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MATHEMATICAL MODEL OF THE GAS FLOW IN THE SHOCK FRONT

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Abstract. In this paper, an approximate solution of the problem of the structure of shock waves in gases is presented. Changes of the average magnitude of density, speed and internal energy of gas along with the width of the shock wave are given. Dependence of the width of the jump on the Mach number is obtained and is compared with results of other authors.

In [1]-[3] there was considered an application in the rarefied gas dynamics of the heuristic flow model previously developed for describing the flow of gas with solid particles [4]. In this model, the molecules are subdivided into two sets, each of which is considered to be described by the continuum. The collision of molecules of the same set generates the pressure and heat flows in this set, whereas the collision of molecules of different sets leads to the transition of particles from one set to another. In these studies, the particles of one of the sets (let us call them s-particles) were moving orderly without the random component of the velocity and naturally all collisions of such particles with particles of another set (t-particles) led to the transition of s-particles to the set of randomly moving t-particles, while t-particles remained to be t-particles.

The advantage of this model is its simplicity, since it is easy to calculate the mass, momentum and energy conducted by the particles in such a transition, and it is not required to use any interaction characteristics of the particles, other than the frequency of collisions. The applications of this model to calculation of the shock-wave structure [1] and the flow around of a sphere by rarefied gas [2, 3] have shown sufficient accuracy.

The assumption that s-particles move orderly (with a negligible component of the random speed) restricts the application of the model to the case of the hypersonic flow of gas.

In this paper, a generalization of the described model is given to the case in which particles of both sets have a random component of the velocity; this gives the opportunity to use the model for all Mach numbers.

As in the above model, we assume that molecules are solid and elastic spheres and that in any collision of particles of different sets the transition of particles from one set to another takes place. We shall also assume that the distribution of particle velocities

is close to Maxwell's one in each of the sets and compute the pressure and heat flows on this basis. However, if previously it was natural to assume that all orderly moving s-particles after the collision are transited to the set of randomly moving t-particles, now, in the case in which there is random motion in both sets, we shall assume that the transition of t-particles to the set of s-particles is also possible. Let us denote the probability of transition of s-particles to the set of t-particles as a result of collision by β and the probability of transition of t-particles to the set of s-particles by $1-\beta$ respectively.

Obviously, the transition probability of particles from one set to another may be associated with the intensity of random motion of particles in the corresponding set. Moreover the greater chance of transition is in the set which has the greater intensity of random motion. This intensity can be described by the average modulus of the velocity of random motion $c_i = \sqrt{16p_i/(3\pi(\kappa-1)\varrho_i)}$, where p_i is the pressure, ϱ_i is the density of the i-th gas ($i = s, t$), κ – the specific heat ratio ($\kappa = 5/3$, if the rotation of the particles is not taken into account, and $\kappa = 4/3$, if the energy of random motion is uniformly distributed over the rotational and translational degrees of freedom).

The most natural assumption is that β proportional to c_t^3 for small c_t^3, as c_t^3 is the volume in the velocity space, which describes the intensity of random motion of particles, and β is the probability of a collided s-particle to be transmitted to the t-set after the collision. Respectively $1-\beta$ is proportional to c_s^3 for small c_s^3, hence we assume that $\beta = c_t^3/(c_t^3 + c_s^3)$.

Now we consider shock-wave structure in gas on the basis of the described model, following [1], where this was done for the case of hypersonic flow prior to the shock-wave. The balance equations of mass, momentum and energy for the s- and t-components of the flow are as follows:

$$\frac{d}{dx}(\varrho_s u_s) = -I\beta + I(1-\beta), \qquad \frac{d}{dx}(\varrho_t u_t) = I\beta - I(1-\beta), \tag{1}$$

$$\frac{d}{dx}\left(\varrho_s u_s^2 + p_s - \frac{4}{3}\mu_s\frac{du_s}{dx}\right) = -I\beta\left(u_s + \frac{p_s}{\varrho_s u_s}\right) + I(1-\beta)\left(u_t + \frac{p_t}{\varrho_t u_t}\right), \tag{2}$$

