{55}$$

where $\psi(\xi) = \sin\pi|\xi|$. By straightforward calculation

$$\widehat{\psi}(x) \;=\; \frac{4\pi^2\cos^2(x/2)}{\pi^2 - x^2}\;; \quad \widehat{\psi}(x) = O(|\,x\,|^{-2}),\; x \to +\infty\,. \tag{56}$$

By means of (49), (53), (54) - (56) we see that all conditions of Theorem 2 are satisfied with $d = 1$, $\delta = 2$ and $\alpha = 1/4$. Applying Theorem 2 to the Fejér-Korovkin kernels we obtain the following result.

Theorem 5. *The family $\{K_{n;\,\lambda}\}$ generated by the Fejér-Korovkin kernels converges in L_p if and only if $1/4 < p \le +\infty$.*

Combining this result with the comparison principle [5], Lemma 2.2, we obtain, in particular, the statement of P. Korovkin on the convergence of the Fourier means generated by $K_n(h)$ in C (cf. [4]).

7 Families generated by Cesaro kernels I: structure of the matrix of multipliers

As known (see, e.g. [15], Vol. 1), the **Cesaro kernels** $C_n^{(\alpha)}(h)$ with index $\alpha > 0$ are defined by (1)-(2), where $A \equiv A_\alpha = \{a_{n,k} : |k| \le n-1,\, n \in \mathbb{N}\}$ and

$$a_{n,k}^{(\alpha)} \;=\; \frac{\Gamma(n-|\,k\,|+\alpha)\,\Gamma(n)}{\Gamma(n-|\,k\,|)\,\Gamma(n+\alpha)}\,,\ \ |\,k\,| \le n-1\,. \tag{57}$$

Obviously, the case $\alpha = 1$ corresponds to the Fejér kernels. Applying the asymptotic formula for the Γ-function

$$\Gamma(z) \;=\; (2\pi)^{1/2}\, e^{-z}\, z^{z-1/2}\, \left(1 + (12z)^{-1} + O(z^{-2})\right)\,,\ \ z \ne 0,\ |\arg z\,| < \pi\,,$$

we get

$$\begin{aligned} a_{n,k}^{(\alpha)} &= \left(1 - \frac{|\,k\,|}{n}\right)^{\alpha} \left(1 + \frac{\alpha}{n-|\,k\,|}\right)^{n-|\,k\,|+\alpha-1/2} \left(\frac{n}{n+\alpha}\right)^{n+\alpha-1/2} \\ &\quad \frac{1 + (\,12(n-|\,k\,|+\alpha)\,)^{-1} + O\left(\,(n-|\,k\,|)^{-2}\,\right)}{1 + (\,12(n-|\,k\,|)\,)^{-1} + O\left(\,(n-|\,k\,|)^{-2}\,\right)}\, \frac{1+O\left(n^{-1}\right)}{1+O\left(n^{-1}\right)} \equiv \\ &\equiv \ \varphi_\alpha\left(\frac{k}{n}\right) \cdot J_1 \cdot J_2 \cdot J_3 \cdot J_4 \end{aligned} \tag{58}$$

for $n \in \mathbb{N}$, $|\,k\,| \le n-1$ from (57). Here $\varphi_\alpha(\xi) = (1 - |\,\xi\,|)_+^\alpha$ and $b_{n,k} = O(a_{n,k})$ means that $|b_{n,k}| \le c a_{n,k}$, where c does not depend on k and n. Using Taylor expansions for $\exp z$, $\ln z$ and $(1-z)^{-1}$ we obtain

$$J_1 \;=\; e^{\alpha} \left(1 \,+\, \frac{\alpha(\alpha-1)}{2(n-|\,k\,|)} \,+\, O\left(\,(n-|\,k\,|)^{-2}\,\right)\right)\,, \tag{59}$$

$$J_2 \;=\; e^{-\alpha}\left(1 \,+\, O\left(n^{-1}\right)\right)\,, \tag{60}$$

$$J_3 \;=\; 1 \,+\, O\left(\,(n-|\,k\,|)^{-2}\,\right)\,, \tag{61}$$

$$J_4 \;=\; 1 \,+\, O\left(\,n^{-1}\,\right)\,. \tag{62}$$

By (58) - (62)

$$a_{n,k}^{(\alpha)} = \varphi_\alpha\left(\frac{k}{n}\right)\left(1 + \frac{\alpha(\alpha-1)}{2(n-|\,k\,|)} + O\left(\,(n-|\,k\,|)^{-2}\,\right)\right)(1+\lambda_n)\,, \tag{63}$$

where $\lambda_n = O(n^{-1})$.

