

**A MAXIMUM PRINCIPLE IN SPECTRAL OPTIMIZATION
PROBLEMS FOR ELLIPTIC OPERATORS SUBJECT TO MASS
DENSITY PERTURBATIONS**

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Communicated by T.V. Tararykova

Key words: high order elliptic operators, eigenvalues, mass density.

AMS Mathematics Subject Classification: 35J40, 35B20, 35P15.

Abstract. We consider eigenvalue problems for general elliptic operators of arbitrary order subject to homogeneous boundary conditions on open subsets of the Euclidean N -dimensional space. We prove stability results for the dependence of the eigenvalues upon variation of the mass density and we prove a maximum principle for extremum problems related to mass density perturbations which preserve the total mass.

1 Introduction

We consider a general class of elliptic partial differential operators

$$\mathcal{L}u = \sum_{0 \leq |\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha (A_{\alpha\beta} D^\beta u)$$

subject to homogeneous boundary conditions on an open subset Ω of \mathbb{R}^N with finite measure. We assume that the coefficients $A_{\alpha\beta}$ are fixed bounded real-valued functions such that $A_{\alpha\beta} = A_{\beta\alpha}$ and such that Gårding's inequality is satisfied. For such operators we consider the eigenvalue problem

$$\mathcal{L}u = \lambda \rho u, \tag{1.1}$$

where ρ is a positive function bounded away from zero and infinity. Problem (1.1) admits a divergent sequence of eigenvalues of finite multiplicity

$$\lambda_1[\rho] \leq \dots \leq \lambda_n[\rho] \leq \dots$$

In this paper we prove few results concerning the dependence of $\lambda_n[\rho]$ upon variation of ρ .

Keeping in mind important problems involving harmonic and bi-harmonic operators in linear elasticity (see e.g., Courant and Hilbert [10]), we shall think of the weight ρ as the mass density of the body Ω and we shall refer to the quantity $M = \int_{\Omega} \rho dx$ as the total mass of Ω . In the study of composite materials it is of interest

to know whether it is possible to minimize or maximize the eigenvalues $\lambda_n[\rho]$ under the assumption that the total mass M is fixed (see e.g., Chanillo et al. [8], Cox and McLaughlin [11, 12, 13], Henrot [18]). In this paper we generalize the results proved in [22] for the Dirichlet Laplacian. In particular, we prove the following maximum principle where we refer to non-zero eigenvalues:

All simple eigenvalues and the symmetric functions of multiple eigenvalues of (1.1) have no points of local maximum or minimum with respect to mass density perturbations preserving the total mass.

See Theorem 4.1 for the precise statement. Moreover, we generalize a result of Cox and McLaughlin [12] and we prove that $\lambda_n[\rho]$ are weakly* continuous functions of ρ , see Theorem 3.1. This, combined with the above mentioned principle, implies that if C is a weakly* compact set of mass densities then for non-zero eigenvalues we have:

All simple eigenvalues and the symmetric functions of multiple eigenvalues of (1.1) admit points of maximum and minimum in C with mass constraint $M = \text{const}$ and such points of maximum and minimum belong to ∂C .

See Corollary 4.1 for the precise statement. The reason why we consider the symmetric functions of multiple eigenvalues and not the eigenvalues themselves is related to well-known bifurcation phenomena which prevent multiple eigenvalues from being differentiable functions of the parameters involved in the equation. Moreover, the symmetric functions of multiple eigenvalues appear to be natural objects in the study of extremum problems, see e.g., [5, 6, 20, 21, 22, 23]. In fact, in this paper we prove that all simple eigenvalues and the symmetric functions of multiple eigenvalues are real-analytic functions of ρ and we compute the appropriate formulas for the Fréchet differentials which we need for our argument, see Theorem 3.2.

Theorem 4.1 and Corollary 4.1 are proved for so-called intermediate boundary conditions in which case one of the boundary conditions is $u = 0$ on $\partial\Omega$ (see condition (4.1) and Example 4). This includes the case of Dirichlet boundary conditions

$$u = \frac{\partial u}{\partial \nu} = \dots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0, \quad \text{on } \partial\Omega. \quad (1.2)$$

On the other hand, Theorems 3.1 and 3.2 are proved for a larger class of homogeneous boundary conditions, including Neumann boundary conditions. See Remark 4.2 for a discussion concerning Neumann-type boundary conditions.

Our work is inspired by the well-known results of Krein [19] and Cox and McLaughlin [11, 12, 13] concerning the description of optimal mass densities for the eigenvalues of the Dirichlet Laplacian under the additional condition $A \leq \rho \leq B$, where A, B are fixed positive constants. Complete solution to this problem for $N = 1$ was given in [19] where explicit formulas for minimizers and maximizers of all eigenvalues were established. In particular, it turns out that optimal mass densities are bang-bang solutions, i.e., minimizers and maximizers satisfy the condition $(\rho - A)(\rho - B) = 0$ on Ω . The case $N > 1$ is discussed in [12, 13] where, among other results, it is proved that minimizers and maximizers of the first eigenvalue of the Dirichlet Laplacian are

bang-bang solutions. Moreover, Friedland [15] proves that the minimizers of suitable functionals of the eigenvalues, in particular of any eigenvalue, are bang-bang as well. In fact, Friedland [15, 16] carries out a deep analysis of extremum problems for the eigenvalues of symmetric compact operators in Hilbert space subject to convex sets of constraints, which in particular allows to prove that optimal mass densities are bang-bang solutions also for higher order operators with Dirichlet boundary conditions on a bounded open interval (cf. [15, Theorem 3.3]). We mention that explicit solutions for the biharmonic operator in spirit to Krein's results can be found in Banks [3, 4] and Schwarz [25], see Henrot [18, § 11.4.1] for related open problems.