$$\frac{d}{dx}\left(\varrho_t u_t^2 + p_t - \frac{4}{3}\mu_t\frac{du_t}{dx}\right) = I\beta\left(u_s + \frac{p_s}{\varrho_s u_s}\right) - I(1-\beta)\left(u_t + \frac{p_t}{\varrho_t u_t}\right), \tag{3}$$

$$\begin{aligned}&\tfrac{d}{dx}\left[\varrho_s u_s\left(U_s + \tfrac{u_s^2}{2} + \tfrac{p_s}{\varrho_s}\right) - \tfrac{4}{3}\mu_s u_s\tfrac{du_s}{dx} - \tfrac{\lambda_s}{c_V}\tfrac{dU_s}{dx}\right]\left(\varrho_s u_s^2 + p_s - \tfrac{4}{3}\mu_s u_s\tfrac{du_s}{dx}\right)\\ &= -I\beta\left(u_s + \tfrac{p_s}{\varrho_s u_s}\right) + I(1-\beta)\left(u_t + \tfrac{p_t}{\varrho_t u_t}\right),\end{aligned} \tag{4}$$

$$\begin{aligned}&\tfrac{d}{dx}\left[\varrho_t u_t\left(U_t + \tfrac{u_t^2}{2} + \tfrac{p_t}{\varrho_t}\right) - \tfrac{4}{3}\mu_t u_t\tfrac{du_t}{dx} - \tfrac{\lambda_t}{c_V}\tfrac{dU_t}{dx}\right]\left(\varrho_t u_t^2 + p_t - \tfrac{4}{3}\mu_t u_t\tfrac{du_t}{dx}\right)\\ &= I\beta\left(u_s + \tfrac{p_s}{\varrho_s u_s}\right) - I(1-\beta)\left(u_t + \tfrac{p_t}{\varrho_t u_t}\right).\end{aligned} \tag{5}$$

Here ϱ_i, u_i ($i = s, t$) are the density and velocity of components (the index indicates the corresponding component), U_i is the kinetic energy of random motion of molecules, c_V is the specific thermal capacity of gas at constant volume, I is the total mass of molecules of one set, collided with molecules of the other set in a unit volume per unit time. We shall use the following approximate formula for I:

$$I = \frac{\pi d^2}{m}\varrho_s\varrho_t\sqrt{(u_s-u_t)^2 + c_s^2 + c_t^2}.$$

Here m is the mass of molecules, d is their diameter; μ_i, λ_i is the viscosity and thermal conductivity of the components, which can be determined as for gas consisting of solid spheres. For each component the equation below is assumed to be valid

$$p_i = (\kappa - 1)\varrho_i U_i.$$

To calculate the shock-wave structure the above equations we shall use the following boundary conditions:

$$u_s \to u_{s-},\ \varrho_s \to \varrho_{s-},\quad U_s \to U_{s-},\quad \varrho_t \to 0 \ \text{ as } \ x \to -\infty$$

and

$$u_t \to u_{t+},\ \varrho_t \to \varrho_{t+},\quad U_t \to U_{t+},\quad \varrho_s \to 0 \ \text{ as } \ x \to +\infty.$$

Due to the conservation of mass, momentum and energy of the flow the following conditions of dynamic compatibility should be fulfilled:

$$\frac{u_{t+}}{u_{s-}} = \frac{\varrho_{s-}}{\varrho_{t+}} = \frac{2}{(\kappa+1)M^2} + \frac{\kappa-1}{\kappa+1}, \frac{p_{t+}}{p_{s-}} = \frac{2\kappa M^2}{\kappa+1} - \frac{\kappa-1}{\kappa+1},$$

where $M = u_{s-}/\sqrt{\kappa p_{s-}/\varrho_{s-}}$ is the Mach number of the flow prior to the shock-wave.

Next we switch to the dimensionless variables, taking as the scales of velocity, density and length $u_{s-}, \varrho_{s-}, \frac{m}{\pi d^2 \varrho_{s-}}$ respectively, and keeping the same notation.