We introduce the matrix $\overline{A}_\alpha = \{\,b_{n,k}\}$, where $b_{n,k}$ are defined by the right-hand side of (63) with $\lambda_n = 0$. By (63) we get

$$\overline{A}_\alpha \;=\; A(\varphi_\alpha) \,+\, R'_\alpha \,+\, R''_\alpha\,, \tag{64}$$

where

$$\mathcal{R}'_\alpha \;=\; \big(r'_{n,k}(\alpha)\big)\ ,\quad r'_{n,k}(\alpha) \;=\; \frac{\alpha(\alpha-1)}{2}\, n^{-1}\,\varphi_{\alpha-1}\left(\frac{k}{n}\right)\,, \tag{65}$$

$$\mathcal{R}''_\alpha \;=\; \big(r''_{n,k}(\alpha)\big)\ ,\quad r''_{n,k}(\alpha) \;=\; O\left(n^{-2}\,\varphi_{\alpha-2}\left(\frac{k}{n}\right)\right). \tag{66}$$

Taking into account that

$$W_n(A_\alpha)(h) \;=\; (1+\lambda_n)\, W_n(\overline{A}_\alpha)(h)$$

it follows from Lemma 1 and Theorem A that the ranges of convergence of the Cesaro families $\{\mathcal{C}^{(\alpha)}_{n;\lambda}\}$ and the families generated by $\overline{A}_\alpha$ coincide. For this reason we deal with the matrix $\overline{A}_\alpha$ in the following.

8 Families generated by Cesaro kernels II: Fourier transform of the generator

As it was shown in the previous section, the matrix $\overline{A}_\alpha$ is of type (7) with $\varphi(\xi) \equiv \varphi_\alpha(\xi) = (1-|\,\xi\,|)^\alpha_+$. Following our approach we have to study the properties of the Fourier transform of the generator φ_α and to dtermine the set $\mathcal{P}_\alpha = \{p \in (0,+\infty] : \widehat{\varphi}_\alpha \in L_p(\mathbb{R})\}$.

Let ψ, ψ_0 and ψ_1 be real valued even infinitely differentiable functions defined on $\mathbb{R}$ and satisfying

$$\psi(\xi) = \begin{cases} 1\,, & \xi \in D_1 \\ 0\,, & \xi \notin D_{5/4} \end{cases};\; \psi_0(\xi) = \begin{cases} 1\,, & \xi \in D_{1/2} \\ 0\,, & \xi \notin D_{3/4} \end{cases};\; \psi_1(\xi) = \psi(\xi) - \psi_0(\xi), \tag{67}$$

where $D_\rho = \{\,\xi : |\,\xi\,| \le \rho\,\}$. It is clear that

$$\varphi_\alpha(\xi) \;=\; (\varphi_\alpha\,\psi_0)(\xi) \;+\; (\varphi_\alpha\,\psi_1)(\xi)\,,\ \xi \in \mathbb{R}. \tag{68}$$

In view of

$$(\,1-y\,)^\alpha \;=\; \sum_{\nu=0}^{3} \frac{(-1)^\nu[\alpha]_\nu}{\nu!}\, y^\nu \;+\; y^4\, g(y)\,,\ -1<y<1\,,$$

where $[\alpha]_\nu = \alpha(\alpha-1)\ldots(\alpha-\nu+1)$ and $g(y)$ is analytic on $(-1,1)$, we obtain

$$(\varphi_\alpha\,\psi_0)(\xi) \;=\; \sum_{\nu=0}^{3} \frac{(-1)^\nu[\alpha]_\nu}{\nu!}\,|\,\xi\,|^\nu\,\psi_0(\xi) \;+\; |\,\xi\,|^4\, g(|\,\xi\,|)\,\psi_0(\xi)\,. \tag{69}$$