Our approach allows to state a maximum principle concerning all eigenvalues of a quite general class of elliptic operators which can be applied to arbitrary sets C of mass densities, not necessarily convex.

2 Preliminaries and notation

Let Ω be an open set in \mathbb{R}^N and $m \in \mathbb{N}$. By $W^{m,2}(\Omega)$ we denote the Sobolev space of functions in $L^2(\Omega)$ with weak derivatives up to order m in $L^2(\Omega)$, endowed with its standard norm defined by

$$\|u\|_{W^{m,2}(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + \sum_{|\alpha|=m} \|D^\alpha u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}, \quad (2.1)$$

for all $u \in W^{m,2}(\Omega)$. By $W_0^{m,2}(\Omega)$ we denote the closure in $W^{m,2}(\Omega)$ of the space of C^∞ -functions with compact support in Ω .

In the sequel, we shall always assume that $V(\Omega)$ is a fixed closed subspace of $W^{m,2}(\Omega)$ containing $W_0^{m,2}(\Omega)$ and such that the embedding $V(\Omega) \subset L^2(\Omega)$ is compact. Moreover, we shall assume that $A_{\alpha\beta} \in L^\infty(\Omega)$ are fixed coefficients such that $A_{\alpha\beta} = A_{\beta\alpha}$ for all $\alpha, \beta \in \mathbb{N}_0^N$ with $|\alpha|, |\beta| \leq m$.

By \mathcal{R} we denote the subset of $L^\infty(\Omega)$ of those functions $\rho \in L^\infty(\Omega)$ such that $\text{ess inf}_\Omega \rho > 0$. Let $\rho \in \mathcal{R}$ be fixed. We consider the following eigenvalue problem

$$\int_\Omega \sum_{0 \leq |\alpha|, |\beta| \leq m} A_{\alpha\beta} D^\alpha u D^\beta \varphi dx = \lambda \int_\Omega u \varphi \rho dx, \quad \forall \varphi \in V(\Omega), \quad (2.2)$$

in the unknowns $u \in V(\Omega)$ (the eigenfunction) and $\lambda \in \mathbb{R}$ (the eigenvalue). Note that problem (2.2) is the weak-formulation of problem (1.1) subject to suitable homogeneous boundary conditions. The choice of the space $V(\Omega)$ is related to the boundary conditions in the classical formulation of the problem. For example, if $V(\Omega) = W_0^{m,2}(\Omega)$ we obtain Dirichlet boundary conditions as in (1.2). If $V(\Omega) = W^{m,2}(\Omega)$ we obtain Neumann boundary conditions. If $V(\Omega) = W^{m,2}(\Omega) \cap W_0^{k,2}(\Omega)$, for some $k < m$, we obtain intermediate boundary conditions. See Example 4 below. See also Necăs [24, Chapter 1].

It is convenient to denote the left-hand side of equation (2.2) by $\mathcal{Q}[u, \varphi]$. It is also convenient to denote by $L_\rho^2(\Omega)$ the space $L^2(\Omega)$ endowed with the scalar product defined by

$$\langle u_1, u_2 \rangle_\rho = \int_\Omega u_1 u_2 \rho dx, \quad \forall u_1, u_2 \in L^2(\Omega).$$

Note that the corresponding norm $\|u\|_{L^2_\rho(\Omega)}$ is equivalent to the standard norm.

We assume that the space $V(\Omega)$ and the coefficients $A_{\alpha\beta}$ are such that Gårding's inequality holds, i.e., we assume that there exist $a, b > 0$ such that

$$a\|u\|_{W^{m,2}(\Omega)}^2 \leq \mathcal{Q}[u, u] + b\|u\|_{L^2(\Omega)}^2, \quad (2.3)$$

for all $u \in V(\Omega)$. Actually, in many cases it will be more convenient to normalize the constants $a, b > 0$ in such a way that

$$a\|u\|_{W^{m,2}(\Omega)}^2 \leq \mathcal{Q}[u, u] + b\|u\|_{L^2_\rho(\Omega)}^2, \quad (2.4)$$

for all $u \in V(\Omega)$. For classical conditions on the coefficients $A_{\alpha\beta}$ ensuring the validity of (2.3) in the case of Dirichlet boundary conditions we refer to Agmon [1, Theorem 7.6]. Moreover, we assume that there exists $c > 0$ such that

$$\mathcal{Q}[u, u] \leq c\|u\|_{W^{m,2}(\Omega)}^2, \quad (2.5)$$

for all $u \in V(\Omega)$. Note that since the coefficients $A_{\alpha\beta}$ are bounded, inequality (2.5) is always satisfied if Ω is a bounded open set with Lipschitz boundary (actually, it is sufficient that Ω is a bounded open set with a quasi-resolved boundary, see Burenkov [7, Theorem 6, p. 160]).

