EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 1, Number 3 (2010), 58 – 96

COERCIVE ESTIMATES AND INTEGRAL REPRESENTATION FORMULAS ON CARNOT GROUPS

D.V. Isangulova, S.K. Vodopyanov

Communicated by V.S. Guliev

Key words: coercive estimate, integral representation, Sobolev space, Carnot group.

AMS Mathematics Subject Classification: 43A80, 46E35, 58J99.

Abstract. For general Carnot groups, we obtain coercive estimates for homogeneous differential operators with constant coefficients, kernels of which have finite dimension. We develop new Sobolev-type integral representations of differentiable functions which are a crucial tool for deriving coercive estimates. Moreover we prove some auxiliary results having independent interest, in particular, Sobolev type embedding and compactness theorems for John domains.

1 Introduction

In the series of well-known papers [33, 34, 35, 36, 37] S. L. Sobolev applied two types of integral representations of functions for proving embedding theorems. In the first of them a function possessing weak derivatives equals the sum of a smooth function and an integral of potential type of weak derivatives of the given function. The second one decomposes a function into two summands: the first one is a polynomial and the second one is an integral of potential type like in the previous case. These representations turned out to be useful both in the theory of function spaces, see for instance [3], and in the theory of PDE, see O. A. Ladyzhenskaya and T. N. Shilkin [17]. They also are applied in the quasiconformal analysis, see Yu. G. Reshetnyak [28], and in the theory of elasticity [21].

Coercive estimates arose in the theory of differential operators as a tool for finding a solution to a differential equation, see [1]. Later a way was found for obtaining coercive estimates by means of special integral representation of functions [2].

It is natural to expect that integral representations of functions may be useful for more complicated metric structures different from the Euclidean one. The main goal of our paper is obtaining coercive estimates for a class of differential operators on Carnot groups. Derivation of these estimates is based on special integral representations of differentiable functions on Carnot groups (see Theorems 2 and 3 below), embedding theorems for the Riesz potentials [41, 6] and a Zygmund–Calderón type theorem (see Lemma 4 below).

We consider a Carnot group $\mathbb{G}$ of topological dimension N with Lie algebra $V = V_1 \oplus \ldots \oplus V_m$, where $[V_1, V_i] = V_{i+1}$ for $i = 1, \ldots, m-1$, $[V_1, V_m] = \{0\}$, $\dim V_1 = n$.

Let left-invariant vector fields $X_1, \dots, X_N$ constitute a basis of Lie algebra V such that the family of vector fields $X_{\dim V_1+\dots+\dim V_{i-1}+1}, \dots, X_{\dim V_1+\dots+\dim V_i}$ is a basis of V_i. Let σ_i be the degree of the vector field X_i: $\sigma_i = \{k \mid X_i \in V_k\}$. We will consider the coordinates of the first type, that is $x = (x_1, \dots, x_N) = \exp\Big(\sum\limits_{i=1}^{N} x_i X_i\Big)(e)$ where e is the unit of the group. The Lebesgue measure on $\mathbb{R}^N$ is the bi-invariant Haar measure on $\mathbb{G}$.

The Carnot–Carathéodory metric d_{cc} is the infimum of lengths of horizontal curves joining two points. (A piece-wise smooth curve γ is horizontal if $\dot{\gamma}(t) \in V_1(\gamma(t))$ for almost all t.) The Hausdorff dimension with respect to the Carnot–Carathéodory metric is $\nu = \sum_{i=1}^{N} \sigma_i$.

To a multi-index $I = (i_1, \dots, i_k) \in \{1, \dots, N\}^k$, it corresponds the differential operator $X^I = X_{i_1} \dots X_{i_k}$ and the weight $d(I) = \sum_{j=1}^{k} \sigma_{i_j}$. By multi-index with subindex h we shall always denote the horizontal multi-index $I_h = (i_1, \dots, i_k)$, $i_j = 1, \dots, n$. Obviously, the length of the horizontal multi-index coincides with its weight: $d(I_h) = k$.

For a multi-index $I = (i_1, \dots, i_k)$, $i_1, \dots, i_k = 1, \dots, N$, set $x^I = x_{i_1} \cdot \ldots \cdot x_{i_k}$. Clearly, x^I is homogeneous of degree $d(I)$, that is $(\delta_t x)^I = t^{d(I)} x^I$ where $\delta_t x = (t^{\sigma_1} x_1, \dots, t^{\sigma_N} x_N)$, $t > 0$, is a *dilation* on Carnot group $\mathbb{G}$. A function f is said to be a *polynomial* on $\mathbb{G}$ if $f(x) = \sum_I a_I x^I$ where all but finitely many of the coefficients a_I vanish. For the polynomial f, the (*homogeneous*) *degree* is said to be $\max\{d(I) \colon a_I \neq 0\}$. Denote by $\mathcal{P}_k$ the linear space of polynomials on $\mathbb{G}$ of homogeneous degree $< k$.

Let Ω be a domain in $\mathbb{G}$, $s, l \in \mathbb{N}$, $1 \le q \le \infty$. The *Sobolev space* $W^l_q(\Omega, \mathbb{R}^s)$ consists of the functions $f = (f_1, \dots, f_s) \colon \Omega \to \mathbb{R}^s$ having the weak derivatives $X^{I_h} f_j$ for $d(I_h) = k$, $k = 1, \dots, l$, $j = 1, \dots, s$, and a finite norm

$$\|f\|_{W^l_q(\Omega,\mathbb{R}^s)} = \|f\|_{q,\Omega} + \sum_{0<d(I_h)\leqslant l} \|X^{I_h} f\|_{q,\Omega}$$

where $\|\cdot\|_{q,\Omega}$ is L_q-norm of a measurable vector-valued function on Ω.

We apply the abovementioned integral representations of functions to obtaining coercive estimates of differential operators with constant coefficients, kernels of which have finite dimension, on John domains. John introduced such domains in the Euclidean case for studying the stability of isometries [14]. We can regard the John domains as a natural extension of the class of Lipschitz domains and the domains satisfying the cone condition. It turned out that the geometry of such domains in $\mathbb{R}^n$ enables us to construct integral representations and to prove a Sobolev-type embedding theorem [27]. The definition of a John domain can be extended easily to the case of metric spaces. In the case of Carnot groups the class of John domains coincides with the class of the so-called Boman chain domains (see [4]). Balls in Carnot–Carathéodory metric are obvious examples of John domains.

A domain $\Omega \subset \mathbb{G}$ is a *John domain* $J(\alpha, \beta)$, $0 < \alpha \le \beta$, if there exists a point $x_0 \in \Omega$ such that every $x \in \Omega$ can be joined in Ω with x_0 by a rectifiable curve γ parameterized by the arc length, such that

$$\gamma(0) = x,\ \gamma(l) = x_0,\ l \le \beta,\ \text{and } \operatorname{dist}(\gamma(s), \partial\Omega) \ge \frac{\alpha s}{l} \text{ for all } s \in [0, l].$$

It is obvious that $B(x_0,\alpha)\subset\Omega\subset B(x_0,\beta)$.

Let Ω be a domain on $\mathbb{G}$ and Q be a homogeneous differential operator of order k with constant coefficients acting from $\mathbb{R}^s$-valued vector functions u of the class $W^k_p(\mathbb{G},\mathbb{R}^s)$ to $\mathbb{R}^m$-valued vector functions:

$$(Qu(x))_j=\sum_{i=1}^{s}\sum_{d(I)=k}c_{ij,I}X^I u_i(x),\quad j=1,\dots,m. \tag{1}$$

The main result of the paper is the following.

Theorem 1. *Let $1<p<\infty$, $p\leqslant q\leqslant\infty$, and Ω be a John domain $J(\alpha,\beta)$ in $\mathbb{G}$, Q be a homogeneous differential operator (1) of order k with constant coefficients and finite-dimensional kernel. Then there exists a projector Π on the kernel of Q and an integer $l\geqslant k$ such that $\ker Q\subset\mathcal{P}_{l+1}$ and*

$$\|X^J(u-\Pi u)\|_{q,\Omega}\leqslant C\Big(\frac{\beta}{\alpha}\Big)^{\theta}\operatorname{diam}(\Omega)^{k-d(J)-\nu/p+\nu/q}\|Qu\|_{p,\Omega}$$

for every function $u\in W^k_p(\Omega,\mathbb{R}^s)$ and every multi-index J, satisfying $d(J)\leqslant k$, with

$$\theta=\begin{cases}l-d(J)+\nu & \text{if } q\neq\infty,\\ l-d(J)+\nu+\nu/p & \text{if } q=\infty,\end{cases}$$

and p,q meeting one of the following conditions:

1) $p\leqslant q\leqslant\frac{\nu p}{\nu-(k-d(J))p}$ for $(k-d(J))p<\nu$, $d(J)<k$;

2) $p\leqslant q<\infty$ for $(k-d(J))p=\nu$;

3) $p\leqslant q\leqslant\infty$ for $(k-d(J))p>\nu$;

4) $q=p$ for $d(J)=k$.

Here $C>0$ is independent of u, Ω, α and β.

The proof of coercive estimates is based on appropriate integral representation formulas (Theorems 2 and 3). In Theorems 2 and 3 we use the following quasimetric: $d_\infty(x,y)=\sup_{i=1,\dots,N}\{|(x^{-1}y)_i|^{1/\sigma_i}\}$, see [22]. Let $c>1$ be a constant in the generalized triangle inequality: $d_\infty(x,y)\leqslant c(d_\infty(x,z)+d_\infty(z,y))$ for all $x,y,z\in\mathbb{G}$. Denote a ball in quasimetric d_∞ by $\operatorname{Box}(a,r)=\{x: d_\infty(a,x)<r\}$. Notice that the quasimetric d_∞ is equivalent to the Carnot–Carathéodory metric d_{cc}: $c_1 d_\infty(x,y)\leqslant d_{cc}(x,y)\leqslant c_2 d_\infty(x,y)$ for all $x,y\in\mathbb{G}$, $0<c_1<c_2<\infty$ are constants.

Theorem 2. *Let an integer $l>0$ and a function $u\in C^\infty(\mathbb{G})$. Then*

$$u(x)=\int\limits_{\mathbb{G}}u(y)\varphi(y^{-1}x)\,dy+\sum_{d(I_h)=l}\int\limits_{\mathbb{G}}X^{I_h}u(y)K_{I_h}(y^{-1}x)\,dy \tag{2}$$

for all $x\in\mathbb{G}$ where

$$\varphi\in C^\infty(\mathbb{G}),\quad \operatorname{supp}\varphi\subseteq\overline{\operatorname{Box}(e,1)}\setminus\operatorname{Box}(e,1/2),$$

$$\int\limits_{\mathrm{Box}(e,1)} \varphi(x)\,dx = 1, \qquad \int\limits_{\mathrm{Box}(e,1)} x_1^{\alpha_1}\dots x_N^{\alpha_N}\varphi(x)\,dx = 0, \quad 0 < \sum_{i=1}^{N}\alpha_i < l,$$

$$K_{I_h} \in C^\infty(\mathbb{G}\setminus\{e\}), \quad \operatorname{supp} K_{I_h} \subseteq \overline{\mathrm{Box}(e,1)},$$

and

$$|X^J K_{I_h}(x)| \leqslant M_{d(J)} d_\infty(x,e)^{-l+d(J)-\nu} \quad \textit{for any multi-index } J.$$

Here $M_{d(J)} > 0$ *is a constant independent of* u *and* $x \in \mathbb{G}$.

Remark 1. *Rewrite formula* (2) *in* $\mathbb{R}^n$*:*

$$u(x) = \int\limits_{\mathbb{R}^n} u(y)\varphi(x-y)\,dy + \sum_{|\alpha|=l}\int\limits_{\mathbb{R}^n} D^\alpha u(y) K_\alpha(x-y)\,dy.$$

Emphasize that such integral representation of a function is new for the Euclidean spaces also since the kernel of the integral operator in Theorem 2 depends only on $x-y$ *but not on* $(x, y-x)$*, as in [37].*

In the second integral representation theorem we want to see a function as the sum of a polynomial and a singular integral of derivatives.

Theorem 3. *Let an integer* $l > 0$, $\varkappa = c + c^2 + 2c^3$ *where* c *is the constant of the generalized triangle inequality of the quasimetric* d_∞*, and a function* $u \in C^\infty(\mathrm{Box}(e,\varkappa))$*. Then for every* $x \in \mathrm{Box}(e,1)$ *the integral representation formula*

$$u(x) = P_l u(x) + \sum_{d(I_h)=l}\ \int\limits_{\mathrm{Box}(e,\varkappa)} X^{I_h}u(y) K'_{I_h}(y,x)\,dy$$

holds where

P_l *is a projection of* $L_1(\mathrm{Box}(e,1))$ *to* $\mathcal{P}_l$,

$K'_{I_h}(y,x) = K_{I_h}(y^{-1}x) + L_{I_h}(y,x)$ *with* K_{I_h} *from Theorem* 2,

and

$L_{I_h} \in C^\infty(\mathbb{G}\times\mathbb{G})$, $\operatorname{supp} L_{I_h}(\cdot,x) \subseteq \mathrm{Box}(e,\varkappa)$ *for* $x \in \mathrm{Box}(e,\varkappa)$.

Notice that, on two-step Carnot groups, Sobolev type integral representation of functions were obtained in [29] and [25]. The method of its proof is based on works by S. Sobolev and Yu. Reshetnyak. However, its generalizations to arbitrary groups meet serious technical obstacles.

In our paper we apply a different method for deriving integral representations. This method was introduced by V. S. Rychkov in [31, 32]. It is based on a representation of functions by means of the convolution with a kernel which is a sum of dyadic dilations. This method can be considered as a counterpart of Calderón-type reproducing formula.

We give below several corollaries of Theorem 1 on coercive estimates.

Consider the differential operator $Qu = \nabla_{\mathcal{L}}^k u = \{X^{I_h}u\}_{d(I_h)=k}$. Then its kernel is just the space $\mathcal{P}_k$ of polynomials of degree $< k$. As a particular case we have Poincaré inequality for higher derivatives. Notice that, by a different method, it was obtained also by G. Lu in [18, 19].

Theorem 4 (Poincaré inequality). *Let $l > 0$, $1 \leqslant p \leqslant q \leqslant \infty$, U be a John domain $J(\alpha,\beta)$ in $\mathbb{G}$. Then there is a projection $P\colon W_p^l(U) \to \mathcal{P}_l$ such that*

$$\|X^J(u - Pu)\|_{q,U} \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{\theta} \operatorname{diam}(U)^{l-d(J)-\nu/p+\nu/q}\|\nabla_{\mathcal{L}}^l u\|_{p,U}$$

for any function u of the Sobolev class $W_p^l(U)$ and every multi-index J, $d(J) < l$, with
1) $p \leqslant q \leqslant \frac{\nu p}{\nu-(l-d(J))p}$ *for* $(l - d(J))p < \nu$;
2) $p \leqslant q < \infty$ *for* $(l - d(J))p = \nu$;
3) $p \leqslant q \leqslant \infty$ *for* $(l - d(J))p > \nu$;
4) $q = \infty$ *for* $l - d(J) \geqslant \nu$,
and $\theta = \begin{cases} l - 1 - d(J) + \nu & \text{if } q \neq \infty, \\ l - 1 - d(J) + \nu + \nu/p & \text{if } q = \infty. \end{cases}$

Here $C > 0$ is independent of u, U, α and β.

The next corollary is an embedding theorem. In Euclidean spaces the embedding theorem on John domains was established by Yu. G. Reshetnyak [26]. The embedding into Orlicz spaces is proved by S. I. Pohozhaev [24] for bounded domains with locally Lipschitz boundary and by B. V. Trushin [39] for domains satisfying flexible σ-cone condition. This class includes the class of John domains.

On Carnot groups the global embedding theorem can be found in [8]. The embedding theorems for $l = 1$ on John domains are obtained in [11]. In the following theorem we state embedding theorem and estimate the norms of embedding operators. For formulating the theorem, define several functional spaces.