As it was shown in [11], Lemma 4.1 and Theorem 4.1, for any $\delta > 0$, $\delta \ne 2k$ for each $k \in \mathbb{N}$ one has

$$\left|\big(|\cdot|^\delta\,\psi_0(\cdot)\big)^\wedge(x)\right| \;\asymp\; |\,x\,|^{-(\delta+1)}\,,\ |\,x\,| \to +\infty\,, \tag{70}$$

where $A \asymp B$ means $A \le c_1 B$ and $B \le c_2 A$ simultaneously for $|x| \ge \rho$ with some positive c_1, c_2 and ρ independent of x. Moreover, by the differentiability properties of the function $|\,\xi\,|^4 g(|\,\xi\,|)$ on $(-1,1)$ we get

$$\left| \left(|\cdot|^5\, g(|\cdot|)\, \psi_0(\cdot) \right)^{\wedge}(x) \right| \;\le\; c\,(1 + |\,x\,|)^{-3}\,,\; x \in \mathbb{R}\,. \tag{71}$$

Since $\psi_0(\xi)$ and $|\,\xi\,|^2\,\psi_0(\xi)$ are infinitely differentiable their Fourier transforms can be also estimated by the right-hand side of (71). Applying (70) and (71) to the right-hand side of (69) we finally obtain

$$\left| \,(\varphi_\alpha\, \psi_0)^{\wedge}\,(x) \right| \;\asymp\; |\,x\,|^{-2}\,,\; |\,x\,| \to +\infty\,, \tag{72}$$

for the Fourier transform of the first summand on the right-hand side of (68).

The Fourier transform of $\varphi_\alpha \psi_1$ can be studied by reduction to the properties of the generator of the Bochner-Riesz kernels $\varphi_{(\beta)}(\xi) = (1 - |\,\xi\,|^2)_+^\beta$ with the index $\beta > 0$ in the one-dimensional case. As is known ([14], Ch. 9, pp. 389-390]),

$$\widehat{\varphi}_{(\beta)}(x) \;=\; \pi^{-\beta}\,\Gamma(\beta+1)\,|\,x\,|^{-\beta-1/2}\, J_{\beta+1/2}(|\,x\,|)\,, \tag{73}$$

where $J_s(x)$, $s > -1/2$, is the Bessel function of order s. Using its properties, in particular, their asymptotic formula and the results on the distribution of their zeros ([14], Ch. 8) it follows from (73)

$$\left| \,\widehat{\varphi}_{(\beta)}(x) \right| \;\le\; c\,(1 + |\,x\,|)^{-(\beta+1)}\,,\; x \in \mathbb{R}\,, \tag{74}$$

$$\left| \,\widehat{\varphi}_{(\beta)}(x) \right| \;\ge\; c'\,|\,x\,|^{-(\beta+1)}\,,\; x \in \Omega(\beta)\,, \tag{75}$$

with some positive constants independent of x, c and c'. Here (for the sake of shortness we omit the index β in our notations, if it does not affect the concepts)

$$\Omega = \bigcup_{k=1}^{+\infty} \{\,\xi : \, a_k \;\le\; |\,\xi\,| \;\le\; b_k\,\}\,, \tag{76}$$

$$1 \;\le\; a_k \;<\; b_k \;\le\; a_{k+1}\,;\; \inf_k\,(b_k - a_k) \;>\; 0\,;\; a_k \;=\; O(k)\,,\; k \to +\infty\,. \tag{77}$$

We shall use the representation

$$\varphi_\alpha\, \psi_1(\xi) \;=\; \varphi_{(\alpha)}(\xi)\, \psi_1(\xi)\, g_\alpha(\,1 - |\,\xi\,|^2\,)\,,\; \xi \in \mathbb{R}\,, \tag{78}$$

where

$$g_\alpha(\eta) \;=\; \begin{cases} \left(\dfrac{1 - (1-\eta)^{1/2}}{\eta} \right)^{\alpha} & ,\quad 0 < |\,\eta\,| < 1 \\ (1/2)^\alpha & ,\quad \eta = 0 \end{cases}\;. \tag{79}$$

The function $g_\alpha(\eta)$ is analytic in $(-1,1)$. By expansion into power series at $\eta = 0$ and considering first $(s-1)$th terms, where $s = [\alpha] + 4$, we conclude

$$2^\alpha \left(\varphi_\alpha \psi_1\right)(\xi) = \varphi_{(\alpha)}(\xi)\,\psi_1(\xi) + \sum_{\nu=1}^{s-1} g_\nu^{(\alpha)} \cdot \left(\varphi_{(\alpha+\nu)}\,\psi_1\right)(\xi) + g_s^{(\alpha)} \cdot$$