Under assumptions (2.4), (2.5), it is easy to prove that problem (2.2) has a divergent sequence of eigenvalues bounded below by $-b$. To do so, we consider the bounded linear operator L from $V(\Omega)$ to its dual $V(\Omega)'$ which takes any $u \in V(\Omega)$ to the functional $L[u]$ defined by $L[u][\varphi] = \mathcal{Q}[u, \varphi]$, for all $\varphi \in V(\Omega)$. Moreover, we consider the bounded linear operator I_ρ from $L^2_\rho(\Omega)$ to $V(\Omega)'$ which takes any $u \in L^2_\rho(\Omega)$ to the functional $I_\rho[u]$ defined by $I_\rho[u][\varphi] = \langle u, \varphi \rangle_\rho$, for all $\varphi \in V(\Omega)$. By inequalities (2.4), (2.5) and by the boundedness of the coefficients $A_{\alpha\beta}$, it follows that the quadratic form defined by the right-hand side of (2.4) induces in $V(\Omega)$ a norm equivalent to the standard norm (2.1). Hence by the Riesz Theorem, it follows that the operator $L + bI_\rho$ is a linear homeomorphism from $V(\Omega)$ onto $V(\Omega)'$. Thus, equation (2.2) is equivalent to the equation

$$(L + bI_\rho)^{(-1)} \circ I_\rho[u] = \mu u \quad (2.6)$$

where

$$\mu = (\lambda + b)^{-1}. \quad (2.7)$$

Thus, it is natural to consider the operator T_ρ from $L^2_\rho(\Omega)$ to itself defined by

$$T_\rho := i \circ (L + bI_\rho)^{(-1)} \circ I_\rho,$$

where i is the embedding of $V(\Omega)$ into $L^2_\rho(\Omega)$. In the sequel, we shall omit i and we shall simply write $T_\rho = (L + bI_\rho)^{(-1)} \circ I_\rho$. Note that

$$\begin{aligned} \langle T_\rho u_1, u_2 \rangle_\rho &= I_\rho[u_2][[(L + bI_\rho)^{(-1)} \circ I_\rho[u_1]]] \\ &= (L + bI_\rho)[[(L + bI_\rho)^{(-1)} \circ I_\rho[u_1]]][(L + bI_\rho)^{(-1)} \circ I_\rho[u_2]], \end{aligned} \quad (2.8)$$

for all $u_1, u_2 \in L^2_\rho(\Omega)$. Thus, since the operator $L + bI_\rho$ is symmetric it follows that T_ρ is a self-adjoint operator in $L^2_\rho(\Omega)$. Moreover, if the embedding $V(\Omega) \subset L^2(\Omega)$ is

compact then the operator T_ρ is compact. By inequality (2.4), T_ρ is injective. It follows that the spectrum of T_ρ is discrete and consists of a sequence of positive eigenvalues of finite multiplicity converging to zero. Then by (2.7) and standard spectral theory, we easily deduce the validity of the following

Lemma 1. *Let $\rho \in \mathcal{R}$. Assume that inequalities (2.4) and (2.5) are satisfied for some $a, b, c > 0$. Then the eigenvalues of equation (2.2) have finite multiplicity and can be represented by means of a divergent sequence $\lambda_n[\rho]$, $n \in \mathbb{N}$ as follows*

$$\lambda_n[\rho] = \min_{\substack{E \subset V(\Omega) \\ \dim E = n}} \max_{\substack{u \in E \\ u \neq 0}} \frac{\int_{\Omega} \sum_{|\alpha|, |\beta| \leq m} A_{\alpha\beta} D^\alpha u D^\beta u dx}{\int_{\Omega} u^2 \rho dx}. \quad (2.9)$$

Each eigenvalue is repeated according to its multiplicity and

$$\lambda_n[\rho] > -b + \frac{a}{\|\rho\|_{L^\infty(\Omega)}}, \quad (2.10)$$

for all $n \in \mathbb{N}$. Moreover, the sequence $\mu_n[\rho] = (b + \lambda_n[\rho])^{-1}$, $n \in \mathbb{N}$, represents all eigenvalues of the compact self-adjoint operator T_ρ .

Example 4. *We consider the case of poly-harmonic operators. Let $m \in \mathbb{N}$. Let $A_{\alpha\beta} = \delta_{\alpha\beta} m! / \alpha!$ for all $\alpha, \beta \in \mathbb{N}^N$ with $|\alpha| = |\beta| = m$, where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ otherwise. Let $k \in \mathbb{N}_0$, $0 \leq k \leq m$ and $V(\Omega) = W^{m,2}(\Omega) \cap W_0^{k,2}(\Omega)$. Note that (2.4) and (2.5) are satisfied for any $b > 0$ where $a, c > 0$ are suitable constants possibly depending on b . Moreover, if $k = m$ and the open set Ω has finite Lebesgue measure then the embedding $V(\Omega) \subset L^2(\Omega)$ is compact. If $0 \leq k < m$ and the open set Ω is bounded and has a Lipschitz continuous boundary then the embedding $V(\Omega) \subset L^2(\Omega)$ is compact (actually it is enough to assume that Ω is a bounded open set with a quasi-continuous boundary, see Burenkov [7, Theorem 8, p.169]). Under these assumptions all corresponding eigenvalues $\lambda_n[\rho]$ are well-defined and non-negative.*

Note that if $k = m$ then $V(\Omega) = W_0^{m,2}(\Omega)$ and by integrating by parts one can easily realize that the bilinear form $\mathcal{Q}[u, \varphi]$ can be written in the more familiar form

$$\mathcal{Q}[u, \varphi] = \begin{cases} \int_{\Omega} \Delta^{\frac{m}{2}} u \Delta^{\frac{m}{2}} \varphi dx, & \text{if } m \text{ is even,} \\ \int_{\Omega} \nabla \Delta^{\frac{m-1}{2}} u \nabla \Delta^{\frac{m-1}{2}} \varphi dx, & \text{if } m \text{ is odd,} \end{cases}$$

for all $u, \varphi \in W_0^{m,2}(\Omega)$. In this case we obtain the classic poly-harmonic operator $\mathcal{L} = (-\Delta)^m$ subject to the Dirichlet boundary conditions (1.2). Recall that the Dirichlet problem arises in the study of vibrating strings for $N = 1$ and $m = 1$, membranes for $N = 2$ and $m = 1$, and clamped plates for $N = 2$ and $m = 2$.