First, we assume that both components are non-viscous and thermally non-conductive $\mu_s = \mu_t = 0$ and $\lambda_s = \lambda_t = 0$. Then equations (1) - (5) can be written as

$$\frac{dA_s}{dx} = I(1-2\beta), \quad \frac{dA_t}{dx} = I(2\beta - 1); \tag{6}$$

$$\frac{d}{dx}(A_s B_s) = I\left[(1-\beta)B_t - \beta B_s\right], \frac{d}{dx}(A_t B_t) = I\left[\beta B_s - (1-\beta)B_t\right]; \tag{7}$$

$$\frac{d}{dx}(A_s C_s) = I\left[(1-\beta)C_t - \beta C_s\right], \frac{d}{dx}(A_t C_t) = I\left[\beta C_s - (1-\beta)C_t\right], \tag{8}$$

where $A_s = \varrho_s u_s$, $A_t = \varrho_t u_t$, $B_s = u_s + \frac{p_s}{\varrho_s u_s}$, $B_t = u_t + \frac{p_t}{\varrho_t u_t}$, $C_s = U_s + \frac{u_s^2}{2} + \frac{p_s}{\varrho_s}$, $C_t = U_t + \frac{u_t^2}{2} + \frac{p_t}{\varrho_t}$.

Adding pairs of equations (6), (7) and (8), after integration and determination of constants by using the boundary conditions, it is easy to obtain that

$$A_s + A_t = 1, \quad A_s B_s + A_t B_t = 1 + \frac{1}{\kappa M^2}$$

$$A_s C_s + A_t C_t = \frac{1}{2} + \frac{1}{(\kappa-1)M^2}.$$

Solutions to equations (6) - (8) show that B_s, B_t, C_s, C_t do not depend on x; u_s, u_t, U_s, U_t and c_s, c_t, β depending on them are also constants. On x depend only the densities of the components ϱ_s, ϱ_t and the quantities p_s, p_t, I determined by them. By the first equation in (6) we have $\frac{d\varrho_s}{dx} = -D\varrho_s(1-\varrho_s)$, whence, by selecting the

reference point x, where $\varrho_s = 0.5$, we have $\varrho_s = \frac{1}{1+e^{Dx}}$, and by the second equation in (6) we obtain $\varrho_t = \frac{e^{Dx}\varrho_{t+}}{1+e^{Dx}}$. Here

$$D = (2\beta - 1)\frac{\sqrt{(M^2-1)^2 + \frac{4}{\pi\kappa}((\kappa M^2+1)^2 - \kappa(M^2-1)^2)}}{\left(\frac{\kappa-1}{2}M^2+1\right)}$$

The fact that the velocity and internal energy of the component in accordance with the obtained result do not depend on x, means that their viscosity and thermal conductivity are non-essential, hence the obtained result is valid also for viscous thermally conductive gas. Then the average values of density, speed and energy for the mixture will be

$$\varrho = \varrho_s + \varrho_t = \frac{1}{1+e^{Dx}}\left(1 + e^{Dx}\frac{M^2(\kappa+1)}{2+M^2(\kappa-1)}\right)$$

$$v = \frac{\varrho_s v_s + \varrho_t v_t}{\varrho} = \frac{1+e^{Dx}}{1+e^{Dx}\left[M^2(\kappa+1)/(2+M^2(\kappa-1))\right]}$$

$$U = \frac{\varrho_s U_s + \varrho_t U_t}{\varrho} = \left[\frac{1}{\kappa(\kappa-1)M^2} + \frac{e^{Dx}}{2\kappa}\left(\frac{\kappa+1}{k-1} - \frac{1}{H}\right)\right]/(1+e^{Dx}H),$$

where

$$H = [M^2(\kappa+1)]/[2+M^2(\kappa-1)].$$

Thus, the above two-component model of solid particles flow, as well as the initial quaternary model of gas suspension [4] can be considered to be applicable even in the cases in which the mean free path of particles is comparable with the gas-dynamic scale. Moreover, it may happen that the concepts, on which the construction of these models are based, will be useful in the aerodynamics of rarefied gases when considering the transient mode from continuum to free molecular flow.

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