Let U be a domain in $\mathbb{G}$. Denote by $C^k(U)$, $k = 0, 1, \dots$, the space of continuous functions $f\colon U \to \mathbb{R}$ with continuous derivatives $X^{I_h} f$ for all multi-indices $d(I_h) \leqslant k$, and with a finite norm $\|f\|_{C^k(U)} = \sup\{|X^{I_h} f(x)|\colon x \in U, d(I_h) \leqslant k\}$.

For defining the class of Hölder functions we introduce the inner metric d_τ^U, $0 < \tau \leqslant 1$, defined on a domain $U \subset \mathbb{G}$:

$$d_\tau^U(x,y) = \inf\Big\{\sum_{i=1}^{m}(d_{cc}(x_i,x_{i-1}))^\tau \mid x = x_0, x_1, \dots, x_m = y \in U,$$
$$d_{cc}(x_i,x_{i-1}) \leqslant \max\{\operatorname{dist}(x_i,\partial U), \operatorname{dist}(x_{i-1},\partial U)\},\ i = 1,\dots,m\Big\}.$$

Notice that $d_1^U(\cdot,\cdot)$ coincides with the infimum of the lengths of all horizontal curves joining two points in U. If U is a John domain $J(\alpha,\beta)$ then its diameter is bounded in the metric d_τ^U for all $\tau \in (0,1]$.

A space $C^{k,\tau}(U)$, $k = 0, 1, \dots$, $0 < \tau < 1$, consists of all functions $f\colon U \to \mathbb{R}$ having continuous derivatives $X^{I_h} f$ for all $d(I_h) \leqslant k$ and the finite norm

$$\|f\|_{C^{k,\tau}(U)} = \sup_{x\in U,\ d(I_h)\le k} |X^{I_h} f(x)| + \sup_{x,y\in U, x\neq y, d(I_h)=k} \frac{|X^{I_h} f(x) - X^{I_h} f(y)|}{d_\tau^U(x,y)}.$$

A space $C^{k,1}_{\mathrm{loc}}(U)$ is a subspace of C^k-smooth functions with the finite norm

$$\|f\|_{C^{k,1}_{\mathrm{loc}}(U)} = \sup_{x\in U,\, d(I_h)\le k} |X^{I_h} f(x)| + \sup_{\substack{B(x,r)\subset U,\, d_{cc}(e,h)=r,\\ d(I_h)=k}} \frac{|X^{I_h} f(xh) + X^{I_h} f(xh^{-1}) - 2X^{I_h} f(x)|}{r}.$$

Set $\Phi(t) = \exp(t^\eta) - 1$, $\eta \geqslant 1$. Let $C^{k,\Phi}(U)$ be the space of functions $f\colon U \to \mathbb{R}$ having bounded continuous derivatives $X^{I_h} f$ for $d(I_h) < k$ and the weak derivatives $X^{I_h} f$ satisfying

$$\int_U \Phi(|X^{I_h} f(x)|)\,dx < \infty \quad \text{for } d(I_h) = k$$

equipped with the following norm:

$$\|f\|_{C^{k,\Phi}(U)} = \sup_{x\in U,\ d(I_h)<k} |X^{I_h} f(x)| + \inf\Big\{\rho > 0 \mid \int_U \Phi\Big(\frac{|X^{I_h} f(x)|}{\rho}\Big)\,dx \leqslant 1,\, d(I_h) = k\Big\}.$$

Obviously $C^{0,\Phi}$ is the Orlicz space L_Φ.

Theorem 5 (Embedding theorem)**.** *Let $l \in \mathbb{N}$, $k \in \mathbb{Z}^+$, $k < l$, $1 \leqslant p < \infty$, and U be a John domain $J(\alpha,\beta)$ in $\mathbb{G}$.*
(1) If $(l-k)p < \nu$ then $W^l_p(U)$ is continuously embedded in $W^k_q(U)$ for $1 \leqslant q \leqslant \frac{\nu p}{\nu-(l-k)p}$ (here we assume $W^0_q = L_q$):

$$W^l_p(U) \overset{i}{\hookrightarrow} W^k_q(U),$$

$$\|i\|_{W^l_p(U)\hookrightarrow W^k_q(U)} \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{l-1-k+\nu} (\operatorname{diam} U)^{-\nu/p+\nu/q} \max\{(\operatorname{diam} U)^{l-k}, 1\}.$$

If $q < \frac{\nu p}{\nu-(l-k)p}$ then the embedding is compact.
(2) If $(l-k)p = \nu$ and $p > 1$ then $W^l_p(U)$ is continuously embedded in $C^{k,\Phi}(U)$ for $\Phi(t) = \exp(t^\eta) - 1$, $\eta \leqslant \frac{p}{p-1}$:

$$W^l_p(U) \overset{i}{\hookrightarrow} C^{k,\Phi}(U),$$

$$\|i\|_{W^l_p(U)\hookrightarrow C^{k,\Phi}(U)} \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{l-k-1+\nu+\nu/p} (\operatorname{diam} U)^{-\nu/p} \max\{(\operatorname{diam} U)^{l-k}, 1\}.$$

If $\eta < \frac{p}{p-1}$ then the embedding is compact.
(3) If $p = 1$ and $l - k = \nu$ then $W^l_p(U)$ is continuously embedded in $C^k(U)$:

$$W^l_p(U) \overset{i}{\hookrightarrow} C^k(U),$$

$$\|i\|_{W^l_p(U)\hookrightarrow C^k(U)} \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{l-1-k+\nu+\nu/p} (\operatorname{diam} U)^{-\nu/p} \max\{(\operatorname{diam} U)^{l-k}, 1\}.$$

Moreover, $W^l_p(U)$ is continuously compactly embedded in $C^{k,\Phi}(U)$ with an arbitrary $\eta \in (1,\infty)$.

(4) *If $(l-k)p > \nu$ and $(l-k-1)p < \nu$ then $W_p^l(U)$ is continuously embedded in $C^{k,\tau}(U)$, $0 < \tau \leqslant l-k-\frac{\nu}{p} < 1$:*

$$W_p^l(U) \overset{i}{\hookrightarrow} C^{k,\tau}(U),$$

$$\|i\|_{W_p^l(U)\hookrightarrow C^{k,\tau}(U)} \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{l-1-k+\nu+\nu/p}(\operatorname{diam} U)^{-\nu/p}\max\{(\operatorname{diam} U)^{l-k},1\}.$$

If $\tau < l-k-\frac{\nu}{p}$ then the embedding is compact. In addition, $W_p^l(U)$ is continuously compactly embedded in $C^k(U)$.
(5) *If $(l-k)p > \nu$ and $(l-k-1)p = \nu$ then $W_p^l(U)$ is continuously embedded in $C^{k,1}_{\mathrm{loc}}(U)$,*

$$W_p^l(U) \overset{i}{\hookrightarrow} C^{k,1}_{\mathrm{loc}}(U),$$

$$\|i\|_{W_p^l(U)\hookrightarrow C^{k,1}_{\mathrm{loc}}(U)} \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{l-1-k+\nu+\nu/p}(\operatorname{diam} U)^{-\nu/p}\max\{(\operatorname{diam} U)^{l-k},1\}.$$

Furthermore, $W_p^l(U)$ is continuously compactly embedded in $C^k(U)$ and $C^{k,\tau}(U)$ with $\tau \in (0,1)$.

Remark 2. *Applying method of paper [40] we can prove that a function $f \in C^{k,\tau}(U)$ can be extended by the continuity to the completion of the domain U with respect to the metric d_τ^U.*

The Sobolev space $\widetilde{W}_q^l(\Omega)$ consists of the functions $f\colon \Omega \to \mathbb{R}$ having the weak derivative $X^{I_h}f$ for $d(I_h) = l$, and a finite norm

$$\|f\|_{\widetilde{W}_q^l(\Omega)} = \|f\|_{q,\Omega} + \sum_{d(I_h)=l} \|X^{I_h}f\|_{q,\Omega}.$$

Like the Euclidean case, the spaces W_p^l and $\widetilde{W}_p^l$ coincide on John domains.

Theorem 6. *Let Ω be a John domain on Carnot group $\mathbb{G}$, $1 \leqslant p \leqslant \infty$, $l \in \mathbb{N}$. Then*

$$W_p^l(\Omega) = \widetilde{W}_p^l(\Omega)$$

and the norms are equivalent.

The proof of Theorem 6 is based essentially on Theorem 4.

One more application of Theorem 1 is an extension theorem of Sobolev-type functions defined on (ε,δ)-domains. Jones [15] proved the extension theorems for the Sobolev space W_p^k beyond such domains in $\mathbb{R}^n$. Lu [19] established extension theorems for the weighted Sobolev spaces on (ε,δ)-domains of Carnot groups. In this paper we introduce and extend functions belonging to some Sobolev-type class. Notice that bounded sets with smooth boundaries are (ε,δ)-domains (e. g., see [7, 29]) on two-step Carnot groups. Besides of this balls in the Heisenberg groups with respect to Carnot–Carathéodory metric are (ε,δ)-domains [42].

An open set Ω is an (ε, δ)-*domain* if for all $x, y \in \Omega$, $d_\infty(x,y) < \delta$, there exists a rectifiable curve γ with endpoints x and y such that γ lies in Ω and

$$\operatorname{length}(\gamma) < \frac{d_\infty(x,y)}{\varepsilon}, \quad d(z, \partial\Omega) \geqslant \frac{\varepsilon d_\infty(x,z) d_\infty(y,z)}{d_\infty(x,y)} \quad \text{for all } z \in \gamma.$$

Radius of an open set Ω is a number

$$\operatorname{rad}\Omega = \sup\{r > 0 : \partial \operatorname{Box}(p,s) \cap \Omega \neq \emptyset \text{ for all } p \in \Omega, 0 \leqslant s < r\}.$$

Let Q be a homogeneous differential operator (1) of order k with constant coefficients and with finite-dimensional kernel. Let Π be a projection of $L_1(\operatorname{Box}(e,1), \mathbb{R}^s)$ to the kernel of Q from Theorem 8.

Consider a domain Ω, $\overline{\operatorname{Box}(e,1)} \subset \Omega$. A locally integrable function $u \colon \Omega \to \mathbb{R}^s$ belongs to the functional space $W_p^Q(\Omega, \mathbb{R}^s)$ if Qu is well-defined in the sense of weak derivatives and the following norm is finite:

$$\|u\|_{W_p^Q(\Omega)} = \|\Pi u\| + \|Qu\|_{p,\Omega}.$$

Notice that, by Theorem 1, the space $W_p^Q(\Omega)$ coincides with the usual Sobolev class $W_p^k(\Omega)$ for each John domain Ω.

Theorem 7 (Extension theorem). *Let* Ω *be a bounded* (ε,δ)*-domain on a Carnot group* $\mathbb{G}$, $\overline{\operatorname{Box}(e,1)} \subset \Omega$, $1 < p \leqslant \infty$. *There is an extension operator*

$$\operatorname{ext} \colon W_p^Q(\Omega) \to W_p^Q(\mathbb{G}),$$

and the norm of the operator ext *depends only on* $\varepsilon, \delta, k, p$ *and radius of the domain* Ω.

We prove first Theorem 2 and then Theorem 3. After doing this we are ready to prove a local version of coercive estimates for the differential operator (1) (Theorem 8). It left to justify only a way from local to global estimates. It is done in Theorem 9.

The structure of the paper is the following. In Section 2 we give necessary definitions and some auxiliary results. Section 3 is devoted to the proof of Theorems 2 and 3. In Section 4 we give embedding theorems for Riesz potentials (Lemma 3), establish Zygmund–Calderón type theorem (Lemma 4), and prove local coercive estimates (Theorem 8). Section 5 is devoted to passing from local estimates to global ones. In Section 6 we prove Theorems 4–7.

Authors thank Anton Parfenov for introducing the method of papers [31, 32].

2 Definitions and auxiliary results

2.1 Left- and right-invariant vector fields

On $\mathbb{G}$ one can choose coordinates $x_1, \dots, x_N$ such that

$$X_i = \frac{\partial}{\partial x_i} + \sum_{\sigma_j > \sigma_i} P_{ij}(x) \frac{\partial}{\partial x_j}$$

where P_{ij} is a polynomial of homogeneous degree $(\sigma_j - \sigma_i)$. (Here we follow Jerison [13].)

We have $x^{-1} = -x$, $X_i^* = -X_i$ and $X_i = \frac{\partial}{\partial x_i}$ for $X_i \in V_m$.

Set $Rf(x) = f(x^{-1})$ and $X^R f = RXRf$. If X is a left-invariant vector field then X^R is right-invariant. Moreover,

$$X_i^R = -\frac{\partial}{\partial x_i} - \sum_{\sigma_j > \sigma_i} P_{ij}(-x)\frac{\partial}{\partial x_j}, \quad i = 1, \dots, N.$$

It is easy to verify using induction on i beginning with the trivial case $\sigma_i = m$ and decreasing to $\sigma_i = 1$ that for some polynomials Q_{ij} with homogeneous degree $\sigma_j - \sigma_i$

$$\frac{\partial}{\partial x_i} = -X_i^R + \sum_{\sigma_j > \sigma_i} X_j^R Q_{ij}.$$

Therefore, $X_i = \sum_{j=1}^n X_j^R A_{ij}$ where A_{ij} are differential operators satisfying $A_{ii} = -\mathrm{id}$ and $\int A_{ij}\varphi\, dx = 0$ for any $\varphi \in C_0^\infty$ and $i \neq j$.

2.2 Convolution

For any integrable functions f and g, the convolution $f * g$ is a function defined as follows:

$$f * g(x) = \int_{\mathbb{G}} f(x^{-1}y)g(y)\, dy = \int_{\mathbb{G}} f(y)g(y^{-1}x)\, dy.$$

It is easy to verify that the following properties holds for any smooth integrable functions f, g, and a left-invariant vector-field X:

1) $f * g \neq g * f$;
2) $X(f * g) = f * Xg$;
3) $(Xf) * g = -f * (X^R g)$.

2.3 Smooth approximations of Sobolev functions

In this subsection we show the density of smooth functions in Sobolev spaces. (Density of smooth functions in the Sobolev spaces on Carnot groups can be found in [11] for $l = 1$.)

Let $\varphi \in C_0^\infty(B(e,1))$ and $a = \int \varphi(x)\, dx$. Set $\varphi_\varepsilon = \frac{1}{\varepsilon^\nu}\varphi \circ \delta_{1/\varepsilon}$. Then for any function $u \in L_p(\mathbb{G})$ we have $\|u_\varepsilon - au\|_{p,\mathbb{G}} \to 0$ as $\varepsilon \to 0$ where $u_\varepsilon = u * \varphi_\varepsilon$ [9].

Lemma 1. *Let Ω be open set on $\mathbb{G}$ with nonempty boundary. Then $C^\infty(\Omega) \cap \widetilde{W}_p^l(\Omega)$ is dense in $\widetilde{W}_p^l(\Omega)$ and $C^\infty(\Omega) \cap W_p^l(\Omega)$ is dense in $W_p^l(\Omega)$.*

Proof. Take $\varphi \in C_0^\infty(B(e,1))$ with $\int \varphi(x)\, dx = 1$ and $f \in \widetilde{W}_p^l(\Omega)$. Then $f_\varepsilon \in C^\infty(\Omega') \cap \widetilde{W}_p^l(\Omega')$ for Ω' compactly supported in Ω.

Consider first $l = 1$. Then

$$X_i f_\varepsilon = f * X_i\varphi_\varepsilon = f * \sum_{j=1}^n X_j^R A_{ij}\varphi_\varepsilon = -\sum_{j=1}^n X_j f * A_{ij}\varphi_\varepsilon$$

and $\int A_{ij}\varphi_\varepsilon = -\delta_{ij}$. It follows

$$\|X_i f_\varepsilon - X_i f\|_{p,\Omega'} \to 0 \quad \text{as } \varepsilon \to 0.$$

Let now l be arbitrary. The same is true for any multi-index I_h, $d(I_h) = l$. Indeed,

$$X^{I_h} f_\varepsilon = f * X^{I_h}\varphi_\varepsilon = f * \sum_{d(J_h)=l} X^{J_h,R} A_{I_h J_h}\varphi_\varepsilon = (-1)^l \sum_{d(J_h)=l} X^{J_h} f * A_{I_h J_h^t}\varphi_\varepsilon$$

where $A_{I_h J_h^t}$ are differential operators with $A_{I_h I_h^t} = (-1)^l\mathrm{id}$ and $\int A_{I_h J_h^t}\varphi_\varepsilon = 0$. (Here $J_h^t = (j_l, \dots, j_1)$ for $J_h = (j_1, \dots, j_l)$.) Therefore, $\|X^{I_h} f_\varepsilon - X^{I_h} f\|_{p,\Omega'} \to 0$ as $\varepsilon \to 0$.