In the general case $k \leq m$, the classic formulation of the eigenvalue problem is

$$\begin{cases} (-\Delta)^m u = \lambda \rho u, & \text{in } \Omega, \\ \frac{\partial^j u}{\partial \nu^j} = 0, \quad \forall j = 0, \dots, k-1, & \text{on } \partial\Omega, \\ \mathcal{B}_j u = 0, \quad \forall j = 1, \dots, m-k, & \text{on } \partial\Omega, \end{cases}$$

where \mathcal{B}_j are uniquely defined ‘complementing’ boundary operators. See Necăs [24] for details. For $N \geq 2$, $m = 2$ and $k = 1$ we obtain the problem

$$\begin{cases} \Delta^2 u = \lambda \rho u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \Delta u - (N - 1)K \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

which is related to the study of a simply supported plate. Here K is the mean curvature of the boundary of Ω . See Gazzola, Grunau and Sweers [17] for further details.

Finally, we note that if $m = 2$ and $k = 0$ then $V(\Omega) = W^{2,2}(\Omega)$ and problem (2.2) is the weak formulation of a Neumann-type problem for the biharmonic operator

$$\begin{cases} \Delta^2 u = \lambda \rho u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \operatorname{div}_{\partial\Omega}[P_{\partial\Omega}[(D^2 u)\nu] + \frac{\partial \Delta u}{\partial \nu}] = 0, & \text{on } \partial\Omega, \end{cases} \quad (2.11)$$

which arises in the study of a vibrating free plate. Here $\operatorname{div}_{\partial\Omega}$ is the tangential divergence and $P_{\partial\Omega}$ the orthogonal projector onto the tangent hyperplane to $\partial\Omega$. See also Chasman [9].

3 Continuity and analyticity

By min-max principle (2.9) it follows that $\lambda_n[\rho]$ is a locally Lipschitz continuous functions of $\rho \in \mathcal{R}$. In fact, one can easily prove that

$$|\lambda_n[\rho_1] - \lambda_n[\rho_2]| \leq \frac{\min\{\lambda_n[\rho_1], \lambda_n[\rho_2]\} + 2b}{\min\{\operatorname{ess\,inf} \rho_1, \operatorname{ess\,inf} \rho_2\}} \|\rho_1 - \rho_2\|_{L^\infty(\Omega)},$$

for all $\rho_1, \rho_2 \in \mathcal{R}$ satisfying $\|\rho_1 - \rho_2\|_{L^\infty(\Omega)} < \min\{\operatorname{ess\,inf} \rho_1, \operatorname{ess\,inf} \rho_2\}$. In fact $\lambda_n[\rho]$ depends with continuity on ρ not only with respect to the strong topology of $L^\infty(\Omega)$ but also with respect to the weak* topology, which is clearly more relevant in optimization problems. The following theorem was proved by Cox and McLaughlin [12] in the case of the Dirichlet Laplacian and mass densities uniformly bounded away from zero and infinity. The proof can be easily adapted to the general case. Moreover, it is possible to replace the uniform lower bound for ρ by a weaker assumption.

Theorem 3.1. *Let $C \subset \mathcal{R}$ be a bounded set. Assume that there exist $a, b, c > 0$ such that inequalities (2.4) and (2.5) are satisfied for all $\rho \in C$. Then the functions from C to \mathbb{R} which take any $\rho \in C$ to $\lambda_n[\rho]$ are weakly* continuous for all $n \in \mathbb{N}$.*

Proof. Since C is bounded in $L^\infty(\Omega)$, it suffices to prove that given $\rho \in C$ and a sequence $\rho_j \in C$, $j \in \mathbb{N}$ such that $\rho_j \rightharpoonup^* \rho$ as $j \rightarrow \infty$ then $\lambda_n[\rho_j] \rightarrow \lambda_n[\rho]$. To do so, we first prove¹ that for each $n \in \mathbb{N}$ there exists $L_n > 0$ such that $\lambda_n[\rho_j] \leq L_n$ for all $j \in \mathbb{N}$. Let $n \in \mathbb{N}$ be fixed and $u_1, \dots, u_n \in V(\Omega)$ be linearly independent eigenfunctions

¹This is clearly trivial if we assume that $0 < \alpha \leq \rho$ for all $\rho \in C$, in which case $\lambda_n[\rho] \leq \lambda_n[\alpha]$.