$$\left(\varphi_{(\alpha+s)}\,\psi_1\right)(\xi)\,h(\xi) = \varphi_{(\alpha)}(\xi) + \sum_{\nu=1}^{s-1} g_\nu^{(\alpha)} \cdot \varphi_{(\alpha+\nu)}(\xi) -$$

$$\sum_{\nu=0}^{s-1} g_\nu^{(\alpha)} \cdot \left(\varphi_{(\alpha+\nu)}\,\psi_0\right)(\xi) + g_s^{(\alpha)} \cdot \left(\varphi_{(\alpha+s)}\,\psi_1\right)(\xi)\,h(\xi) \equiv \tag{80}$$

$$\varphi_{(\alpha)}(\xi) + I(\xi)\,,\ \xi \in \mathbb{R}\,,$$

by (68) and (78). Here $g_\nu^{(\alpha)}$, $\nu = 1, \ldots, s$, are certain coefficients and the function h is infinitely differentiable in $D_{\sqrt{2}} \setminus \{0\}$.

By (74) we obtain

$$\left|\,\widehat{\varphi}_{(\alpha+\nu)}(x)\,\right| \;\le\; c\,(\,1 + |\,x\,|\,)^{-(\alpha+\nu+1)}\,,\ x \in \mathbb{R},\ \nu = 0, 1, \ldots, s-1\,. \tag{81}$$

The functions $\varphi_{(\alpha+\nu)}\,\psi_0$, $\nu = 0, 1, \ldots, s-1$, are infinitely differentiable and they have a compact support. Hence, in particular,

$$\left|\,\left(\varphi_{(\alpha+\nu)}\,\psi_0\right)^\wedge(x)\,\right| \;\le\; c\,(\,1 + |\,x\,|\,)^{-(\alpha+2)}\,,\ x \in \mathbb{R},\ \nu = 0, 1, \ldots, s-1\,. \tag{82}$$

Recall that $s = [\alpha] + 4$. By direct calculation we find that the function $\varphi_{\alpha+s}\,\psi_1 h$ is $([\alpha]+3)$-times continuously differentiable. This yields the estimate

$$\left|\,\left(\varphi_{(\alpha+s)}\,\psi_1\,h\right)^\wedge(x)\,\right| \;\le\; c\,(\,1 + |\,x\,|\,)^{-([\alpha]+3)}\,,\ x \in \mathbb{R}\,. \tag{83}$$

By (81) (for $\nu \neq 0$), (82) and (83) we get

$$|\,\widehat{I}(x)\,| \;\le\; c\,|\,x\,|^{-(\alpha+2)}\,,\ |\,x\,| \ge 1\,, \tag{84}$$

for the Fourier transform of the remainder $I(\xi)$ in (80).

Combining (74) and (75) with $\beta = \alpha$, (80), (81) (for $\nu = 0$) and (84) we obtain

$$\left|\,(\varphi_\alpha\,\psi_1)^\wedge(x)\,\right| \;\le\; c\,(\,1 + |\,x\,|\,)^{-(\alpha+1)}\,,\ x \in \mathbb{R}\,, \tag{85}$$

$$\left|\,(\varphi_\alpha\,\psi_1)^\wedge(x)\,\right| \;\ge\; c'\,|\,x\,|^{-(\alpha+1)}\,,\ x \in \Omega\,, \tag{86}$$

where Ω is of type (76)-(77) and c, c' are positive constants independent of x.

Now we are able to formulate and prove the main result of this section.

Lemma 5. *It holds $\mathcal{P}_\alpha = (q_*, +\infty]$, where $q_* \equiv q_*(\alpha) = \max(1/2, 1/(\alpha+1))$, $\alpha > 0$.*

Proof. Let first $0 < \alpha \le 1$. By (68), (72) and (85)

$$|\,\widehat{\varphi}_\alpha(x)\,| \;\le\; c\,(\,1 + |\,x\,|\,)^{-(\alpha+1)}\,,\;\; x \in \mathbb{R}\,. \tag{87}$$

Applying (68), (72) and (86) we get

$$\begin{aligned} |\,\widehat{\varphi}_\alpha(x)\,| \;&\ge\; \left|\,(\varphi_\alpha\,\psi_1)^\wedge\,(x)\,\right| \;-\; \left|\,(\varphi_\alpha\,\psi_0)^\wedge\,(x)\,\right| \\ &\ge\; |\,x\,|^{-(\alpha+1)}\;(\,c' \;-\; c\,|\,x\,|^{\alpha-1}\,) \\ &\ge\; c''\,|\,x\,|^{-(\alpha+1)}\,,\;\; x \in \Omega\,,\;\; |\,x\,| \ge (\,c'/(2c)\,)^{1/(1-\alpha)}\;. \end{aligned} \tag{88}$$