Passing from Ω' to Ω repeats word-by-word the proof in Euclidean setting (see, for example, [20]). □

2.4 Construction of the function φ from Theorem 2

For a function φ, define

$$T\varphi(x) = 2^\nu \varphi(\delta_2 x).$$

Obviously,
1) $T^k\varphi(x) = 2^{k\nu}\varphi(\delta_{2^k} x) = T^i T^j \varphi(x)$ if $i + j = k$, $\quad i, j, k \in \mathbb{N}$;
2) $X_i T^k = 2^{\sigma_i k} T^k X_i$ and $X_i^R T^k = 2^{\sigma_i k} T^k X_i^R$, $i = 1, \dots, N$.

Lemma 2. *Let $l > 0$ be an integer. There exists a smooth function φ supported in the annuli $\overline{\mathrm{Box}(e,1)} \setminus \mathrm{Box}(e,1/2)$ such that*

$$\int_{\mathrm{Box}(e,1)} \varphi(x)\,dx = 1,$$

$$\int_{\mathrm{Box}(e,1)} x_1^{\alpha_1} \dots x_N^{\alpha_N} \varphi(x)\,dx = 0, \quad 0 < \sum \alpha_i < l,\ \alpha_i \in \mathbb{Z}^+,$$

and

$$T\varphi - \varphi = \sum_{d(I_h)=l} X^{I_h,R} \zeta_{I_h}$$

where the smooth functions ζ_{I_h} are supported in the annuli $\overline{\mathrm{Box}(e,1)} \setminus \mathrm{Box}(e,1/4)$.

In the Euclidean case the analog of the lemma can be found in [23, Lemma 3.7].

For proving Lemma 2, we construct a smooth function $\psi\colon \mathbb{R} \to \mathbb{R}$ such that

$$\operatorname{supp}\psi \subseteq [a,b], \quad \int_{-\infty}^{\infty} \psi(s)\,ds = 1,$$

$$\int_{-\infty}^{\infty} s^k \psi(s)\,ds = 0 \quad \text{for all } k = 1, \dots, l-1.$$

It is sufficient to consider l test functions $\psi_1, \dots, \psi_l$ supported on the interval $[a,b]$ satisfying $\det A \neq 0$ where

$$A = \{a_{ij}\}_{ij=1}^l, \quad a_{ij} = \int_{-\infty}^{\infty} s^{i-1}\psi_j(s)\,ds, \quad i, j = 1, \dots, l.$$

Put a vector

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_l \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then $\psi = \sum_{i=1}^{l} c_i \psi_i$ satisfies all the properties we need.

Let a function $f \in C_0^\infty(\mathbb{R})$ satisfy

$$\int_{-\infty}^{\infty} f(s)\,ds = 0, \quad \int_{-\infty}^{\infty} s^k f(s)\,ds = 0 \quad \text{for all } k = 1, \dots, l-1.$$

Then $f(s) = \frac{d^l}{ds^l}\xi(s)$ where

$$\xi(s) = \frac{1}{(l-1)!}\int_{-\infty}^{s} (s-t)^{l-1} f(t)\,dt.$$

Moreover, $\operatorname{supp}\xi \subseteq \operatorname{supp} f$.

Proof of Lemma 2. Consider N functions $\psi_i \in C_0^\infty(\mathbb{R})$, $i = 1, \dots, N$, satisfying

$$\int_{-\infty}^{\infty} \psi_i(s)\,ds = 1, \quad \int_{-\infty}^{\infty} s^k \psi_i(s)\,ds = 1 \quad \text{for all } k = 1, \dots, l-1,$$

and $\operatorname{supp}\psi_i \subseteq [2^{-\sigma_i}, 1]$.

Set $\varphi(x) = \psi_1(x_1)\psi_2(x_2)\dots\psi_N(x_N)$. Then $\varphi \in C_0^\infty(\mathbb{G})$ and $\operatorname{supp}\varphi \subseteq \overline{\operatorname{Box}(e,1)} \setminus \operatorname{Box}(e,1/2)$. Furthermore, $T\varphi(x) = \prod_{i=1}^{N} \mathbf{t}^{\sigma_i}\psi_i(x_i)$ where the operator $\mathbf{t}$ is defined as follows: $\mathbf{t}\psi(s) = 2\psi(2s)$. Therefore

$$\begin{aligned} T\varphi(x) - \varphi(x) = {} & (\mathbf{t}\psi_1(x_1) - \psi_1(x_1))\psi_2(x_2)\cdots\psi_N(x_N) \\ & + \mathbf{t}\psi_1(x_1)(\mathbf{t}\psi_2(x_2) - \psi_2(x_2))\psi_3(x_3)\cdots\psi_N(x_N) \\ & + \dots + \mathbf{t}\psi_1(x_1)\cdots\mathbf{t}^{\sigma_{i-1}}\psi_{i-1}(x_{i-1})(\mathbf{t}^{\sigma_i}\psi_i(x_i) - \psi_i(x_i))\psi_{i+1}(x_{i+1})\cdots\psi_N(x_N) \\ & + \dots + \mathbf{t}\psi_1(x_1)\cdots\mathbf{t}^{\sigma_{N-1}}\psi_{N-1}(x_{N-1})(\mathbf{t}^{\sigma_N}\psi_N(x_N) - \psi_N(x_N)). \end{aligned}$$

We have $T\varphi - \varphi = \sum_{i=1}^{N} \frac{\partial^l}{\partial x_i^l}\zeta_i$ with

$$\zeta_i(x) = \mathbf{t}\psi_1(x_1)\dots\mathbf{t}^{\sigma_{i-1}}\psi_{i-1}(x_{i-1})\xi_i(x_i)\psi_{i+1}(x_{i+1})\dots\psi_N(x_N),$$

$\operatorname{supp}\zeta_i \subseteq \overline{\operatorname{Box}(e,1)} \setminus \operatorname{Box}(e,1/4)$,

$$\xi_i(x_i) = \frac{1}{(l-1)!}\int_{-\infty}^{x_i} (x_i - t)^{l-1}(\mathbf{t}^{\sigma_i}\psi_i(t) - \psi_i(t))\,dt.$$

Show that

$$\frac{\partial^l}{\partial x_i^l}\zeta_i = \sum_{d(I_h)=l} X^{I_h,R}\zeta_{i,I_h}.$$

It suffices to verify it for $l = 1$. Recall that $X_i^R = -\frac{\partial}{\partial x_i} - \sum\limits_{\sigma_j > \sigma_i} P_{ij}(-x)\frac{\partial}{\partial x_j}$, and the polynomial $P_{ij}(-x)$ has homogeneous degree $\sigma_j - \sigma_i$. It follows that $P_{ij}(-x)$ is independent of x_j since the polynomial x_j has homogeneous degree σ_j. Thus,

$$\begin{aligned}\frac{\partial}{\partial x_i}\zeta_i(x) = -X_i^R\zeta_i(x) + \sum_{\sigma_j>\sigma_i}\frac{\partial}{\partial x_j}(P_{ij}(-x)\zeta_i(x))\\ = -X_i^R\zeta_i(x) - \sum_{\sigma_j>\sigma_i} X_j^R(P_{ij}(-x)\zeta_i(x))\\ + \sum_{\sigma_k>\sigma_j>\sigma_i}\frac{\partial}{\partial x_k}(P_{jk}(-x)P_{ij}(-x)\zeta_i(x))\dots.\end{aligned}$$

Taking into account $\frac{\partial}{\partial x_k} = -X_k^R$ for $X_k \in V_m$ and proceeding as follows, we finally come to the following equality:

$$\frac{\partial}{\partial x_i}\zeta_i(x) = \sum_{j=i}^{N} X_j^R\zeta_{ij}(x), \quad \zeta_{ij} \in C_0^\infty(\mathrm{Box}(e,1)).$$

Here ζ_{ij} equals ζ_i multiplied by the polynomial of degree $(\sigma_j - \sigma_i)$.

Horizontal vector fields $X_1^R, \dots, X_n^R$ generate the whole Lie algebra V. Hence

$$X_j^R = \sum_{\sigma_s+\sigma_k=\sigma_j} c_{sk}^j[X_s^R, X_k^R] = \sum_{d(I_h)=\sigma_j} c_{I_h}^j X^{I_h,R}.$$

The desired property follows immediately. □

2.5 Stratified Taylor formula with integral remainder

The Taylor formula in $\mathbb{R}^1$ with integral remainder is known: $f \in C^l(\mathbb{R})$, $l > 0$,

$$f(t) = \sum_{k=0}^{l-1}\frac{1}{k!}\frac{d^k f(0)}{dt^k}t^k + \frac{1}{l!}\int_0^t \frac{d^l f(s)}{dt^l}(t-s)^{l-1}\,ds. \tag{3}$$

Fix points $x_0, z \in \mathbb{G}$. Consider a curve $\gamma(t) = x_0\delta_t z$, $t \geqslant 0$. We have

$$\dot{\gamma}(t) = \sum_{i=1}^{N}\sigma_i t^{\sigma_i-1} z_i X_i(\gamma(t)).$$

Let $u \in C^l(\mathbb{G}, \mathbb{R})$. Put $f(s) = u(x_0\delta_s z)$. Then

$$\frac{d^k f(s)}{ds^k} = \sum_{j=1}^{k}\sum_{\substack{I\in\{1,\dots,N\}^j,\\ d(I)\geqslant k}} s^{d(I)-k}Q_{k,I}(z)X^I u(x_0\delta_s z)$$

where $Q_{k,I}(z)$ is a homogeneous polynomial of degree $d(I)$.

Obviously,

$$\frac{d^k f(0)}{ds^k} = \sum_{j=1}^{k} \sum_{\substack{I\in\{1,\dots,N\}^j,\\ d(I)=k}} Q_{k,I}(z) X^I u(x_0) = \sum_{d(I_h)=k} Q_{I_h}(z) X^{I_h} u(x_0).$$

Here we replaced X_j of degree $\sigma_j > 1$ by a linear combination of σ_j horizontal vector fields. Notice that $Q_{I_h}(z)$ is the homogeneous polynomial of degree $d(I_h) = k$.

Applying Taylor formula (3) with $t = 1$ we obtain

$$u(x_0 z) = \sum_{k=0}^{l-1} \frac{1}{k!} \sum_{d(I_h)=k} Q_{I_h}(z) X^{I_h} u(x_0) \\ + \sum_{j=1}^{l} \sum_{\substack{I\in\{1,\dots,N\}^j,\\ d(I)\geqslant l}} \frac{Q_{l,I}(z)}{l!} \int_0^1 s^{d(I)-l} X^I u(x_0 \delta_s z)(1-s)^{l-1}\, ds. \quad (4)$$

Example 10. *Heisenberg group $\mathbb{H}^1$ has topological dimension $N = 3$ and 2-dimensional horizontal subspace V_1 spanned by $X_1 = \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3}$ and $X_2 = \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3}$, $[X_1, X_2] = -4\frac{\partial}{\partial x_3} = -4X_3$. The group law is given by the following rule: $(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 - 2x_1 y_2 + 2x_2 y_1)$.*

Taylor formula (4) can be rewritten as

$$u(x_0 z) = u(x_0) + \int_0^1 \big(z_1 X_1 + z_2 X_2 + 2s z_3 X_3\big) u(x_0 \delta_s z)\, ds,$$

for $l = 1$, and

$$u(x_0 z) = u(x_0) + z_1 X_1 u(x_0) + z_2 X_2 u(x_0) \\ + \int_0^1 \frac{1-s}{2} \big(2z_3 X_3 + z_1^2 X_1 X_1 + z_1 z_2 (X_1 X_2 + X_2 X_1) + z_2^2 X_2 X_2 \\ + 4s z_1 z_3 X_1 X_3 + 4s z_2 z_3 X_2 X_3 + 4s^2 z_3^2 X_3 X_3\big) u(x_0 \delta_s z)\, ds$$

for $l = 2$.

3 Integral representation formulas

3.1 Proof of Theorem 2

Let φ be a function constructed in Lemma 2. Since $\int T^k \varphi = 1$ and $\operatorname{supp} T^k \varphi \subseteq \overline{\operatorname{Box}(e, 2^{-k})}$ we have $T^k \varphi \to \delta$ as $k \to \infty$ in the sense of distributions. Notice $T^k \varphi =$

$\varphi + \sum_{i=0}^{k-1} T^i(T\varphi - \varphi)$. From here we have pointwise equality

$$\begin{aligned} u(x) &= \lim_{k\to\infty}(u * T^k\varphi)(x) = u*\varphi(x) + u * \sum_{k=0}^{\infty} T^k(T\varphi - \varphi)(x) \\ &= u*\varphi(x) + u*\sum_{k=0}^{\infty} T^k \sum_{d(I_h)=l} X^{I_h,R}\zeta_{I_h}(x) \\ &= u*\varphi(x) + u*\sum_{d(I_h)=l}\sum_{k=0}^{\infty} 2^{-lk} X^{I_h,R}T^k\zeta_{I_h}(x) \\ &= u*\varphi(x) + (-1)^l \sum_{d(I_h)=l} X^{I_h} u * \sum_{k=0}^{\infty} 2^{-lk}T^k\zeta_{I_h^t}(x). \end{aligned}$$

Here we have denoted $I_h^t = (i_k, \dots, i_1)$ for the multi-index $I_h = (i_1, \dots, i_l)$.

Fix $I_h = (i_1, \dots, i_l) \in \{1, \dots, n\}^l$. Define

$$K_{I_h} = (-1)^l \sum_{k=0}^{\infty} 2^{-lk} T^k \zeta_{I_h^t}.$$

We have $\operatorname{supp} K_{I_h} \subseteq \overline{\mathrm{Box}(e,1)}$. Take $x \in \overline{\mathrm{Box}(e,1)} \setminus \{0\}$. Set an integer $j \geqslant 0$ such that

$$2^{-j-1} \leqslant d_\infty(x,e) < 2^{-j}.$$

Obviously $T^k\zeta_{I_h^t}(x) = 0$ for all $k \geqslant j+1$ and $k \leqslant j-2$. Thus, $K_{I_h} \in C^\infty(\mathbb{G} \setminus \{0\})$.

Fix multi-index J. Estimate $|X^J K_{I_h}(x)|$. If $j > 0$ we have

$$\begin{aligned} X^J K_{I_h}(x) &= (-1)^l 2^{-l(j-1)} X^J T^{j-1} \zeta_{I_h^t}(x) + (-1)^l 2^{-lj} X^J T^j \zeta_{I_h^t}(x) \\ &= (-1)^l 2^{(-l+d(J))(j-1)} T^{j-1} X^J \zeta_{I_h^t}(x) + (-1)^l 2^{(-l+d(J))j} T^j X^J \zeta_{I_h^t}(x) \end{aligned}$$

and

$$|X^J K_{I_h}(x)| \leqslant \big(2^{(-l+d(J)+\nu)(j-1)} + 2^{(-l+d(J)+\nu)j}\big) \sup_{\mathrm{Box}(e,1)} |X^J \zeta_{I_h^t}|.$$

There are three cases:

$$|X^J K_{I_h}(x)| \leqslant \sup_{\mathrm{Box}(e,1)} |X^J\zeta_{I_h^t}| \times \begin{cases} 2^{(-l+d(J)+\nu)(j-1)+1} & \text{if } l - d(J) > \nu, \\ 2 & \text{if } l - d(J) = \nu, \\ 2^{(-l+d(J)+\nu)j+1} & \text{if } l - d(J) < \nu. \end{cases}$$

If $j = 0$ then $|X^J K_{I_h}(x)| \leqslant \sup_{\mathrm{Box}(e,1)} |X^J \zeta_{I_h^t}|$.