associated with the eigenvalues $\lambda_1[\rho], \dots, \lambda_n[\rho]$, normalized by $\langle u_r, u_s \rangle_\rho = \delta_{rs}$ for all $r, s = 1, \dots, n$. Note that

$$\lim_{j \rightarrow \infty} \int_{\Omega} u_r u_s \rho_j dx = \int_{\Omega} u_r u_s \rho dx,$$

for all $r, s = 1, \dots, n$. Thus

$$\lim_{j \rightarrow \infty} \int_{\Omega} \left(\sum_{r=1}^n \gamma_r u_r \right)^2 \rho_j dx = \int_{\Omega} \left(\sum_{r=1}^n \gamma_r u_r \right)^2 \rho dx, \quad (3.1)$$

uniformly with respect to $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$ with $|\gamma| \leq 1$. Let E be the linear space generated by u_1, \dots, u_n . By (3.1) it follows that for any $\epsilon > 0$ there exists $j_\epsilon \in \mathbb{N}$ such that

$$\frac{\int_{\Omega} \sum_{|\alpha|, |\beta| \leq m} A_{\alpha\beta} D^\alpha u D^\beta u dx}{\int_{\Omega} u^2 \rho_j dx} \leq \frac{\int_{\Omega} \sum_{|\alpha|, |\beta| \leq m} A_{\alpha\beta} D^\alpha u D^\beta u dx}{\int_{\Omega} u^2 \rho dx} + \epsilon(\lambda_n[\rho] + 2b) \leq \lambda_n[\rho] + \epsilon(\lambda_n[\rho] + 2b) \quad (3.2)$$

for all $u \in E$, $j \geq j_\epsilon$. By combining (2.9) and (3.2) we deduce that $\lambda_n[\rho_j] \leq \lambda_n[\rho] + \epsilon(\lambda_n[\rho] + 2b)$ for all $j \geq j_\epsilon$, which implies the existence of a uniform bound L_n as claimed above. The rest of the proof follows the lines of Cox [12]. Let $u_n[\rho_j]$, $n \in \mathbb{N}$ be a sequence of eigenfunctions associated with the eigenvalues $\lambda_n[\rho_j]$ normalized by $\langle u_n[\rho_j], u_l[\rho_j] \rangle_{\rho_j} = \delta_{nl}$ for all $n, l \in \mathbb{N}$. Note that $\mathcal{Q}[u_n[\rho_j], u_n[\rho_j]] = \lambda_n[\rho_j]$ for all $j \in \mathbb{N}$. By inequality (2.4), the sequence $u_n[\rho_j]$, $j \in \mathbb{N}$ is bounded in the space $V(\Omega)$ equipped with the norm (2.1). It follows that possibly passing to subsequences, there exists $\bar{u}_n \in V(\Omega)$ such that $u_n[\rho_j]$ weakly converges to \bar{u}_n as $j \rightarrow \infty$ in $V(\Omega)$, and there exists $\bar{\lambda}_n \in \mathbb{R}$ such $\lambda_n[\rho_j]$ converges to $\bar{\lambda}_n$ as $j \rightarrow \infty$. Moreover, since the embedding $V(\Omega) \subset L^2(\Omega)$ is compact we can directly assume that $u_n[\rho_j]$ converges to \bar{u}_n strongly in $L^2(\Omega)$ as $j \rightarrow \infty$. By passing to the limit in the weak equation

$$\mathcal{Q}[u_n[\rho_j], \varphi] = \lambda_n[\rho_j] \langle u_n[\rho_j], \varphi \rangle_{\rho_j}, \quad \forall \varphi \in V(\Omega),$$

it follows that $\bar{\lambda}_n$ is an eigenvalue and of problem (2.2) and \bar{u}_n a corresponding eigenfunction. Note that $\langle \bar{u}_n, \bar{u}_l \rangle_\rho = \delta_{nl}$ for all $n, l \in \mathbb{N}$, hence $\bar{\lambda}_n$, $n \in \mathbb{N}$ is a divergent sequence. It remains to prove that $\bar{\lambda}_n = \lambda_n[\rho]$ for all $n \in \mathbb{N}$. To do so, assume by contradiction that there exists an eigenfunction $\bar{u} \in V(\Omega)$ associated with an eigenvalue $\bar{\lambda}$ of the weak problem (2.2) such that $\langle \bar{u}, \bar{u}_n \rangle_\rho = 0$ for all $n \in \mathbb{N}$. Assume that \bar{u} is normalized by $\|\bar{u}\|_\rho = 1/(b + \bar{\lambda})$. By the Auchmuty principle [2] applied to the operator $L + bI_\rho$, we have

$$-\frac{1}{2(b + \lambda_n[\rho_j])} \leq \frac{\mathcal{Q}[u, u] + b\|u\|_{L_{\rho_j}^2(\Omega)}^2}{2} - \|u - P_{n-1, \rho_j} u\|_{L_{\rho_j}^2(\Omega)}, \quad (3.3)$$

for all $u \in V(\Omega)$ and $n, j \in \mathbb{N}$. Here $P_{n-1, \rho_j} u$ denotes the orthogonal projection in $L_{\rho_j}^2(\Omega)$ of u onto the space generated by $u_1[\rho_j], \dots, u_{n-1}[\rho_j]$ for all $n \geq 2$ and $P_{0, \rho_j} u \equiv 0$. By setting $u = \bar{u}$ and passing to the limit in (3.3) as $j \rightarrow \infty$, we obtain

$$-\frac{1}{2(b + \bar{\lambda}_n)} \leq \frac{\mathcal{Q}[\bar{u}, \bar{u}] + b\|\bar{u}\|_{L_\rho^2(\Omega)}^2}{2} - \|\bar{u}\|_{L_\rho^2(\Omega)} = -\frac{1}{2(b + \bar{\lambda})}$$

for all $j \in \mathbb{N}$, which contradicts the fact that $\bar{\lambda}_n \rightarrow \infty$ as $n \rightarrow \infty$. \square