Let $0 < p < +\infty$. In view of (76)-(77) and by means of (87), (88) we find

$$c_1 \sum_{k=k_0}^{+\infty} k^{-p(\alpha+1)} \;\le\; c_1' \sum_{k=k_0}^{+\infty} \int_{a_k}^{b_k} x^{-p(\alpha+1)}\,dx \;\le\; \|\,\widehat{\varphi}_\alpha\,\|_p^p \;\le\; c_2 \int_1^{+\infty} x^{-p(\alpha+1)}\,dx$$

for and some $k_0 \in \mathbb{N}$. This yields $\mathcal{P}_\alpha = (1/(\alpha+1), +\infty]$.

Let now $\alpha > 1$. In this case the summand $\varphi_\alpha\psi_0$ on the right-hand side of (68) is dominating. By (68), (72) and (85) we get

$$|\,\widehat{\varphi}_\alpha(x)\,| \;\le\; c\,(\,1 + |\,x\,|\,)^{-2}\,,\;\; x \in \mathbb{R}\,, \tag{89}$$

and

$$\begin{aligned} |\,\widehat{\varphi}_\alpha(x)\,| \;&\ge\; \left|\,(\varphi_\alpha\,\psi_0)^\wedge\,(x)\,\right| \;-\; \left|\,(\varphi_\alpha\,\psi_1)^\wedge\,(x)\,\right| \;\ge \\ &\ge\; |\,x\,|^{-2}\;(\,c_1 \;-\; c\,|\,x\,|^{1-\alpha}\,) \;\ge \\ &\ge\; c_2\,|\,x\,|^{-2}\,,\;\; x \in \Omega\,,\;\; |\,x\,| \ge (\,c_1/(2c)\,)^{1/(\alpha-1)}\;. \end{aligned} \tag{90}$$

In view of (89) and (90) we have $\mathcal{P}_\alpha = (1/2, +\infty]$.

To complete the proof we recall that for $\alpha = 1$ the function φ_α is the generator of the Fejér kernels and $\mathcal{P}_1 = (1/2, +\infty]$ (see, e.g. [5]). □

9 Families generated by Cesaro kernels III: influence of remainders and convergence result

Now we consider the contributions of the matrices $\mathcal{R}'_\alpha$ and $\mathcal{R}''_\alpha$ given by (65) and (66).

Lemma 6. *Suppose* $\nu \in \mathbb{N}$ *and* $0 < \alpha < \nu - 1/2$. *If the matrix* $\mathcal{R}_\alpha(\nu) = \{r_{n,\,k}(\alpha, \nu)\}$ *satisfies*

$$r_{n,\,k}(\alpha, \nu) \;=\; O\left(n^{-\nu}\,\varphi_{\alpha-\nu}\left(\frac{k}{n}\right)\right)\,,\;\; |\,k\,| \le n-1,\;\; n \in \mathbb{N}\,, \tag{91}$$

then $\mathcal{M}_{1/(\alpha+1);\,R_\alpha(\nu)} < +\infty$.

Proof. Indeed, assumption (91) leads to

$$\begin{aligned} n^{2(1/q-1)} \sum_{|k|\le n-1} |\, r_{n,k}(\alpha;\nu)\,|^2 &\lesssim n^{2(1/q-1-\alpha)} \sum_{|k|\le n-1} (n - |\,k\,|)^{2(\alpha-\nu)} \\ &\lesssim n^{2(1/q-1-\alpha)} \sum_{j=1}^{n} j^{2(\alpha-\nu)} \\ &\asymp n^{2(1/q-1-\alpha)} \cdot \begin{cases} 1 & , \quad \alpha < \nu - 1/2 \\ \ln(n+1) & , \quad \alpha = \nu - 1/2 \\ n^{2(\alpha-\nu)+1} & , \quad \alpha > \nu - 1/2 \end{cases} \end{aligned} \tag{92}$$

for any $0 < q \le 1$. Applying (92) to $q = 1/(\alpha+1)$ we obtain the desired result. □