In all cases we have

$$|X^J K_{I_h}(x)| \leqslant M_{d(J)} d_\infty(x,e)^{l-d(J)-\nu}$$

where

$$M_s = 2^{|l-s-\nu|} \sup\{|X^J \zeta_{I_h}(z)| : z \in \mathrm{Box}(e,1), d(I_h) = l, d(J) = s\}.$$

3.2 Proof of Theorem 3

For proving Theorem 3 we have to show that the function $u*\varphi$ equals polynomial plus integral of derivatives of u of order l.

Fix points $x, x_0 \in \mathrm{Box}(e,1)$. Substitute Taylor formula for u with respect to x_0 to the convolution $u*\varphi$. It follows

$$
\begin{aligned}
u*\varphi(x) &= \int_{\mathrm{Box}(x,1)} u(y)\varphi(y^{-1}x)\,dy \\
&= \sum_{k=0}^{l-1} \sum_{d(I_h)=k} \frac{X^{I_h}u(x_0)}{k!} \int_{\mathrm{Box}(x,1)} Q_{I_h}(x_0^{-1}y)\varphi(y^{-1}x)\,dy \\
&\quad + \sum_{j=1}^{l} \sum_{\substack{I\in\{1,\dots,N\}^j \\ d(I)\geqslant l}} \int_{\mathrm{Box}(x,1)} \int_0^1 \frac{Q_{l,I}(x_0^{-1}y)}{l!} s^{d(I)-l} \\
&\quad \times X^I u(x_0\delta_s(x_0^{-1}y))(1-s)^{l-1}\varphi(y^{-1}x)\,ds\,dy.
\end{aligned}
$$

Since all the moments of the function φ vanish and the polynomial $Q_{I_h}(x_0^{-1}y)$ has homogeneous degree $d(I_h) < l$, it follows that

$$
\int_{\mathrm{Box}(x,1)} Q_{I_h}(x_0^{-1}y)\varphi(y^{-1}x)\,dy = Q_{I_h}(x_0^{-1}x).
$$

In the second summand make a change of variables: $z = x_0\delta_s(x_0^{-1}y)$. Then $y = x_0\delta_{1/s}(x_0^{-1}z)$ and $dy = s^{-\nu}dz$. We have

$$
\begin{aligned}
&\int\limits_{\mathrm{Box}(x,1)} \int\limits_0^1 \frac{Q_{l,I}(x_0^{-1}y)}{l!} s^{d(I)-l} X^I u(x_0\delta_s(x_0^{-1}y))(1-s)^{l-1}\varphi(y^{-1}x)\,ds\,dy \\
&= \int\limits_{\mathbb{G}} X^I u(z) \int\limits_0^1 \frac{Q_{l,I}(\delta_{1/s}(x_0^{-1}z))}{l!} s^{d(I)-l-\nu} \\
&\qquad \times (1-s)^{l-1}\varphi(\delta_{1/s}(z^{-1}x_0)x_0^{-1}x)\,ds\,dz \\
&= \int\limits_{\mathbb{G}} X^I u(z) \frac{Q_{l,I}(x_0^{-1}z)}{l!} \int\limits_0^1 \frac{(1-s)^{l-1}}{s^{l+\nu}} \varphi(\delta_{1/s}(z^{-1}x_0)x_0^{-1}x)\,ds\,dz.
\end{aligned}
$$

In the last equality we used the homogeneity of the polynomial $Q_{l,I}$.

Consider the last integral over s:

$$
\begin{aligned}
\mathcal{I}(z,x,x_0) &= \int\limits_0^1 \frac{(1-s)^{l-1}}{s^{l+\nu}} \varphi(\delta_{1/s}(z^{-1}x_0)x_0^{-1}x)\,ds \\
&= \int\limits_1^\infty \frac{(t-1)^{l-1}}{t^{l-1}} t^{l+\nu-2} \varphi(\delta_t(z^{-1}x_0)x_0^{-1}x)\,dt.
\end{aligned}
$$

The function φ is supported in $\mathrm{Box}(e,1)$. It follows, $d_\infty(\delta_t(z^{-1}x_0)x_0^{-1}x, 0) < 1$. From here, $td_\infty(z,x_0) = d_\infty(\delta_t(z^{-1}x_0),0) \leqslant c(1+d_\infty(x_0,x)) < c(1+2c)$. If $\frac{c(1+2c)}{d_\infty(z,x_0)} > 1$ then

$$|\mathcal{I}(z,x,x_0)| \leqslant \sup_{\mathrm{Box}(e,1)} |\varphi| \int_1^{\frac{c(1+2c)}{d_\infty(z,x_0)}} t^{l+\nu-2}\,dt \leqslant C \sup_{\mathrm{Box}(e,1)} |\varphi| \frac{1}{d_\infty(z,x_0)^{l+\nu-1}}.$$

If $\frac{c(1+2c)}{d_\infty(z,x_0)} \leqslant 1$ then $\mathcal{I} = 0$.

Differentiating $\mathcal{I}(z,x,x_0)$ with respect to the variable z, we obtain

$$X_z^J \mathcal{I}(z,x,x_0) = \int\limits_1^\infty \frac{(t-1)^{l-1}}{t^{l-1}} t^{l+\nu+d(J)-2}(X^{J,R}\varphi)(\delta_t(z^{-1}x_0)x_0^{-1}x)\,dt.$$

It follows

$$|X_z^J \mathcal{I}(z,x,x_0)| \leqslant C\frac{1}{d_\infty(z,x_0)^{l+\nu+d(J)-1}} \sup_{\mathrm{Box}(e,1)} |X^{J,R}\varphi|.$$

Differential operator X^I has degree $d(I) \geqslant l$. It can be rewritten as a linear combination of $d(I)$ horizontal vector fields. Differentiating by parts $d(I)-l$ times we finally obtain

$$\begin{aligned} u*\varphi(x) &= \sum_{k=0}^{l-1} \sum_{d(I_h)=k} \frac{X^{I_h}u(x_0)}{k!} Q'_{I_h}(x_0^{-1}x) \\ &+ \sum_{d(I_h)=l} \int\limits_{\mathrm{Box}(x_0,c+2c^2)} X^{I_h}u(z) L_{I_h}(z^{-1}x_0; x_0^{-1}x)\,dz \end{aligned}$$

where

$$|L_{I_h}(z^{-1}x_0; x_0^{-1}x)| \leqslant C\frac{1}{d_\infty(x_0,z)^{\nu-1}}.$$

Multiply $u*\varphi(x)$ by $\varphi(x_0)$ and integrate over x_0. It follows

$$\begin{aligned} u*\varphi(x) &= \int_{\mathrm{Box}(e,1)} u*\varphi(x)\,\varphi(x_0)\,dx_0 \\ &= \int_{\mathrm{Box}(e,1)} \sum_{k=0}^{l-1} \sum_{d(I_h)=k} \frac{X^{I_h}u(x_0)}{k!} Q_{I_h}(x_0^{-1}x)\varphi(x_0)\,dx_0 \\ &+ \sum_{d(I_h)=l} \int_{\mathrm{Box}(e,1)} \int\limits_{\mathrm{Box}(x_0,c+2c^2)} X^{I_h}u(z) L_{I_h}(z^{-1}x_0; x_0^{-1}x)\varphi(x_0)\,dz\,dx_0 \\ &= \int_{\mathrm{Box}(e,1)} \sum_{k=0}^{l-1} \sum_{d(I_h)=k} \frac{(-1)^k u(x_0)}{k!} X_{x_0}^{I_h}(Q_{I_h}(x_0^{-1}x)\varphi(x_0))\,dx_0 \\ &+ \sum_{d(I_h)=l} \int\limits_{\mathrm{Box}(e,c+c^2+2c^3)} X^{I_h}u(z) \int_{\mathrm{Box}(e,1)} L_{I_h}(z^{-1}x_0; x_0^{-1}x)\varphi(x_0)\,dx_0\,dz \end{aligned}$$

$$= P_{l-1}u(x) + \sum_{d(I_h)=l} \int\limits_{\mathrm{Box}(e,c+c^2+2c^3)} X^{I_h}u(z)L'_{I_h}(z,x)\,dz$$

where P_{l-1} is a projection on polynomials of degree $< l$, $L'_{I_h}(z,x) \in C^\infty$ and $\operatorname{supp} L'_{I_h}(\cdot,x) \subseteq \mathrm{Box}(e,c+c^2+2c^3)$ for all $x \in \mathrm{Box}(e,1)$. (Here we used $\mathrm{Box}(x_0,c+2c^2) \subset \mathrm{Box}(e,c+c^2+2c^3)$ for all $x_0 \in \mathrm{Box}(e,1)$.)

Now it rests to insert the term $u * \varphi$ in the integral representation formula from Theorem 2.

4 Local coercive estimates

4.1 Singular integrals

In what follows, C denotes various positive constants. They may differ even in a same string of estimates. Set $\varkappa = c + c^2 + 2c^3$.

Introduce the following fractional integral operator (analog of the Riesz potential):

$$\mathcal{R}^\gamma v(x) = \int_{\mathrm{Box}(e,\varkappa)} v(y)\, d_\infty(x,y)^{\gamma-\nu}\, dy, \quad 0 \leqslant v \in L_p(\mathrm{Box}(e,\varkappa)), \quad \gamma > 0.$$

Lemma 3. *Let $1 \leqslant p < \infty$, $\gamma p < \nu$ and $q = \frac{\nu p}{\nu - \gamma p}$. Then there exists a constant C such that for every nonnegative function $v \in L_p(\mathrm{Box}(e,\varkappa))$ the following inequality holds:*

$$\|\mathcal{R}^\gamma v\|_{q,\mathbb{G}} \leqslant C_1 \|v\|_{p,\mathrm{Box}(e,\varkappa)} \quad \text{if } p > 1,$$

$$|\{x \in \mathrm{Box}(e,\varkappa) \mid \mathcal{R}^\gamma v(x) > t\}| \leqslant C_2 \left(\frac{\|v\|_{1,\mathrm{Box}(e,\varkappa)}}{t}\right)^q, \quad t > 0, \quad \text{if } p = 1.$$

Constants C_1 and C_2 are independent of v. Moreover, $C_1 \leqslant C_3 q^{1-\frac{\gamma}{\nu}}$ where the constant C_3 is independent of q.

In the case $p > 1$ Lemma 3 can be found in [41, Theorem 10], for $p = 1$ see [6, Theorem 4.1].

Lemma 4. *Let a function η satisfy the following conditions:*
(i) $\eta \in C^\infty(\mathbb{G})$, $\operatorname{supp}\eta \subseteq \overline{\mathrm{Box}(e,1)} \setminus \mathrm{Box}(e,1/4)$;
(ii) $\int_{\mathbb{G}} \eta(x)\,dx = 0$.
For $v \in L_p(\mathbb{G})$, $1 < p < \infty$, set

$$K(x) = \sum_{k=0}^{\infty} T^k \eta(x)$$

and

$$\mathcal{K}_\varepsilon v(x) = \int_{\mathbb{G}\setminus\mathrm{Box}(x,\varepsilon)} K(y^{-1}x)\, v(y)\, dy.$$

Then

$$\|\mathcal{K}_\varepsilon v\|_p \leqslant A_p \|v\|_p$$

where A_p is independent of v and ε. Moreover, for each function $v \in L_p(\mathbb{G})$, there exists $\lim_{\varepsilon\to 0} \mathcal{K}_\varepsilon v \overset{L_p}{=} \mathcal{K}v$ and $\|\mathcal{K}v\|_p \leqslant A_p \|v\|_p$.

Proof of Lemma 4 is based on the following technical lemma.

Lemma 5 ([16, Lemma 11]). *Let $\varphi(n) \geqslant 0$ be a function on the integers $n \in \mathbb{Z}$ with $\Phi = \sum_{n=-\infty}^{\infty} \varphi(n)^{1/2} < \infty$. If $T_1, \dots, T_N$ are linear operators on a Hilbert space with $\|T_i T_j^*\| \leqslant \varphi(i-j)$ and $\|T_i^* T_j\| \leqslant \varphi(i-j)$ for all i and j, then $\|T_1 + \dots + T_N\| \leqslant \Phi$, independently of N.*

Proof of Lemma 4. Consider $p = 2$. The proof of the boundedness of the operator $\mathcal{K}$ in L_2 follows arguments of paper [16]. Meanwhile we cannot use it directly since our kernel is not homogeneous.

Set
$$\mathcal{K}_k f(x) = f * T^k \eta(x) = \int f(xy^{-1})\, T^k \eta(y)\, dy.$$
Then
$$\mathcal{K}_k^* f(x) = \int f(xy)\, T^k \eta(y)\, dy$$
and
$$\mathcal{K}_j \mathcal{K}_k^* f = f * K_{jk}, \quad K_{jk}(x) = \int T^j \eta(yx)\, T^k \eta(y)\, dy,$$
$$\mathcal{K}_j^* \mathcal{K}_k f = f * K'_{jk}, \quad K'_{jk}(x) = \int T^j \eta(y)\, T^k \eta(xy)\, dy.$$
Analog of classical Young inequality ($\|f * g\|_2 \leqslant \|g\|_1 \|f\|_2$) yields
$$\|\mathcal{K}_j \mathcal{K}_k^*\|_{L_2 \to L_2} \leqslant \|K_{jk}\|_1.$$
It suffices to estimate $\|K_{jk}\|_1$ with $k > j$ since $K_{jk}(x) = K_{kj}(x^{-1})$.

Since $T^k \eta$ has the mean value 0, then
$$K_{jk}(x) = \int (T^j \eta(yx) - T^j \eta(x))\, T^k \eta(y)\, dy,$$
and
$$\begin{aligned} \|K_{jk}\|_1 &\leqslant \iint |T^j \eta(yx) - T^j \eta(x)|\, |T^k \eta(y)|\, dx\, dy \quad (x_1 = \delta_{2^j} x,\ y_1 = \delta_{2^j} y) \\ &= 2^{(k-j)\nu} \iint |\eta(y_1 x_1) - \eta(x_1)|\, |\eta(\delta_{2^{k-j}} y_1)|\, dx_1\, dy_1 \\ &\leqslant 2^{(k-j)\nu} \iint d_{cc}(y_1, e) \sup_z |\nabla_{\mathcal{L}} \eta(z)|\, |\eta(\delta_{2^{k-j}} y_1)|\, dx_1\, dy_1 \\ &\leqslant C 2^{(k-j)\nu} 2^{j-k} \int\limits_{d_\infty(x_1,e) < c(1+2^{j-k})} dx_1 \int\limits_{2^{j-k-2} < d_\infty(y_1,0) < 2^{j-k}} dy_1 \leqslant C 2^{j-k}. \end{aligned}$$
The norm $\|K'_{jk}\|_1$ can be estimated in a similar way.

Now, using Lemma 5 we can obtain boundedness of the operator $\mathcal{K}$ in L_2. For proving the boundedness of $\mathcal{K}$ in L_p, $1 < p < 2$, one can use Marcinkiewicz interpolation theorem. For doing this one needs to prove that the mapping $f \mapsto Kf$ is of weak-type

$(1,1)$. The latter can be verified by standard argument (see, e. g. [38, Theorem 3, Ch. I, § 5]).

For $p > 2$ we use adjoint operator $\mathcal{K}^*$. It is easy to see, that the boundedness of the operator $\mathcal{K}$ in L_p follows from the boundedness of the operator $\mathcal{K}^*$ in $L_{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$. □

Remark 3. *In Lemma 4 we can consider function η supported on any annuli centered at e and being Lipschitzian with respect to both left-invariant and right-invariant metrics.*

4.2 Coercive estimate. Local version

In this subsection we consider differential operator Q defined in (1).