By classical results in perturbation theory, one can prove that $\lambda_n[\rho]$ depends real-analytically on ρ as long as ρ is such that $\lambda_n[\rho]$ is a simple eigenvalue. This is no longer true if the multiplicity of $\lambda_n[\rho]$ varies. In the case of multiple eigenvalues, analyticity can be proved for the symmetric functions of the eigenvalues. Namely, given a finite set of indexes $F \subset \mathbb{N}$, we set

$$\mathcal{R}[F] \equiv \{\rho \in \mathcal{R} : \lambda_j[\rho] \neq \lambda_l[\rho], \forall j \in F, l \in \mathbb{N} \setminus F\}$$

and

$$\Lambda_{F,h}[\rho] = \sum_{\substack{j_1, \dots, j_h \in F \\ j_1 < \dots < j_h}} \lambda_{j_1}[\rho] \cdots \lambda_{j_h}[\rho], \quad h = 1, \dots, |F|. \quad (3.4)$$

Moreover, in order to compute formulas for the Frechét differentials, it is also convenient to set

$$\Theta[F] \equiv \{\rho \in \mathcal{R}[F] : \lambda_{j_1}[\rho] = \lambda_{j_2}[\rho], \forall j_1, j_2 \in F\}.$$

Then we have the following result

Theorem 3.2. *Assume that there exist $a, b, c > 0$ such that inequalities (2.3) and (2.5) are satisfied. Let F be a finite subset of \mathbb{N} . Then $\mathcal{R}[F]$ is an open set in $L^\infty(\Omega)$ and the functions $\Lambda_{F,h}$ are real-analytic in $\mathcal{R}[F]$. Moreover, if $F = \cup_{k=1}^n F_k$ and $\rho \in \cap_{k=1}^n \Theta[F_k]$ is such that for each $k = 1, \dots, n$ the eigenvalues $\lambda_j[\rho]$ assume the common value $\lambda_{F_k}[\rho]$ for all $j \in F_k$, then the differentials of the functions $\Lambda_{F,h}$ at the point ρ are given by the formula*

$$d\Lambda_{F,h}[\rho][\dot{\rho}] = - \sum_{k=1}^n c_k \sum_{l \in F_k} \int_{\Omega} u_l^2 \dot{\rho} dx, \quad (3.5)$$

for all $\dot{\rho} \in L^\infty(\Omega)$, where

$$c_k = \sum_{\substack{0 \leq h_1 \leq |F_1| \\ \dots \\ 0 \leq h_n \leq |F_n| \\ h_1 + \dots + h_n = h}} \binom{|F_k| - 1}{h_k - 1} \lambda_{F_k}^{h_k}[\rho] \prod_{\substack{j=1 \\ j \neq k}}^n \binom{|F_j|}{h_j} \lambda_{F_j}^{h_j}[\rho],$$

and for each $k = 1, \dots, n$, $\{u_l\}_{l \in F_k}$ is an orthonormal basis in $L_\rho^2(\Omega)$ of the eigenspace associated with $\lambda_{F_k}[\rho]$.

Proof. We set

$$\tilde{\Lambda}_{F,h}[\rho] = \sum_{\substack{j_1, \dots, j_h \in F \\ j_1 < \dots < j_h}} (\lambda_{j_1}[\rho] + b) \cdots (\lambda_{j_h}[\rho] + b),$$

for all $\rho \in \mathcal{R}[F]$. Note that by elementary combinatorics, we have

$$\Lambda_{F,h}[\rho] = \sum_{k=0}^h (-b)^{h-k} \binom{|F| - k}{h - k} \tilde{\Lambda}_{F,k}[\rho], \quad (3.6)$$

where we have set $\Lambda_{F,0} = \tilde{\Lambda}_{F,0} = 1$.

By adapting to the operator $L + bI_\rho$ the same argument used in [22] for the Dirichlet Laplacian, one can prove that $\mathcal{R}[F]$ is an open set in $L^\infty(\Omega)$ and that $\tilde{\Lambda}_{F,h}[\rho]$ depends real-analytically on $\rho \in \mathcal{R}[F]$. Thus, by (3.6) we deduce the real-analyticity of the functions $\Lambda_{F,h}$.

We now prove formula (3.5). First we assume that $n = 1$, hence $F = F_1$ and $\rho \in \Theta[F]$. For simplicity, we write $\lambda_F[\rho]$ rather than $\lambda_{F_1}[\rho]$. The same computations used in [22] yields the following formula for the Frechét differential $d\tilde{\Lambda}_{F,h}[\rho]$ of $\tilde{\Lambda}_{F,h}$ at the point $\rho \in \mathcal{R}[F]$:

$$d\tilde{\Lambda}_{F,h}[\rho][\dot{\rho}] = -(\lambda_F[\rho] + b)^{h+1} \binom{|F| - 1}{h - 1} \sum_{l \in F} \langle dT_\rho[\dot{\rho}][u_l], u_l \rangle_\rho, \quad \forall \dot{\rho} \in L^\infty(\Omega). \quad (3.7)$$