Lemma 7. *It holds $\mathcal{M}_{q_*;\,R''_\alpha} < +\infty$, where q_* has the meaning of Lemma 5.*

Proof. For $0 < \alpha < 3/2$ this statement follows immediately from Lemma 6. Applying (92) with $q = 2/5$ we conclude that $\mathcal{M}_{2/5+\varepsilon;\,R''_\alpha} < +\infty$ for each $\varepsilon > 0$ if $\alpha = 3/2$ and $\mathcal{M}_{2/5;\,R''_\alpha} < +\infty$ if $\alpha > 3/2$. In both these cases $q_* = 1/2$ and

$$\mathcal{M}_{q_*;\,R''_\alpha} \le \mathcal{M}_{2/5;\,R''_\alpha} \le \mathcal{M}_{2/5+\varepsilon;\,R''_\alpha} < +\infty\,.$$

The proof is complete. □

Lemma 8. *It holds $M_{q_*;\,R'_\alpha} < +\infty$, where q_* has the meaning of Lemma 5 .*

Proof. If $0 < \alpha < 1/2$ one has

$$M_{1/(\alpha+1);\,R'_\alpha} \le (2\pi)^{\alpha+1/2} \mathcal{M}_{1/(\alpha+1);\,R'_\alpha} < +\infty$$

by (14) and Lemma 6.

Let now $1/2 \le \alpha < 1$. In view of (65) we get by Abel's identity for $n \ge 2$

$$\begin{aligned} W_n(\mathcal{R}'_\alpha)(h) &= \alpha(\alpha-1)\, n^{-\alpha} \left(\frac{n^{\alpha-1}}{2} + \sum_{k=1}^{n-1} (n-k)^{\alpha-1} \cos kh \right) \\ &= \alpha(\alpha-1)\, n^{-\alpha} \Bigg(D_{n-1}(h) + \sum_{k=1}^{n-2} (\,(n-k)^{\alpha-1} - \\ &- \quad (n-k-1)^{\alpha-1}\,)\, D_k(h) \Bigg) = \alpha(\alpha-1)\, n^{-\alpha} \cdot \\ &\left(D_{n-1}(h) + (\alpha-1) \sum_{k=1}^{n-2} (\,\xi_{k,n}(\alpha)\,)^{\alpha-2} D_k(h) \right) , \end{aligned} \tag{93}$$

where $\xi_{k,n}(\alpha) \in [\,n-k-1, n-k\,]$ and $D_k(h)$ are the Dirichlet kernels. Since $1/(\alpha+1) \le 1/(2-\alpha) < 1$ for $1/2 \le \alpha < 1$, we can choose q such that

$$1/(1+\alpha) \le 1/(2-\alpha) < q < 1\,. \tag{94}$$

Applying the well-known estimates for the norms of Dirichlet kernels and using (93), (94) we obtain

$$\begin{aligned}\| W_n(\mathcal{R}'_\alpha) \|_{1/(1+\alpha)} &\lesssim \| W_n(\mathcal{R}'_\alpha) \|_q \\ &\lesssim n^{-\alpha} \left(\| D_{n-1} \|_q^q + \sum_{k=1}^{n-2} (n-k)^{q(\alpha-2)} \| D_{n-1} \|_q^q \right)^{1/q} \\ &\lesssim n^{-\alpha} \left(\sum_{\nu=1}^{n} \nu^{q(\alpha-2)} \right) \lesssim n^{-\alpha} .\end{aligned}$$

This yields $M_{1/(\alpha+1);\, R'_\alpha} < +\infty$.

For $\alpha = 1$ the matrix R'_α is identical 0 by (65). In the case $1 < \alpha < 2$ one has $\widehat{\varphi_{\alpha-1}}(x) = O(|\, x\,|^{-\alpha})$, $(x \to +\infty)$ by (87). Applying now Lemma 4 with $d = 1$, $p = 1/2$, $\delta = \alpha$ and using (65) we find

$$M_{1/2;\, R'_\alpha} \lesssim n^{-1} M_{1/2;\, \varphi_{\alpha-1}} \lesssim n^{-1} n^{2-\alpha} = n^{1-\alpha}$$

This implies $M_{1/2;\, R'_\alpha} < +\infty$.