Theorem 8. *Let $1 < p < \infty$, Q be a differential operator of order k with constant coefficients and finite-dimensional kernel. Then there is a projector $\Pi \colon W_p^k(\mathrm{Box}(e,\varkappa),\mathbb{R}^s) \to \ker Q$ such that*

$$\|X^J(u - \Pi u)\|_{q,\mathrm{Box}(e,1)} \leqslant C\|Qu\|_{p,\mathrm{Box}(e,\varkappa)} \qquad (5)$$

for every function $u \in W_p^k(\mathrm{Box}(e,\varkappa),\mathbb{R}^s)$ and every multi-index J, $d(J) \leqslant k$, with q satisfying

1) $p \leqslant q \leqslant \frac{\nu p}{\nu-(k-d(J))p}$ *for* $(k-d(J))p < \nu$, $d(J) < k$;
2) $p \leqslant q < \infty$ *for* $(k-d(J))p = \nu$;
3) $p \leqslant q \leqslant \infty$ *for* $(k-d(J))p > \nu$;
4) $q = p$ *for* $d(J) = k$.

Here $C > 0$ is independent of u.

Proof. The idea of the proof is given in [29] for Heisenberg groups. (The case of two-step Carnot groups see in [25].) We give briefly a short sketch of this idea. It suffices to prove the theorem for the smooth function u. Then, by standard arguments we can pass to the Sobolev function u.

STEP 1. Denote by $\mathcal{P}_{h,i}$ the linear space of polynomials on $\mathbb{G}$ of homogeneous degree i. Since the kernel of Q is finite-dimensional there is a number $l \geqslant k$ such that $\mathcal{P}_{h,l} \cap \ker Q = \{0\}$. It follows $\mathcal{P}_l \cap \ker \nabla_{\mathcal{L}}^{l-k} Q = \{0\}$. (Here $\nabla_{\mathcal{L}}^{l-k} = \{X^{I_h} \colon d(I_h) = l-k\}$ is the homogeneous differential operator of order $l-k$.)

Therefore there is a matrix A with constant coefficient such that $\nabla_{\mathcal{L}}^{l-k} Q = A\nabla_{\mathcal{L}}^l$. Thus, the matrix A is reversible and $\nabla_{\mathcal{L}}^l = A^{-1}\nabla_{\mathcal{L}}^{l-k} Q$.

By integral representation theorem we have

$$u(x) - Pu(x) = \sum_{d(I_h)=l} \int\limits_{\mathrm{Box}(e,\varkappa)} X^{I_h}u(y)K'_{I_h}(y,x)\,dy = \int\limits_{\mathrm{Box}(e,\varkappa)} K'(y,x)\nabla_{\mathcal{L}}^l u(y)\,dy$$

for $x \in \mathrm{Box}(e,1)$ where P is a projection operator on $\mathcal{P}_l$ and $K'(y,x)$ is a matrix-valued kernel. Integrating by parts we obtain

$$u(x) - Pu(x) = \int\limits_{\mathrm{Box}(e,\varkappa)} K'(y,x)\,A^{-1}\nabla_{\mathcal{L}}^{l-k} Qu(y)\,dy = \int\limits_{\mathrm{Box}(e,\varkappa)} H(y,x)\,Qu(y)\,dy$$

where $H(y,x)$ is matrix-valued function such that $H(y,x) = H_1(y,x) + H_2(y^{-1}x)$, $H_1 \in C_O^\infty(\mathbb{G}\times\mathbb{G})$, $\operatorname{supp} H_1(\cdot,x) \subseteq \operatorname{Box}(e,\varkappa)$, $H_2 \in C_O^\infty(\mathbb{G}\setminus\{0\})$, $\operatorname{supp} H_2 \subseteq \overline{\operatorname{Box}(e,1)}$, and

$$|X_x^J H_2(y^{-1}x)| \leqslant M d_\infty(x,y)^{k-d(J)-\nu} \quad \text{for any multi-index } J.$$

First, we prove

$$\|X^J(u-Pu)\|_{q,\operatorname{Box}(e,1)} \leqslant C\|Qu\|_{p,\operatorname{Box}(e,\varkappa)}, \quad d(J)\leqslant k. \tag{6}$$

Step 2. Set $\gamma = k - d(J)$. If $\gamma = 0$ then (6) holds by Lemma 4 with $p = q > 1$. The case $\gamma \geqslant \nu$ is trivial since $X_x^J H(y,x)$ is a smooth function. Consider the case $0<\gamma<\nu$.

In the case $\gamma p > \nu$ we use Hölder inequality for obtaining (6):

$$\begin{aligned}\|X^J(u-Pu)\|_{\infty,\operatorname{Box}(e,1)} \\ \leqslant C\|Qu\|_{p,\operatorname{Box}(e,\varkappa)} \sup_{x\in\operatorname{Box}(e,1)} \left(\int_{\operatorname{Box}(e,\varkappa)} d_\infty(x,y)^{p'(\gamma-\nu)}\,dy\right)^{1/p'} \\ \leqslant C\|Qu\|_{p,\operatorname{Box}(e,\varkappa)} \quad \left(p' = \frac{p}{p-1}\right).\end{aligned}$$

For $\gamma p < \nu$ and $q = \frac{\nu p}{\nu-\gamma p}$, Lemma 3 yields (6).

It rests to consider the case $\gamma p = \nu$. Consider a number $p_1 = \frac{q\nu}{\nu+\gamma q}$. Obviously, $1 < p_1 < p$ if $q > \frac{\nu}{\nu-\gamma}$. By Lemma 3

$$\|X^J(u-Pu)\|_{q,\operatorname{Box}(e,1)} \leqslant C\|Qu\|_{p_1,\operatorname{Box}(e,\varkappa)} \leqslant C\|Qu\|_{p,\operatorname{Box}(e,\varkappa)}.$$

Step 3. Now we need to replace Pu by projection to the kernel of Q. Consider any projection $P_1\colon \mathcal{P}_l \to \ker Q$ such that $\ker(Id - P_1) = \ker Q$. Then $\|g - P_1 g\| \leqslant C\|Qg\|$ for any polynomial $g\in\mathcal{P}_l$ since in finite-dimensional space all the norms are equivalent. Existing of such a projecting P_1 is rather evident: one just needs to take a basis of $ker Q$ and to complete it till the basis of $\mathcal{P}_l$.

Set $\Pi = P_1\circ P$. Then Π is a projection of $L_1(\operatorname{Box}(e,1))$ on the kernel of Q. We have

$$\begin{aligned}QPu &= Q\left(u - \int_{\operatorname{Box}(e,\varkappa)} H(y,x)Qu(y)\,dy\right) \\ &= Qu - \int_{\operatorname{Box}(e,\varkappa)} Q_x H_1(y,x)Qu(y)\,dy - \int_{\operatorname{Box}(e,\varkappa)} Q_x H_2(y^{-1}x)Qu(y)\,dy \\ &= Qu - \mathcal{K}_1(Qu) - \mathcal{K}_2(Qu).\end{aligned}$$

Operators $\mathcal{K}_1$ and $\mathcal{K}_2$ are bounded in L_p for all $p\in(1,\infty)$. The first one is bounded since the function H_1 is C^∞-smooth, the second is bounded by Lemma 4. Indeed all the conditions of Lemma 4 hold. All the elements of the matrix-valued function QH_2 have the form $\sum_{k=0}^\infty T^k\eta$, where the smooth function η is supported in the annuli $\overline{\operatorname{Box}(e,1)}\setminus \operatorname{Box}(e,1/4)$, and the mean value vanishes. Therefore, $\|QPu\| \leqslant \|Qu\|_{p,\operatorname{Box}(e,\varkappa)}$.

Finally,

$$\|X^J(u-\Pi u)\|_{q,\mathrm{Box}(e,1)} \leqslant \|X^J(u-Pu)\|_{q,\mathrm{Box}(e,1)} + \|X^J(Pu-P_1Pu)\|_{q,\mathrm{Box}(e,1)} \\ \leqslant \|Qu\|_{p,\mathrm{Box}(e,\varkappa)} + \|Pu-P_1Pu\| \leqslant C\|Qu\|_{p,\mathrm{Box}(e,\varkappa)} + C\|QPu\|_{p,\mathrm{Box}(e,\varkappa)}$$

and the theorem follows. □

5 From local to global

Theorem 1 on coercive estimate on John domains follows from the local result (Theorem 8). The same concerns the Poincaré inequality. In order to get the global result from the local, we apply the well-known technique (see, e. g. [19, 25, 11]). This method is based on nice covering by balls of the John domain. We will give the proof in order to write down the explicit dependence of the John coefficients α and β.

Passing from local to global does not explore any group structure of $\mathbb{G}$. So, throughout this section we will consider metric space $\mathbb{X}$ with a metric d and a Borel measure μ. We assume also that μ is *doubling* which means that $\mu(2B) \leqslant C_d\mu(B)$ for every ball $B \subset \mathbb{X}$. A standard iteration of the doubling condition yields $\mu(B) \leqslant C_b\big(\frac{r}{r'}\big)^{\nu}\mu(B')$ whenever B is an arbitrary ball of radius r and $B' = B(x', r')$, $x' \in B$, $r' \leqslant r$. The exponent ν depends only on the doubling constant C_d. If Ω is a bounded subset of $\mathbb{X}$ then $\mu(B(x,r)) \geqslant \frac{\mu(\Omega)}{(2\operatorname{diam}\Omega)^{\nu}} r^{\nu}$ for every $x \in \Omega$ and $r \leqslant \operatorname{diam}\Omega$ [12, Lemma 14.6].

We begin, though, with some preliminary lemmas. Our first lemma is a variant of a rather well-known lemma (e. g. see [11, 5]); we include a proof for completeness.

Lemma 6. *Suppose that $1 \leqslant p < \infty$, $h > 1$, Ω is a domain in $\mathbb{X}$. Let $\mathcal{F}$ be a family of balls contained in Ω, and let a_B be a non-negative number for each ball $B \in \mathcal{F}$. Then*

$$\Big\|\sum_{B\in\mathcal{F}} a_B\chi_{hB}\Big\|_{p,\Omega} \leqslant Ch^{\nu}p\Big\|\sum_{B\in\mathcal{F}} a_B\chi_{B}\Big\|_{p,\Omega}$$

where C is independent of h, p and Ω.

Proof. Let g be a non-negative function in $L_{p'}(\Omega)$, $p' = p/(p-1)$. Then

$$I = \int_{\Omega}\Big(\sum_{B\in\mathcal{F}} a_B\chi_{hB}\Big) g\,d\mu \leqslant h^{\nu}\sum_{B\in\mathcal{F}} a_B\Big[C_b\frac{1}{\mu(hB)}\int_{hB\cap\Omega} g\,d\mu\Big]\mu(B).$$

The bracketed quantity is dominated by maximal function $M(g\chi_\Omega)(y)$ for every $y \in B$. Now recall that $\|M(g\chi_\Omega)\|_{p',\Omega} \leqslant \|M(g\chi_\Omega)\|_{p',\mathbb{X}} \leqslant C_Mp\|g\chi_\Omega\|_{p',\mathbb{X}}$ (e. g. see [5]) where the constant C_M is independent of p. Therefore, using Hölder inequality and the boundedness of the maximal operator in $L_{p'}$ we obtain

$$I \leqslant C_bh^{\nu}\sum_{B\in\mathcal{F}} a_B\int_B M(g\chi_\Omega)\,d\mu = C_bh^{\nu}\int_{\Omega} M(g\chi_\Omega)\sum_{B\in\mathcal{F}} a_B\chi_B\,d\mu \\ \leqslant C_bh^{\nu}\|M(g\chi_\Omega)\|_{p',\Omega}\Big\|\sum_{B\in\mathcal{F}} a_B\chi_{hB}\Big\|_{p,\Omega} \leqslant C_bC_Mh^{\nu}p\|g\|_{p',\Omega}\Big\|\sum_{B\in\mathcal{F}} a_B\chi_{hB}\Big\|_{p,\Omega}.$$

Taking a supremum over all $g \geqslant 0$ in the unit ball of $L_{p'}(\Omega)$, the lemma follows by duality. $\square$

Since the definition of the John domains is given only in metric terms we do not repeat it for metric space setting. The next lemma is a well-known result stating that John domains satisfies the so-called Boman chain condition (see, for example, [4]). Here and in the sequel we denote $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$ for a point x of a domain Ω.

Lemma 7. *Let $\varkappa \geqslant 1$, Ω be a John domain $J(\alpha, \beta)$ with distinguished point $x_0 \in \Omega$, $B_0 = B\big(x_0, \frac{d_\Omega(x_0)}{\varkappa}\big)$. Then there is a countable family of balls $\mathcal{F}$ such that*

1) *$B_0 \in \mathcal{F}$, $\cup_{B\in\mathcal{F}} \frac{1}{2}B = \Omega$, $\varkappa B \subset \Omega$ for all $B \in \mathcal{F}$, $\{\frac{1}{10}B\}_{B\in\mathcal{F}}$ is a pairwise disjoint collection of balls;*

2) *for each ball $B \in \mathcal{F}$ there is a positive integer $m = m(B)$ and a chain $\{B_0, \dots, B_m = B\} \subset \mathcal{F}$ satisfying the following properties for all $i = 0, \dots, m-1$:*

(i) $\frac{2\varkappa-1}{2\varkappa+1} r(B_i) \leqslant r(B_{i+1}) \leqslant \frac{2\varkappa+1}{2\varkappa-1} r(B_i)$;

(ii) *$B_j \cap B_{j+1}$ contains a ball G_j of radius $\frac{1}{2}\min\{r(B_j), r(B_{j+1})\}$;*

(iii) *$B \subset h_1 G_j$ and $B \subset h_2 B_j$ with $h_1 = \frac{\beta}{\alpha}(2\varkappa + 21)$, $h_2 = \frac{\beta}{\alpha}(\varkappa + 10)$;*

3) *$\{B_0, \dots, B_j\}$ is a chain for the ball B_j fulfilling* (i)–(iii), *$j = 0, \dots, m$.*

Proof. 1) Consider a covering of Ω by balls $\{B(x, r) : x \in \Omega, r = d_\Omega(x)/\varkappa\}$. Choose a countable family $\mathcal{F}$ such that $\cup_{B\in\mathcal{F}}\frac{1}{2}B = \Omega$ and $\{\frac{1}{10}B\}_{B\in\mathcal{F}}$ is a disjoint family. Without loss of generality we may assume that the ball $B_0 = B\big(x_0, \frac{d_\Omega(x_0)}{\varkappa}\big) \in \mathcal{F}$.

2)–3) Fix a ball $B \in \mathcal{F}$. By definition of John domain there is a curve γ joining center of the ball B and x_0. Consider a chain of balls $\{B_0, \dots, B_{m=m(B)} = B\} \subset \mathcal{F}$ such that $\bigcup_i \frac{1}{2}B_i \supset \gamma$ and $\frac{1}{2}B_i \cap \frac{1}{2}B_{i+1} \neq \emptyset$, $i = 0, \dots, m-1$. Let the chain $\{B_0, \dots, B_m\}$ be minimal in the following sense: $\bigcup_{i\neq j} \frac{1}{2}B_i \not\supset \gamma$ for any j. Hence, for every $i = 0, \dots, m-1$, there is a number $s_i < l$ such that $\gamma(s_i) \in \frac{1}{2}B_i$.

Denote $B_i = B(x_i, r_i)$, $i = 0, \dots, m$. Consider $0 \leqslant i < k \leqslant m$. Obviously,

$$
\begin{aligned}
d(x_k, x_i) \leqslant d(x_k, \gamma(s_k)) + d(\gamma(s_k), \gamma(s_i)) + d(\gamma(s_i), x_i) &\leqslant \frac{r_k}{2} + \frac{\beta}{\alpha} d_\Omega(\gamma(s_i)) + \frac{r_i}{2} \\
\leqslant \frac{r_k}{2} + \frac{\beta}{\alpha}(d(\gamma(s_i), x_i) + d_\Omega(x_i)) + \frac{r_i}{2} &\leqslant \frac{r_k}{2} + \frac{\beta(\varkappa+1)}{\alpha} r_i
\end{aligned}
$$

and $\varkappa r_k = d_\Omega(x_k) \leqslant d(x_k, x_i) + \varkappa r_i$. From here $r_k \leqslant 6\frac{\beta}{\alpha} r_i$. Then, for every $y \in B_k$, we have $d(x_i, y) \leqslant d(x_i, x_k) + r_k \leqslant \frac{\beta}{\alpha}(10 + \varkappa) r_i$.