By standard calculus and by recalling that $T_\rho u_l = (\lambda_F[\rho] + b)^{-1} u_l$ for all $l \in F$, we have

$$\begin{aligned} \langle dT_\rho[\dot{\rho}][u_l], u_l \rangle_\rho &= -b \langle (L + bI_\rho)^{-1} dI_\rho[\dot{\rho}](L + bI_\rho)^{-1} I_\rho u_l, u_l \rangle_\rho \\ &+ \langle (L + bI_\rho)^{-1} dI_\rho[\dot{\rho}] u_l, u_l \rangle_\rho = \frac{\lambda_F[\rho]}{\lambda_F[\rho] + b} \langle (L + bI_\rho)^{-1} dI_\rho[\dot{\rho}] u_l, u_l \rangle_\rho \\ &= \frac{\lambda_F[\rho]}{(\lambda_F[\rho] + b)^2} \int_\Omega u_l^2 \dot{\rho} dx \end{aligned} \quad (3.8)$$

hence

$$d\tilde{\Lambda}_{F,h}[\rho][\dot{\rho}] = -\lambda_F[\rho](\lambda_F[\rho] + b)^{h-1} \binom{|F| - 1}{h - 1} \sum_{l \in F} \int_\Omega u_l^2 \dot{\rho} dx, \quad (3.9)$$

for all $\dot{\rho} \in L^\infty(\Omega)$. By (3.6) and (3.9) we get

$$\begin{aligned} d\Lambda_{F,h}[\rho][\dot{\rho}] &= - \sum_{k=1}^h \lambda_F[\rho](\lambda_F[\rho] + b)^{k-1} (-b)^{h-k} \binom{|F| - 1}{k - 1} \binom{|F| - k}{h - k} \sum_{l \in F} \int_\Omega u_l^2 \dot{\rho} dx \\ &= -\lambda_F[\rho] \binom{|F| - 1}{h - 1} \sum_{k=0}^{h-1} \binom{h - 1}{k} (\lambda_F[\rho] + b)^k (-b)^{h-1-k} \sum_{l \in F} \int_\Omega u_l^2 \dot{\rho} dx, \end{aligned}$$

which immediately implies (3.5) for $n = 1$. We now consider the case $n > 1$. By means of a continuity argument, one can easily see that there exists an open neighborhood \mathcal{W} of ρ in $\mathcal{R}[F]$ such that $\mathcal{W} \subset \cap_{k=1}^n \mathcal{R}[F_k]$. Thus,

$$\Lambda_{F,h} = \sum_{\substack{0 \leq h_1 \leq |F_1|, \dots, 0 \leq h_n \leq |F_n| \\ h_1 + \dots + h_n = h}} \prod_{k=1}^n \Lambda_{F_k, h_k} \quad (3.10)$$

on \mathcal{W} . By differentiating equality (3.10) at the point ρ and applying formula (3.5) for $n = 1$ to each function Λ_{F_k, h_k} , we deduce the validity of formula (3.5) for arbitrary values of $n \in \mathbb{N}$. \square \square

4 Maximum principle

In this section we consider the case of general intermediate boundary conditions. This means that we assume that $V(\Omega)$ is a closed subspace of $W^{m,2}(\Omega)$ satisfying the inclusion

$$V(\Omega) \subset W_0^{1,2}(\Omega). \quad (4.1)$$

Assume that Ω has finite measure. For all $M > 0$ we set

$$L_M = \left\{ \rho \in L^\infty(\Omega) : \int_\Omega \rho dx = M \right\} \quad (4.2)$$

The following theorem is a generalization of [22, Theorem 4.4] to the case of intermediate boundary conditions.

Theorem 4.1. *Let all assumptions of Theorem 3.2 hold. Assume in addition that Ω has finite measure and inclusion (4.1) holds. Then for all $h = 1, \dots, |F|$ the map $\Lambda_{F,h}$ of $\mathcal{R}[F] \cap L_M$ to \mathbb{R} which takes any $\rho \in \mathcal{R}[F] \cap L_M$ to $\Lambda_{F,h}[\rho]$ has no points of local maximum or minimum $\tilde{\rho}$ such that $\lambda_j[\tilde{\rho}]$ have the same sign and $\lambda_j[\tilde{\rho}] \neq 0$ for all $j \in F$.*

Proof. It is convenient to consider the real-valued function M defined on $L^\infty(\Omega)$ by $M[\rho] = \int_\Omega \rho dx$ for all $\rho \in L^\infty(\Omega)$. Assume by contradiction the existence of $\tilde{\rho}$ as in the statement. Then $\tilde{\rho}$ is a critical point for the function $\Lambda_{F,h}$ subject to the mass constraint $M[\rho] = M$. This implies the existence of a Lagrange multiplier which means that there exists $c \in \mathbb{R}$ such that $d\Lambda_{F,h}[\tilde{\rho}] = cdM[\tilde{\rho}]$ (see e.g., Deimling [14, Theorem 26.1]). By formula (3.5), it follows that

$$\int_\Omega \left(\sum_{k=1}^n c_k \sum_{l \in F_k} u_l^2 \right) \dot{\rho} dx = c \int_\Omega \dot{\rho} dx,$$

for all $\dot{\rho} \in L^\infty(\Omega)$. Note that c_k are non-zero real numbers of the same sign. Since $\dot{\rho}$ is arbitrary, it follows that

$$\left(\sum_{k=1}^n c_k \sum_{l \in F_k} u_l^2 \right) = c, \quad \text{a.e. in } \Omega. \quad (4.3)$$