If $\alpha \geq 2$ we apply (89) with $\alpha - 1$ in place of α. In view of (68), (72) and (85) it remains valid also for $\alpha - 1 = 1$. Then $\widehat{\varphi_{\alpha-1}}(x) = O(|\, x\,|^{-2})$, $(x \to +\infty)$. Applying now Lemma 4 with $d = 1$, $p = 1/2$, $\delta = 2$ and (65) derive the estimates

$$M_{1/2;\, R'_\alpha} \lesssim n^{-1} M_{1/2;\, \varphi_{\alpha-1}} \lesssim n^{-1} (\ln(n+1))^{1/2}.$$

It follows $M_{1/2;\, R'_\alpha} < +\infty$. □

Combining Theorem 1, Lemma 5 and Lemmas 7 - 8 we obtain the following convergence criterion.

Theorem 6. *The family* $\{ \mathcal{C}^{(\alpha)}_{n;\,\lambda} \}$, $\alpha > 0$, *generated by the Cesaro kernels with index* α *converges in* L_p *if and only if* $q_*(\alpha) < p \leq +\infty$, *where* $q_* \equiv q_*(\alpha) = \max(1/2, 1/(\alpha+1))$.

In view of the comparison principle [5], Lemma 2.2, this theorem implies the classical M. Riesz theorem on the convergence of the Cesaro means in the space C (see, e.g. [15], Vol. 1, Ch. 3).

References

[1] Z. Burinska, K. Runovski, H.-J. Schmeisser, *On the method of approximation by families of linear polynomial operators.* J. of Anal. and Appl., 19 (2000), 677 – 693.

[2] R. DeVore, G. Lorentz, *Constructive Approximation.* Springer, Berlin-Heidelberg-New York, 1993.

[3] L. Grafakos, *Classical and Modern Fourier Analysis.* Pearson Education, Inc., Upper Saddle River, New Jersey, 2004.

[4] P. Korovkin, *Linear operators and approximation theory.* Fizmatgiz, Moscow, 1959 (in Russian).

[5] V. Rukasov, K. Runovski, H.-J. Schmeisser, *On convergence of families of linear polynomial operators.* Funct. et Approx., 41, no. 1 (2008), 41 – 54.

[6] K. Runovski, *On families of linear polynomial operators in* L_p*-spaces,* $0 < p < 1$. Ross. Akad. Nauk Matem. Sbornik, 184 (1993), 33 – 42 (in Russian). English transl. Russian Acad. Sci. Sb. Math., 78 (1994), 165 – 173.

[7] K. Runovski, *On approximation by families of linear polynomial operators in* $L_p -$ *spaces,* $0 < p < 1$. Ross. Akad. Sci. Matem. Sbornik, 185 (1994), 81 – 102 (in Russian). English transl. Russian Acad. Sci. Sb. Math., 82 (1995), 441 – 459.

[8] K. Runovski, I. Rystsov, H.-J. Schmeisser, *Computational aspects of a method of stochastic approximation.* J. of Anal. and Appl., 25 (2006), 367 – 383.

[9] K. Runovski, H.-J. Schmeisser, *On the convergence of Fourier means and interpolation means.* Journal of Comp. Anal. and Appl., 6, no. 3 (2004), 211 – 220.

[10] K. Runovski, H.-J. Schmeisser, *On the families of operators generated by the Bochner-Riesz kernels.* J. of Fourier Anal. and Appl., 14 (2008), 16 – 38.

[11] K. Runovski, H.-J. Schmeisser, *On some extensions of Bernstein's inequalities for trigonometric polynomials.* Funct. et Approx., XXIX (2001), 125 – 142.

[12] H.-J. Schmeisser, H. Triebel: *Topics in Fourier Analysis and Function Spaces.* Geest & Portig, Leipzig, 1987; and Wiley, Chichester, 1987.

[13] S.B. Stechkin, *On the order of approximation of continuous functions.* Izv. Akad. Nauk USSR, 15 (1951), 219 – 242 (in Russian).

[14] E.M. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality and Oscillatory Integrals.* Princeton Univ. Press, Princeton, 1993.

[15] A. Zygmund, *Trigonometric series.* Vol. 1 & 2. Cambridge Univ. Press, Cambridge, 1990.

Konstantin Runovski
Sevastopol Branch of Moscow State University
Ger. Sevastopolya Str., 7
99001 Sevastopol, Ukraine E-mail: k_runov@mail.ru

Hans-Jurgen Schmeisser
Mathematisches Institut
Friedrich-Schiller University
D-07737 Jena, Germany
E-mail: mhj@uni-jena.de

Received: 04.06.2010