Since $\frac{1}{2}B_i \cap \frac{1}{2}B_{i+1} \neq \emptyset$ and $\varkappa r_i = d_\Omega(x_i)$ it follows that $\frac{2\varkappa-1}{2\varkappa+1} \leqslant \frac{r_i}{r_{i+1}} \leqslant \frac{2\varkappa+1}{2\varkappa-1}$ and there is a ball $G_i = B(y_i, \rho_i) \subset B_i \cap B_{i+1}$ with radius $\rho_i = \frac{1}{2}\min\{r_i, r_{i+1}\}$ and centre $y_i \in \frac{1}{2}B_i \cap \frac{1}{2}B_{i+1}$.

Suppose $\rho_i = \frac{r_j}{2}$, where j equals either i or $i+1$. Then $d(y_i, y) \leqslant d(y_i, x_j) + d(x_j, y) \leqslant \rho_i + r_j\frac{\beta}{\alpha}(\varkappa + 10) \leqslant \rho_i \frac{\beta}{\alpha}(2\varkappa + 21)$ for all $y \in B_k$. $\square$

Let $\mathcal{P}$ be a vector space of $\mathbb{R}^s$-valued functions on $\mathbb{X}$ with the following properties:

$$
\sup_{x\in sB} |P(x)| \leqslant Cs^l \sup_{x\in B} |P(x)|, \quad \sup_{x\in B} |P(x)| \leqslant \frac{C}{\mu(B)} \int_B |P(x)|\, d\mu(x) \tag{7}
$$

for every ball $B \in \mathbb{X}$, number $s \geqslant 1$, and every function $P \in \mathcal{P}$, where the number $l \geqslant 0$ and the constant C are independent of the function P and the ball B.

Example 11. *1. Obviously, the family of constant functions satisfies the conditions* (7)*.*

2. In the case of Carnot groups the set of polynomials of degree $< k$ *(*$\mathbb{X} = \mathbb{G}$ *and* $\mathcal{P} = \mathcal{P}_k$*) satisfies* (7) *with* $l = k - 1$ *(for example, see [19, Lemma 2.1]).*

The next theorem is the main result of this section.

Theorem 9. *Let* $(\mathbb{X}, d, \mu)$ *be a doubling space,* $\mathcal{P}$ *be a vector space of functions on* $\mathbb{X}$ *satisfying* (7)*,* Ω *be a John domain* $J(\alpha, \beta)$ *with distinguished point* x_0*, and* f *and* g *be measurable functions defined on* Ω*. Suppose* $\varkappa \geqslant 1$*,* $1 \leqslant p < \infty$*,* $p \leqslant q \leqslant \infty$*,* $\lambda \geqslant 0$*, and, for each ball* B *with* $\varkappa B \subset \Omega$*, there exists a function* $P(B) \in \mathcal{P}$ *such that*

$$\|f - P(B)\|_{q,B} \leqslant C r(B)^{\lambda} \|g\|_{p,\varkappa B}.$$

Then

$$\|f - P(B_0)\|_{q,\Omega} \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{\theta} (\operatorname{diam}\Omega)^{\lambda} \|g\|_{p,\Omega}$$

where $B_0 = B(x_0, \frac{d_\Omega(x_0)}{\varkappa})$ *and* $\theta = \begin{cases} l + \nu & \text{if } q \neq \infty, \\ l + \nu + \nu/p & \text{if } q = \infty. \end{cases}$

If $\mathbb{X}$ is a Carnot group and $\mathcal{P}$ is the space of polynomials of homogeneous degree $< l$ the theorem is formulated in [19, Lemma 4.2].

The proof of Theorem 9 is based on the following lemma.

Lemma 8. *Let* $\mathbb{X}, \mathcal{P}, \Omega, f, g, B_0, \theta, p, q, \lambda$ *be as in Theorem* 9*. Then*

$$\|P(B) - P(B_0)\|_{q,B} \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{\theta} (\operatorname{diam}\Omega)^{\lambda} \|g\|_{p,\Omega}$$

for any ball B *with* $r(B) = \frac{\operatorname{dist}(x(B), \partial\Omega)}{\varkappa}$*.*

Proof. Consider a ball B with $r(B) = \frac{\operatorname{dist}(x(B), \partial\Omega)}{\varkappa}$. Construct a chain of balls $\{B_0, \dots, B_{m(B)} = B\}$ satisfying conditions (i)–(iii) of Lemma 7.

Step 1: $\mathbf{q} = \infty$. We have

$$\begin{aligned}
\|P(B) - P(B_0)\|_{\infty,B} &\leqslant \sum_{i=0}^{m(B)-1} \|P(B_{i+1}) - P(B_i)\|_{\infty,B} \\
&\leqslant \sum_{i=0}^{m(B)-1} \sup_{x \in h_1 G_i} |P(B_{i+1})(x) - P(B_i)(x)| \\
&\leqslant C h_1^l \sum_{i=0}^{m(B)-1} \|P(B_{i+1}) - P(B_i)\|_{\infty,G_i} \\
&\leqslant C h_1^l \sum_{i=0}^{m(B)-1} (\|f - P(B_{i+1})\|_{\infty,B_{i+1}} + \|f - P(B_i)\|_{\infty,B_i})
\end{aligned}$$

$$\leqslant Ch_1^l \sum_{i=0}^{m(B)} r(B_i)^\lambda \|g\|_{p,\varkappa B_i} = Ch_1^l \sum_{i=0}^{m(B)} a_{B_i},$$

where $a_B = r(B)^\lambda \|g\|_{p,\varkappa B}$.

For every $x \in B$, we have $\sum_{i=0}^{m(B)} a_{B_i} \leqslant \sum_i a_{B_i}\chi_{h_2 B_i}(x)$. Moreover, for every $x \in B_j$, $j = 0, \dots, m(B)$, we have

$$\|f - P(B_0)\|_{\infty,B_j} \leqslant Ch_1^l \sum_{i=0}^{j} a_{B_i} \leqslant Ch_1^l \sum_{i=0}^{j} a_{B_i}\chi_{h_2 B_i}(x) \leqslant Ch_1^l \sum_{i=0}^{m(B)} a_{B_i}\chi_{h_2 B_i}(x).$$

Set $\Omega' = \bigcup\limits_{i=1}^{m(B)} B_i \supset B_0$. It is easy to see that $|\Omega'| \geqslant |B_0| \geqslant C\alpha^\nu$. Then, by Lemma 6,

$$\begin{aligned}
\|P(B) - P(B_0)\|_{\infty,B}^p &\leqslant \|P(B) - P(B_0)\|_{\infty,\Omega'}^p \\
&\leqslant Ch_1^{lp} \frac{1}{|\Omega'|} \int_{\Omega'} \Big(\sum_i a_{B_i}\chi_{h_2 B_i}(x)\Big)^p d\mu(x) \\
&\leqslant Ch_1^{lp} h_2^{\nu p} \frac{1}{|\Omega'|} \int_{\Omega'} \Big(\sum_i a_{B_i}\chi_{\frac{1}{10}B_i}(x)\Big)^p d\mu(x) \\
&\leqslant Ch_1^{lp} h_2^{\nu p} \frac{1}{|\Omega'|} \int_{\Omega'} \sum_i a_{B_i}^p \chi_{\frac{1}{10}B_i}(x) d\mu(x) \leqslant Ch_1^{lp} h_2^{\nu p} \frac{1}{|\Omega'|} \sum_i a_{B_i}^p \mu(B_i) \\
&\leqslant Ch_1^{lp} h_2^{\nu p} \frac{1}{\alpha^\nu} (\operatorname{diam}\Omega)^{\lambda p + \nu} \sum_i \int_{\varkappa B_i} |g(x)|^p \, dx \\
&\leqslant Ch_1^{lp} h_2^{\nu p} \frac{\beta^\nu}{\alpha^\nu} (\operatorname{diam}\Omega)^{\lambda p} \|g\|_{p,\Omega}^p.
\end{aligned}$$

Step 2: $\mathbf{q \neq \infty}$. We have

$$\begin{aligned}
\|P(B_0) - P(B)\|_{q,B} &\leqslant \sum_{i=0}^{m(B)-1} \|P(B_{i+1}) - P(B_i)\|_{q,B} \\
&\leqslant \mu(B)^{1/q} \sum_{i=0}^{m(B)-1} \sup_{x \in h_1 G_i} |P(B_{i+1})(x) - P(B_i)(x)| \\
&\leqslant \mu(B)^{1/q} \sum_{i=0}^{m(B)-1} Ch_1^l \frac{1}{\mu(G_i)^{1/q}} \|P(B_{i+1}) - P(B_i)\|_{q,G_i} \\
&\leqslant C\mu(B)^{1/q} h_1^l \sum_{i=0}^{m(B)-1} \bigg(\frac{\|f - P(B_{i+1})\|_{q,B_{i+1}}}{\mu(B_{i+1})^{1/q}} + \frac{\|f - P(B_i)\|_{q,B_i}}{\mu(B_i)^{1/q}} \bigg) \\
&\leqslant C\mu(B)^{1/q} h_1^l \sum_{i=0}^{m(B)} \frac{r(B_i)^\lambda}{\mu(B_i)^{1/q}} \|g\|_{p,\varkappa B_i} = C\mu(B)^{1/q} h_1^l \sum_{i=0}^{m(B)} a_B
\end{aligned}$$

where

$$a_B = \frac{r(B)^\lambda}{\mu(B)^{1/q}} \|g\|_{p,\varkappa B}.$$

It is evident, for every $x \in B$, we have

$$\sum_{i=0}^{m(B)} a_{B_i} \leqslant \sum_{\widetilde{B}\in\mathcal{F}} a_{\widetilde{B}} \chi_{h_2\widetilde{B}}(x).$$

Then, by Lemma 6,

$$\begin{aligned}
\|P(B) - P(B_0)\|^p_{q,B} &\leqslant Ch_1^{lp} \int_{\frac{1}{10}B} \Big(\sum_{\widetilde{B}\in\mathcal{F}} a_{\widetilde{B}} \chi_{h_2\widetilde{B}}(x)\Big)^p d\mu(x) \\
&\leqslant Ch_1^{lp} \int_{\Omega} \Big(\sum_{\widetilde{B}\in\mathcal{F}} a_{\widetilde{B}} \chi_{h_2\widetilde{B}}(x)\Big)^p d\mu(x) \leqslant Ch_1^{lp} h_2^{\nu p} \int_{\Omega} \sum_{\widetilde{B}\in\mathcal{F}} a^p_{\widetilde{B}} \chi_{\frac{1}{10}\widetilde{B}}(x) d\mu(x) \\
&\leqslant Ch_1^{lp} h_2^{\nu p} \sum_{\widetilde{B}\in\mathcal{F}} a^p_{\widetilde{B}} \mu(\widetilde{B}) \leqslant Ch_1^{lp} h_2^{\nu p} (\operatorname{diam}\Omega)^{\lambda p} \sum_{\widetilde{B}\in\mathcal{F}} \int_{\varkappa\widetilde{B}} |g(x)|^p \, d\mu(x) \\
&\leqslant Ch_1^{lp} h_2^{\nu p} (\operatorname{diam}\Omega)^{\lambda p} \int_{\Omega} \sum_{\widetilde{B}\in\mathcal{F}} |g(x)|^p \chi_{\varkappa B}(x) \, d\mu(x) \\
&\leqslant Ch_1^{lp} h_2^{\nu p} (\operatorname{diam}\Omega)^{\lambda p} \|g\|^p_{p,\Omega}. \qquad (8)
\end{aligned}$$

□

Proof of Theorem 9**.** Let $\mathcal{F}$ be a family of balls covering Ω from Lemma 7.

Step 1: $\mathbf{q} = \infty$. For every $\varepsilon > 0$, there is a ball $B \in \mathcal{F}$ such that

$$\|f - P(B_0)\|_{\infty,\Omega} \leqslant \|f - P(B_0)\|_{\infty,B} + \varepsilon.$$

In view of Lemma 8 we obtain the desired estimate:

$$\begin{aligned}
\|f - P(B_0)\|_{q,B} &\leqslant \|f - P(B)\|_{q,B} + \|P(B_0) - P(B)\|_{q,B} \\
&\leqslant Cr(B)^\lambda \|g\|_{p,B} + C\Big(\frac{\beta}{\alpha}\Big)^\theta (\operatorname{diam}\Omega)^\lambda \|g\|_{p,\Omega} \leqslant C\Big(\frac{\beta}{\alpha}\Big)^\theta (\operatorname{diam}\Omega)^\lambda \|g\|_{p,\Omega}.
\end{aligned}$$

Step 2: $\mathbf{q} < \infty$. Consider a ball $B \in \mathcal{F}$ and its chain $B_0, B_1, \dots, B_{m(B)} = B$. By Lemma 8 we have

$$\begin{aligned}
\|f - P(B_0)\|_{q,B} &\leqslant \|f - P(B)\|_{q,B} + \|P(B_0) - P(B)\|_{q,B} \\
&\leqslant Cr(B)^\lambda \|g\|_{p,B} + C\mu(B)^{1/q} h_1^l \sum_{i=0}^{m(B)} \frac{r(B_i)^\lambda}{\mu(B_i)^{1/q}} \|g\|_{p,\varkappa B_i} \leqslant C\mu(B)^{1/q} h_1^l \sum_{i=0}^{m(B)} a_B
\end{aligned}$$

where as before $a_B = \dfrac{r(B)^\lambda}{\mu(B)^{1/q}} \|g\|_{p,\varkappa B}$. Since

$$\sum_{i=0}^{m(B)} a_{B_i} \leqslant \sum_{\widetilde{B}\in\mathcal{F}} a_{\widetilde{B}} \chi_{h_2\widetilde{B}}(x) \quad \text{for every } x \in B,$$

we have by (8)

$$\|f-P(B_0)\|^p_{q,\Omega} \leqslant Ch_1^{lp}\sum_{B\in\mathcal{F}}\int_{\frac{1}{10}B}\Big(\sum_{\widetilde{B}\in\mathcal{F}} a_{\widetilde{B}}\chi_{h_2\widetilde{B}}(x)\Big)^p d\mu(x)$$
$$\leqslant Ch_1^{lp}\int_{\Omega}\Big(\sum_{\widetilde{B}\in\mathcal{F}} a_{\widetilde{B}}\chi_{h_2\widetilde{B}}(x)\Big)^p d\mu(x) \leqslant Ch_1^{lp}h_2^{\nu p}(\operatorname{diam}\Omega)^{\lambda p}\|g\|^p_{p,\Omega}.$$

□

Proof of Theorem 1. We have $\mathbb{X}=\mathbb{G}$, John domain Ω, operator Q of order k with constant coefficients and finite dimensional kernel, and function $u\in W^k_p(\Omega,\mathbb{R}^s)$. Let $\ker Q\subset\mathcal{P}_l$. Recall that Carnot–Carathéodory metric d_{cc} and quasimetric d_∞ are equivalent: $c_1 d_\infty(x,y)\leqslant d_{cc}(x,y)\leqslant c_2 d_\infty(x,y)$ for all $x,y\in\mathbb{G}$. By local coercive estimate (Theorem 8) for every ball $B=B_{cc}(a,r)$ with $B_{cc}(a,\varkappa r c_2/c_1)\subset\Omega$ there is a polynomial $P(B)\in\ker Q$ of order $<l$ such that

$$\|X^J(u-P(B))\|_{q,B_{cc}(a,r)} \leqslant \|X^J(u-P(B))\|_{q,\mathrm{Box}(a,r/c_1)}$$
$$\leqslant Cr^{k-d(J)-\nu/p+\nu/q}\|Qu\|_{p,\mathrm{Box}(a,\varkappa r/c_1)} \leqslant Cr^\lambda\|Qu\|_{p,B_{cc}(a,\varkappa r c_2/c_1)}.$$

(Here p,q,J satisfy conditions of Theorem 1 and $\lambda=k-d(J)-\nu/p+\nu/q$.)

Applying Theorem 9 with $\mathcal{P}=\mathcal{P}_{l-d(J)}$, $f=X^J u$, $g=Qu$ we obtain Theorem 1.