Since $u_l \in W_0^{1,2}(\Omega)$, then by a standard argument one can prove that the function $(\sum_{k=1}^n \sum_{l \in F_k} (\sqrt{|c_k|} u_l)^2)^{1/2}$ belongs to the space $W_0^{1,2}(\Omega)$ and equals $\sqrt{|c|}$ almost everywhere in Ω . As is well-known the space $W_0^{1,2}(\Omega)$ does not contain constant functions apart from the function identically equal to zero. Thus $c = 0$ and accordingly $u_l = 0$ for all $l \in F$, a contradiction. \square

Remark 4.1. *Theorem 4.1 concerns mass densities $\tilde{\rho}$ such that $\lambda_j[\tilde{\rho}]$ do not vanish and have the same sign for all $j \in F$. This assumption is clearly guaranteed for positively defined operators. Moreover, we note that the sign of the eigenvalues is preserved by small perturbations of ρ . Hence our assumption is not much restrictive in the analysis of bifurcation phenomena associated with multiple eigenvalues different from zero.*

Finally, by Theorems 3.1 and 4.1 we deduce the following

Corollary 4.1. *Let all assumptions of Theorem 4.1 hold. Let $C \subset \mathcal{R}[F]$ be a weakly* compact set in $L^\infty(\Omega)$. Assume that there exist $a, b > 0$ such that inequality (2.4) is satisfied for all $\rho \in C$. Let $M > 0$ be such that $C \cap L_M$ is not empty. Assume that the eigenvalues $\lambda_j[\rho]$ have the same sign and do not vanish for all $j \in F$, $\rho \in C$. Then for all $h \in \{1, \dots, |F|\}$ the map $\Lambda_{F,h}$ from $C \cap L_M$ to \mathbb{R} which takes $\rho \in C \cap L_M$ to $\Lambda_{F,h}[\rho]$ admits points of maximum and minimum and all such points belong to $\partial C \cap L_M$.*

Proof. Recall that weakly* compact sets are bounded. Thus, by Theorem 3.1 the functions $\Lambda_{F,h}$ are weakly* continuous on C hence they admit both maximum and minimum on the weakly* compact subset $C \cap L_M$ of C . By Corollary 4.1 the corresponding points of maximum and minimum cannot be interior points of C , hence they belong to $\partial C \cap L_M$. \square

Example 5. *Consider the poly-harmonic operators subject to Dirichlet or intermediate boundary conditions as described in Example 4. Let $A, B \in L^\infty(\Omega)$ be functions satisfying the condition*

$$0 < \operatorname{ess\,inf}_{x \in \Omega} A(x) < \operatorname{ess\,sup}_{x \in \Omega} B(x) < \infty.$$

Let

$$C = \{\rho \in L^\infty(\Omega) : A \leq \rho \leq B\}.$$

Clearly, C is a weakly* compact set. Moreover, since all mass densities ρ are uniformly bounded away from zero and infinity, inequality (2.4) is satisfied for suitable constants $a, b > 0$ not depending on $\rho \in C$. Thus Corollary 4.1 is applicable to all non-zero eigenvalues. It turns out that points of maximum and minimum $\tilde{\rho}$ should coincide with $A(x)$ or $B(x)$ in a set of positive measure.

Remark 4.2. *Condition (4.1) was used only to guarantee that $V(\Omega) \setminus \{0\}$ does not contain constant functions. Thus, one may replace condition (4.1) by slightly more general conditions. For example one may assume that*

$$V(\Omega) \subset W_{0,\Gamma}^{1,2}(\Omega),$$

where $W_{0,\Gamma}^{1,2}(\Omega)$ is the closure in $W^{1,2}(\Omega)$ of C^∞ -functions vanishing in an open neighborhood of a suitable subset of Γ of $\partial\Omega$. In this case, one would talk about mixed-intermediate boundary conditions.

If $V(\Omega)$ is a closed subspace of $W^{m,2}(\Omega)$ containing constant functions different from zero, then we could argue as in the proof on Theorem 4.1 up to condition (4.3). Thus, in the general case one could simply characterize the critical mass densities of the functions $\Lambda_{F,h}$ as those mass densities for which condition (4.3) is satisfied. Clearly, in the case of simple eigenvalues condition (4.3) reduces to $u = \operatorname{const}$ in Ω which implies that $\lambda = 0$. Thus, we conclude that the maximum principle stated in the introduction holds for all simple eigenvalues and all homogeneous boundary conditions under consideration. As for multiple eigenvalues we note that the analysis of condition (4.3) is not straightforward as it may appear at a first glance. Under suitable regularity

assumptions on the eigenfunctions u_1, u_2 associated with a double eigenvalue λ of the Neumann Laplacian, one may prove that the condition $u_1^2 + u_2^2 = \text{const}$ in Ω implies that $\lambda = 0$. However, we do not include such arguments here since we plan to perform a deeper analysis of Neumann and other boundary conditions in a forthcoming paper.

Acknowledgments

We acknowledge financial support by the research project “Singular perturbation problems for differential operators” Progetto di Ateneo of the University of Padova. The first author acknowledges financial support from the research project PRIN 2008 “Aspetti geometrici delle equazioni alle derivate parziali e questioni connesse”. The second author was partially supported by the Grant MTM2008-03541 of the MICINN, Spain, and the ERC Advanced Grant NUMERIWAVES.

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Received: 25.07.2013