□

6 Corollaries

6.1 Poincaré inequality (proof of Theorem 4)

To prove Poincaré inequality on John domains (Theorem 4), it suffices to prove it on balls (Theorem 10 below) and apply Theorem 9.

Theorem 10. *Let $l>0$, $1\leqslant p\leqslant q\leqslant\infty$. Then there is a projection $P\colon W^l_p(\mathrm{Box}(e,1))\to\mathcal{P}_l$ such that*

$$\|X^J(u-Pu)\|_{q,\mathrm{Box}(e,1)}\leqslant C\|\nabla^l_{\mathcal{L}}u\|_{p,\mathrm{Box}(e,\varkappa)}$$

for any function u of the Sobolev class $W^l_p(\mathrm{Box}(e,\varkappa))$ and every multi-index J, $d(J)<l$, with

1) $p\leqslant q\leqslant\frac{\nu p}{\nu-(l-d(J))p}$ *for* $(l-d(J))p<\nu$;
2) $p\leqslant q<\infty$ *for* $(l-d(J))p=\nu$;
3) $p\leqslant q\leqslant\infty$ *for* $(l-d(J))p>\nu$;
4) $q=\infty$ *for* $l-d(J)\geqslant\nu$.

Constant C depends only on p, q, l and $d(J)$.

Proof. Theorem 10 is a version of Theorem 8 for the operator $Q=\nabla^l_{\mathcal{L}}$ except of the case $p=1$. Therefore it suffices to prove inequality (6) for $p=1$, $q=\frac{\nu}{\nu-\gamma}$ and $\gamma=l-d(J)<\nu$.

Set

$$S_k = \{x \in \operatorname{Box}(e, \varkappa) : 2^k < |X^J(u(x) - Pu(x))| \leqslant 2^{k+1}\}, \quad k \in \mathbb{Z},$$

$$u_k(x) = \begin{cases} 2^k & \text{if } |X^J(u(x) - Pu(x))| \leqslant 2^k, \\ |X^J(u(x) - Pu(x))| & \text{if } 2^k < |X^J(u(x) - Pu(x))| < 2^{k+1}, \\ 2^{k+1} & \text{if } |X^J(u(x) - Pu(x))| \geqslant 2^{k+1}, \end{cases}$$

and

$$2^M < \|\nabla^l_{\mathcal{L}} u\|_{1,\operatorname{Box}(e,\varkappa)} \leqslant 2^{M+1}.$$

Since $|\nabla^l_{\mathcal{L}} u_k| \leqslant |\nabla^l_{\mathcal{L}} u|\chi_{S_k}$ it follows

$$u_k(x) \leqslant C\mathcal{R}^\gamma(|\nabla^l_{\mathcal{L}} u|\chi_{S_k})(x) \quad \text{for } x \in \operatorname{Box}(e, 1).$$

We have

$$\begin{aligned} \int_{\operatorname{Box}(e,1)} &|X^J(u(x) - Pu(x))|^q \, dx \\ &\leqslant \int_{\operatorname{Box}(e,1)\cap\{|X^J(u-Pu)|\leqslant 2^M\}} |X^J(u(x) - Pu(x))|^q \, dx \\ &+ \int_{\operatorname{Box}(e,1)\cap\{|X^J(u-Pu)|> 2^M\}} |X^J(u(x) - Pu(x))|^q \, dx = I_1 + I_2. \end{aligned}$$

The first summand can be estimated as

$$I_1 \leqslant |\operatorname{Box}(e, 1)| 2^{Mq} \leqslant C\|\nabla^l_{\mathcal{L}} u\|^q_{1,\operatorname{Box}(e,\varkappa)}.$$

To estimate the second summand we recall

$$S_k \cap \operatorname{Box}(e, 1) \subset \{x \in \operatorname{Box}(e, 1) : \mathcal{R}^\gamma(|\nabla^l_{\mathcal{L}} u|\chi_{S_k}) > 2^k/C\}.$$

Applying Lemma 3 we obtain

$$\begin{aligned} I_2 = \sum_{k=M}^{\infty} \int_{S_k\cap\operatorname{Box}(e,1)} |X^J(u(x) - Pu(x))|^q \, dx &\leqslant \sum_{k=M}^{\infty} |S_k \cap \operatorname{Box}(e, 1)| 2^{(k+1)q} \\ &\leqslant C \sum_{k=M}^{\infty} \left(\int_{S_k} |\nabla^l_{\mathcal{L}} u(x)| \, dx \right)^q \leqslant C\|\nabla^l_{\mathcal{L}} u\|^q_{1,\operatorname{Box}(e,\varkappa)}. \end{aligned}$$

□

6.2 Embedding theorem (proof of Theorem 5)

For proving Theorem 5, investigate first projection $P_m \colon L_1(\operatorname{Box}(e, 1)) \to \mathcal{P}_m$ defined in Theorem 3.

Consider John domain $\Omega \in J(\alpha, \beta)$ with distinguished point x_0 and the Sobolev space $W^m_p(\Omega)$. The projection operator $P \colon W^m_p(\Omega) \to \mathcal{P}_m$ from Poincaré inequality

(Theorem 4) is just the translated projection P_m from $\mathrm{Box}(e,1)$ to $\mathrm{Box}(x_0,r)$ where $r = \frac{d_\Omega(x_0)}{\varkappa c_2} \geqslant \frac{\alpha}{\varkappa c_2}$.

Estimate $\|P_m g\|_{q,\Omega}$ for $g \in W_p^m(\Omega)$. For $z \in \mathrm{Box}(e,1)$, set $y = x_0\delta_r z \in \mathrm{Box}(x_0,r)$ and $u(z) = g(y) \in W_p^m(\mathrm{Box}(e,1))$. The polynomial $P_m u \in \mathcal{P}_m$ equals

$$\sum_{d(J)=0}^{m-1} z^J \int_{\mathrm{Box}(e,1)} \varphi_J(z_0)u(z_0)\,dz_0.$$

Making the change-of-variable formula ($z_0 = \delta_{1/r}(x_0^{-1}y_0)$) we obtain

$$P_m g(y) = \sum_{d(J)=0}^{m-1} (\delta_{1/r}(x_0^{-1}y))^J \int_{\mathrm{Box}(x_0,r)} \varphi_J(\delta_{1/r}(x_0^{-1}y_0))g(y_0)\,\frac{dy_0}{r^\nu}$$
$$= \sum_{d(J)=0}^{m-1} \frac{(x_0^{-1}y)^J}{r^{d(J)+\nu}} \int_{\mathrm{Box}(x_0,r)} \varphi_J(\delta_{1/r}(x_0^{-1}y_0))g(y_0)\,dy_0.$$

Thus,

$$\|P_m g\|_{q,\Omega} \leqslant C(\mathrm{diam}\,\Omega)^{\nu/q} \sum_{d(J)<m} \frac{(\mathrm{diam}\,\Omega)^{d(J)}}{r^{d(J)+\nu}} \|g\|_{1,\mathrm{Box}(x_0,r)}$$
$$\leqslant C\Big(\frac{\beta}{\alpha}\Big)^{m-1+\nu/p} (\mathrm{diam}\,\Omega)^{\nu/q-\nu/p}\|g\|_{p,\Omega} \tag{9}$$

and

$$\|X^{I_h}P_m g\|_{q,\Omega} \leqslant C(\mathrm{diam}\,\Omega)^{\nu/q} \sum_{d(J)=d(I_h)}^{m-1} \frac{(\mathrm{diam}\,\Omega)^{d(J)-d(I_h)}}{r^{d(J)+\nu}} \|g\|_{1,\mathrm{Box}(x_0,r)}$$
$$\leqslant C\Big(\frac{\beta}{\alpha}\Big)^{m-1+\nu/p} (\mathrm{diam}\,\Omega)^{-d(I_h)+\nu/q-\nu/p}\|g\|_{p,\Omega}. \tag{10}$$

Proof of Theorem 5. We have a John domain $\Omega \in J(\alpha,\beta)$, a function $f \in W_p^l(\Omega)$ and a nonnegative integer $k < l$.

Consider a multi-index I_h with $d(I_h) \in [0,\dots,k]$. Set $m = l-k$ and $g = X^{I_h}f \in W_p^m(\Omega)$. Poincaré inequality (Theorem 4) yields

$$\|g - P_m g\|_{q,\Omega} \leqslant C\Big(\frac{\beta}{\alpha}\Big)^\theta (\mathrm{diam}\,\Omega)^{m-\nu/p+\nu/q}\|\nabla_{\mathcal{L}}^m g\|_{p,\Omega} \tag{11}$$

where

$$\theta = \begin{cases} m-1+\nu = l-1-k+\nu & \text{if } q \neq \infty,\\ m-1+\nu+\nu/p = l-1-k+\nu+\nu/p & \text{if } q = \infty. \end{cases}$$

(1) $(l-k)p<\nu$**.** Consider q such that $p\leqslant q\leqslant \frac{\nu p}{\nu-(l-k)p}$. From (9) and (11) it follows

$$\begin{aligned}\|X^{I_h}f\|_{q,\Omega} =&\|g\|_{q,\Omega}\leqslant C\Big(\frac{\beta}{\alpha}\Big)^{m-1+\nu}(\operatorname{diam}\Omega)^{m-\nu/p+\nu/q}\|\nabla^m_{\mathcal L} g\|_{p,\Omega}\\ &+C\Big(\frac{\beta}{\alpha}\Big)^{m-1+\nu/p}(\operatorname{diam}\Omega)^{\nu/q-\nu/p}\|g\|_{p,\Omega}\\ \leqslant& C\Big(\frac{\beta}{\alpha}\Big)^{l-k-1+\nu}(\operatorname{diam}\Omega)^{\nu/q-\nu/p}\max\{(\operatorname{diam}\Omega)^{l-k},1\}\|f\|_{W^l_p(\Omega)}.\end{aligned}$$

To prove compactness we use the compactness of the Riesz-type potential $\mathcal{R}^\gamma\colon L_p\to L_q$ on balls for $q<\frac{\nu p}{\nu-\gamma p}$ defined in Subsection 4.1 (it can be derived, for example, from [41, Theorem 13]). Observe that the projection operator P_m is compact on any ball. Indeed, it maps bounded family of functions in L_1 to the bounded family of polynomials. Since the space $\mathcal{P}_m$ is finite-dimensional, there is converging sequence in the uniform norm.

Consider $(l-k)p<\nu$, $q<q^*=\frac{\nu p}{\nu-(l-k)p}$, and the countable family of balls $\{B_j\}$ from Lemma 7 covering Ω. Take any bounded sequence $\{f_i\}$ in $W^l_p(\Omega)$. Then it will be also bounded in $W^k_{q^*}$-norm: $\|f_i\|_{W^k_{q^*}(\Omega)}<C_0$. It suffices to show that there is a Cauchy subsequence in W^k_q-norm on Ω. Recall that the operator $\mathcal{R}^{l-k}$ and the projection operator P_m are compact on balls. Therefore, on any ball B_j we can obtain Cauchy subsequence $\{f_{i_s}\}$ in the W^k_q-norm. Applying Cantor's diagonal method we can extract a subsequence $\{f_{i_s}\}$ which is the Cauchy sequence in W^k_q-norm on each ball B_j, $j=1,2,\dots$. Denote this subsequence again by f_i.

Fix $\varepsilon>0$. Since $\sum_{j=1}^\infty|B_j|<\infty$, there is number $M_1>0$ such that $|\bigcup_{j=M_1}^\infty B_j|<\varepsilon^{\frac{q*}{q*-q}}$. Define $\Omega_\varepsilon=\bigcup_{j=M_1}^\infty B_j$.

There is a number $M_2>0$ such that $\|f_{i_1}-f_{i_2}\|_{W^k_q(B_j)}\leqslant\frac{\varepsilon}{2^j}$ for any $i_1,i_2>M_2$ and $j<M_1$. Then

$$\begin{aligned}\|f_{i_1}-f_{i_2}\|_{W^k_q(\Omega)}&\leqslant\sum_{j=1}^{M_1-1}\|f_{i_1}-f_{i_2}\|_{W^k_q(B_j)}+\|f_{i_1}-f_{i_2}\|_{W^k_q(\Omega_\varepsilon)}\\ &\leqslant\varepsilon+\|f_{i_1}-f_{i_2}\|_{W^k_{q*}(\Omega_\varepsilon)}|\Omega_\varepsilon|^{1-q/q*}<\varepsilon(1+2C_0)\quad\text{for any } i_1,i_2>M_2.\end{aligned}$$

Hence, the sequence $\{f_i\}$ is the Cauchy sequence in W^k_q-norm on Ω.

(4) $(l-k)p>\nu$ **and** $(l-k-1)p<\nu$**.** Consider $d(I_h)\in[0,\dots,k]$, $m=l-k$ and $g=X^{I_h}f$. Equations (9) and (11) yield

$$\begin{aligned}\|X^{I_h}f\|_{\infty,\Omega}=\|g\|_{\infty,\Omega}\leqslant& C\Big(\frac{\beta}{\alpha}\Big)^{m-1+\nu+\nu/p}(\operatorname{diam}\Omega)^{l-k-\nu/p}\|\nabla^m_{\mathcal L}g\|_{p,\Omega}\\ &+C\Big(\frac{\beta}{\alpha}\Big)^{m-1+\nu/p}(\operatorname{diam}\Omega)^{-\nu/p}\|g\|_{p,\Omega}\\ \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{l-k-1+\nu+\nu/p}&(\operatorname{diam}\Omega)^{-\nu/p}\max\{(\operatorname{diam}\Omega)^{l-k},1\}\|f\|_{W^l_p(\Omega)}.\end{aligned}$$

Continuity. Show that $X^{I_h}f$ is continuous on Ω for $d(I_h)=0,\dots,k$.

Consider a sequence of C^k-smooth functions f_j ($f_j \in C^k(\Omega)$) such that $\|f_j - f\|_{W_p^l(\Omega)} \to 0$ as $j \to \infty$. As before, we have

$$\|f_m - f_j\|_{C^k(\Omega)} = \sup_{d(I_h)=0,\dots,k} \|X^{I_h}(f_m - f_j)\|_{\infty,\Omega} \leqslant C(\Omega, k, l, p)\|f_m - f_j\|_{W_p^l(\Omega)}.$$

This means that $X^{I_h} f_j$ converges to $X^{I_h} f$ uniformly on Ω as $j \to \infty$ for all $d(I_h) = 0, \dots, k$. Hence, $X^{I_h} f$ is continuous.

Hölder continuity. Now we show that $g = X^{I_h} f$ is Hölder continuous ($g \in C^{0,\tau}(\Omega)$, $\tau = l - k - \nu/p$) for all multi-indices I_h with $d(I_h) = k$. By Poincaré inequality (Theorem 4) there is a polynomial $\Pi_B \in \mathcal{P}_m$ of homogeneous degree strictly less than $l - k$ such that (11) is fullfilled on the ball B:

$$\|g - \Pi_B\|_{\infty,B} \leqslant Cr^\tau \|\nabla_{\mathcal{L}}^m g\|_{p,B}. \tag{12}$$

Fix two points $x, y \in U$ such that $r = d_{cc}(x, y) \leqslant \max\{\operatorname{dist}(x, \partial\Omega), \operatorname{dist}(y, \partial\Omega)\}$. Without loss of generality we may assume that a ball $B = B(x, r) \subset \Omega$. Set $B' = B(x, \operatorname{dist}(x, \partial\Omega))$ and $B_0 = B(x_0, \operatorname{dist}(x_0, \partial\Omega))$ where x_0 is the distinguished point in Ω from the definition of the John domain.

We have

$$\begin{aligned}|g(x) - g(y)| \leqslant{} & |g(x) - g(y) - \Pi_B(x) + \Pi_B(y)| \\ & + |\Pi_B(x) - \Pi_B(y) - \Pi_{B'}(x) + \Pi_{B'}(y)| \\ & + |\Pi_{B'}(x) - \Pi_{B'}(y) - \Pi_{B_0}(x) + \Pi_{B_0}(y)| \\ & + |\Pi_{B_0}(x) - \Pi_{B_0}(y)| = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4.\end{aligned}$$

Estimate each summand separately.

Obviously, from (12) it follows

$$\mathcal{A}_1 = |g(x) - \Pi_B(x) - g(y) + \Pi_B(y)| \leqslant 2Cr^\tau \|\nabla_{\mathcal{L}}^m g\|_{p,B}.$$

If $l - k - 1 = 0$ then all the polynomials Π_B, $\Pi_{B'}$ and Π_{B_0} are constants and $\mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4 = 0$. Assume $l - k - 1 > 0$. Estimate $\mathcal{A}_2$. By Poincaré inequality, for $q = \frac{\nu p}{\nu - (l-k-1)p}$, we have

$$\begin{aligned}\mathcal{A}_2 \leqslant r\|\nabla_{\mathcal{L}}(\Pi_B - \Pi_{B'})\|_{\infty,B} &\leqslant \frac{Cr}{|B|^{1/q}}\|\nabla_{\mathcal{L}}(\Pi_B - \Pi_{B'})\|_{q,B} \\ &\leqslant Cr^{1-\nu/q}\big(\|\nabla_{\mathcal{L}}(g - \Pi_B)\|_{q,B} + \|\nabla_{\mathcal{L}}(g - \Pi_{B'})\|_{q,B'}\big) \\ &\leqslant Cr^{1-\nu/q}\big(\|\nabla_{\mathcal{L}}^m g\|_{p,B} + \|\nabla_{\mathcal{L}}^m g\|_{p,B'}\big) \leqslant Cr^\tau \|\nabla_{\mathcal{L}}^m g\|_{p,B'}.\end{aligned}$$

By Lemma 8 we obtain

$$\begin{aligned}\mathcal{A}_3 = |\Pi_{B_0}(x) - \Pi_{B_0}(y) - \Pi_{B'}(x) + \Pi_{B'}(y)| &\leqslant r\|\nabla_{\mathcal{L}}(\Pi_{B'} - \Pi_{B_0})\|_{\infty,B} \\ &\leqslant \frac{Cr}{|B|^{1/q}}\|\nabla_{\mathcal{L}}(\Pi_{B_0} - \Pi_{B'})\|_{q,B} \leqslant Cr^\tau \Big(\frac{\beta}{\alpha}\Big)^{l-k-2+\nu} \|\nabla_{\mathcal{L}}^m g\|_{p,\Omega}\end{aligned}$$

where $q = \frac{\nu p}{\nu-(l-k-1)p}$.

For evaluating $\mathcal{A}_4$, we apply inequality (10). It follows

$$\mathcal{A}_4 \leqslant r\|\nabla_{\mathcal{L}} P_{B_0}\|_{\infty,B} \leqslant Cr\Big(\frac{\beta}{\alpha}\Big)^{l-k-1+\nu/p} \operatorname{diam}(\Omega)^{-1-\nu/p}\|g\|_{p,\Omega}.$$

Assume for simplicity $\operatorname{diam}\Omega = 1$. Then $r < 1$ and we finally obtain

$$|g(x)-g(y)| \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{l-k-1+\nu}(r^\tau + r)\|fg\|_{W_p^m(\Omega)} \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{l-k-1+\nu} r^\tau\|g\|_{W_p^m(\Omega)}.$$

It rests to consider two arbitrary points $x, y \in \Omega$. Then for any sequence $\{x_i\}_{i=0}^j \subset \Omega$, $x_0 = x$, $x_j = y$ and $d_{cc}(x_i, x_{i-1}) \leqslant \max\{\operatorname{dist}(x_i, \partial\Omega), \operatorname{dist}(x_{i-1}, \partial\Omega)\}$ for all $i = 1, \dots, j$, we have

$$|g(x)-g(y)| \leqslant \sum_{i=1}^j |g(x_i) - g(x_{i-1})| \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{l-k-1+\nu}\|g\|_{W_p^m(\Omega)}\sum_{i=1}^j (d_{cc}(x_i, x_{i-1}))^\tau.$$

Passing to the infimum over all sequences $\{x_i\}$ we get

$$|g(x)-g(y)| \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{l-k-1+\nu} d_\tau^\Omega(x,y)\|g\|_{W_p^m(\Omega)}.$$

Let now $R = \operatorname{diam}(\Omega)$ be arbitrary. Define $\widetilde{\Omega} = \{y = x_0\delta_{1/R}(x_0^{-1}x),\ x \in \Omega\}$. Then $\operatorname{diam}(\widetilde{\Omega}) = 1$ and $\widetilde{\Omega} \in J(\alpha, \beta)$. Set $\widetilde{g}(y) = g(x)$. Then $\widetilde{g} \in W_p^m(\widetilde{\Omega})$ and

$$\begin{aligned}\|\widetilde{g}\|_{W_p^m(\widetilde{\Omega})} = \sum_{d(I_h)\leqslant m}\|X^{I_h}\widetilde{g}\|_{p,\widetilde{\Omega}} = R^{-\nu/p}\sum_{d(I_h)\leqslant m} R^{d(I_h)}\|X^{I_h}g\|_{p,\Omega} \\ \leqslant CR^{-\nu/p}\max\{R^m, 1\}\|g\|_{W_p^m(\Omega)}.\end{aligned}$$

Compactness of the embedding into $C^k(\Omega)$ is obvious. Indeed, take any sequence $\{f_i\}_{i=1}^\infty \subset W_p^l(\Omega)$ bounded in W_p^l-norm. Then $\{f_i\}$ is uniformly bounded in C^k-norm: $\|f_i\|_{C^k(\Omega)} \leqslant C_0$ for all i. Consider $X^{I_h}f_i = g_i$, $d(I_h) \leqslant k$. If $d(I_h) < k$ then

$$|g_i(x) - g_i(y)| \leqslant d_1^\Omega(x,y)\|\nabla_{\mathcal{L}} g_i\|_{\infty,\Omega} \leqslant C_0 d_1^\Omega(x,y) \quad \text{for all } x, y \in \Omega.$$

If $d(I_h) = k$ then by Hölder continuity

$$|g_i(x) - g_i(y)| \leqslant C d_\tau^\Omega(x,y) C_0 \quad \text{for all } x, y \in \Omega.$$

It means that the sequence $\{g_i\}_{i=1}^\infty$ is uniformly bounded and equicontinuous. Hence, there is a convergent subsequence.

Show the compactness of the embedding $W_p^l(\Omega) \hookrightarrow C^{k,t}(\Omega)$ for any $t < \tau = l - k - \nu/p$. Continuity of this embedding is obvious. Consider any bounded sequence $\{f_i\}$ in W_p^l-norm. By compactness of the embedding $W_p^l(\Omega) \hookrightarrow C^k(\Omega)$ shown above, there is a subsequence $\{f_{i_s}\}$ converging to some function f_0 in C^k-norm. Denote $g_s =$

$X^{I_h}(f_{i_s}-f_0)\in W_p^{l-k}(\Omega)\subset C(\Omega)$, $d(I_h)=k$. Fix $\varepsilon>0$. It suffices to show that there is a number $M=M(\varepsilon)>0$ such that

$$\frac{|g_s(x)-g_s(y)|}{d_t^{\Omega}(x,y)}<\varepsilon \quad \text{for any } s>M \text{ and } x,y\in\Omega.$$

Since the sequence $\{g_s\}$ is uniformly bounded in the W_p^{l-k}-norm on Ω, it follows $\{g_s\}\subset C^{k,\tau}(\Omega)$ and $|g_s(x)-g_s(y)|\leqslant C_0 d_\tau^{\Omega}(x,y)$ for all $s\in\mathbb{N}$ and all points $x,y\in\Omega$ where C_0 is independent of s and x,y.

We have $g_s\to 0$ as $s\to\infty$ uniformly on Ω. Therefore, there is a number $M>0$ such that $\sup\limits_{x\in\Omega}|g_s(x)|<\varepsilon^{\frac{\tau}{\tau-t}}$ for each $s>M$. Take any two points $x,y\in\Omega$. Consider sequence of points $x_0=x,\dots,x_m=y$ such that $d_{cc}(x_i,x_{i-1})\leqslant\max\{d_\Omega(x_i),d_\Omega(x_{i-1})\}$ for all $i=1,\dots,m$. Then

$$\begin{aligned}|g_s(x)-g_s(y)|&\leqslant\sum_{i=1}^m|g_s(x_i)-g_s(x_{i-1})|\\&\leqslant\sum_{i=1}^m\biggl(\frac{|g_s(x_i)-g_s(x_{i-1})|}{(d_{cc}(x_i,x_{i-1}))^\tau}\biggr)^{t/\tau}(d_{cc}(x_i,x_{i-1}))^t|g_s(x_i)-g_s(x_{i-1})|^{1-t/\tau}\\&\leqslant\sum_{i=1}^m C_0^{t/\tau}(d_{cc}(x_i,x_{i-1}))^t 2\varepsilon\end{aligned}$$

for any $s>M$. Passing to the infimum over all sequences $\{x_i\}$ we obtain the necessary estimate.

(5) $(l-k)p>\nu$ **and** $(l-k-1)p=\nu$**.** Continuity of $X^{I_h}f$ for $d(I_h)\in[0,k]$ can be shown as in the item (4). Verify that $g=X^{I_h}f\in C^{0,1}_{\mathrm{loc}}(\Omega)$, $d(I_h)=k$. Take $B=B(x,r)\subset\Omega$ and h with $d_{cc}(h,e)=r$. Then

$$\begin{aligned}&|g(xh)+g(xh^{-1})-2g(x)|\\&\quad\leqslant|g(xh)+g(xh^{-1})-2g(x)-\Pi_B(xh)-\Pi_B(xh^{-1})+2\Pi_B(x)|\\&\quad+|\Pi_B(xh)+\Pi_B(xh^{-1})-2\Pi_B(x)-\Pi_{B'}(xh)-\Pi_{B'}(xh^{-1})+2\Pi_{B'}(x)|\\&\quad+|\Pi_{B'}(xh)+\Pi_{B'}(xh^{-1})-2\Pi_{B'}(x)-\Pi_{B_0}(xh)-\Pi_{B_0}(xh^{-1})+2\Pi_{B_0}(x)|\\&\quad+|\Pi_{B_0}(xh)+\Pi_{B_0}(xh^{-1})-2\Pi_{B_0}(x)|=\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3+\mathcal{C}_4.\end{aligned}$$

Estimate each summand separately. From (12) with $\tau=1$ it follows

$$\mathcal{C}_1\leqslant 4Cr\|\nabla_{\mathcal{L}}^m g\|_{p,B}.$$

If $l-k-2=0$ then all the polynomials Π_B, $\Pi_{B'}$ and Π_{B_0} are linear and $\mathcal{C}_2=\mathcal{C}_3=\mathcal{C}_4=0$. Assume $l-k-2>0$. To estimate $\mathcal{C}_2$–$\mathcal{C}_4$ we use Lemma 9. By Lemma 9 and Poincaré inequality (12) we have

$$\begin{aligned}\mathcal{C}_2&\leqslant Cr^2\|\nabla_{\mathcal{L}}^2(\Pi_B-\Pi_{B'})\|_{\infty,B}\leqslant Cr^{2-\nu/q}\|\nabla_{\mathcal{L}}^2(\Pi_B-\Pi_{B'})\|_{q,B}\\&\leqslant Cr^{2-\nu/q}(\|\nabla_{\mathcal{L}}^2(g-\Pi_B\|_{q,B}+\|\nabla_{\mathcal{L}}^2(g-\Pi_{B'})\|_{q,B'})\leqslant Cr\|\nabla_{\mathcal{L}}^m g\|_{p,B'}.\end{aligned}$$

Here we set $B' = B(x, \operatorname{dist}(x, \partial\Omega))$, $B_0 = B(x_0, \operatorname{dist}(x_0, \partial\Omega))$, $q = \frac{\nu p}{\nu-(l-k-2)p}$.

Applying Lemma 8 we get

$$\mathcal{C}_3 \leqslant Cr^2 \|\nabla^2_{\mathcal{L}}(\Pi_{B'} - \Pi_{B_0})\|_{\infty,B} \leqslant Cr^{2-\nu/q} \|\nabla^2_{\mathcal{L}}(\Pi_{B'} - \Pi_{B_0})\|_{q,B} \leqslant Cr\Big(\frac{\beta}{\alpha}\Big)^{l-k-3+\nu} \|\nabla^m_{\mathcal{L}} g\|_{p,\Omega}.$$

In view of (10) we obtain

$$\mathcal{C}_4 \leqslant Cr^2 \|\nabla^2_{\mathcal{L}} \Pi_{B_0}\|_{\infty,B} \leqslant Cr^2 \Big(\frac{\beta}{\alpha}\Big)^{l-k-1+\nu/p} \operatorname{diam}(\Omega)^{-2-\nu/p} \|g\|_{p,\Omega}.$$

Finally, assuming $\operatorname{diam}\Omega = 1$ we have

$$|g(xh) + g(xh^{-1}) - 2g(x)| \leqslant Cr\Big(\frac{\beta}{\alpha}\Big)^{l-k-1+\nu} \|g\|_{W^m_p(\Omega)}.$$

Since $W^l_p(\Omega)$ is continuously embedded in $C^{k,\tau}(\Omega)$ for all $\tau \in (0,1)$, the compactness follows from the item (4).

(2) $(l-k)p = \nu$ **and** $p > 1$**.** If $k > 0$ then $p = \frac{\nu}{l-k} > \frac{\nu}{l-k+1}$ and $X^{I_h} f$ is continuous for all $d(I_h) < k$ (see item (4)). Moreover,

$$\sup_{x\in\Omega,\, d(I_h)<k} |X^{I_h} f(x)| \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{l-k-1+\nu+\nu/p} (\operatorname{diam}\Omega)^{-\nu/p} \max\{(\operatorname{diam}\Omega)^{l-k}, 1\} \|f\|_{W^l_p(\Omega)}.$$

Consider the multi-index I_h of the weight k. As before, $m = l - k$ and $g = X^{I_h} f$. Rewrite equation (11) for $g \in W^m_p(\Omega) \subset W^m_{p^*}(\Omega)$ with $p^* < p$ and $q = \frac{\nu p^*}{\nu - mp^*}$:

$$\begin{aligned}\|g - P_m g\|_{q,\Omega} &\leqslant Cq^{1-\frac{m}{\nu}} \Big(\frac{\beta}{\alpha}\Big)^{m-1+\nu} \|\nabla^m_{\mathcal{L}} g\|_{p^*,\Omega} \\ &\leqslant Cq^{\frac{p-1}{p}} \Big(\frac{\beta}{\alpha}\Big)^{m-1+\nu} \|\nabla^m_{\mathcal{L}} g\|_{p,\Omega}. \qquad (13)\end{aligned}$$

Here we used Lemma 3, proof of Theorem 9, and assumed $\operatorname{diam}(\Omega) = 1$.

Equations (9) and (13) imply

$$\begin{aligned}\|X^{I_h} f\|_{q,\Omega} = \|g\|_{q,\Omega} &\leqslant C\Big(\frac{\beta}{\alpha}\Big)^{m-1+\nu} q^{\frac{p-1}{p}} \|\nabla^m_{\mathcal{L}} g\|_{p,\Omega} + C\Big(\frac{\beta}{\alpha}\Big)^{m-1+\nu/p} \|g\|_{p,\Omega} \\ &\leqslant C_0 \Big(\frac{\beta}{\alpha}\Big)^{l-k-1+\nu} q^{\frac{p-1}{p}} \|f\|_{W^l_p(\Omega)}.\end{aligned}$$

For a constant $\rho = 2^{\frac{p-1}{p}} \mathrm{e} C_0 \big(\frac{\beta}{\alpha}\big)^{l-k-1+\nu} \|f\|_{W^l_p(\Omega)}$, we have

$$\begin{aligned}\int_\Omega \Phi\Big(\frac{|g(x)|}{\rho}\Big)\, dx &= \sum_{j=1}^\infty \int_\Omega \frac{1}{j!}\Big(\frac{|g(x)|}{\rho}\Big)^{j\frac{p}{p-1}} \\ &\leqslant \sum_{j=1}^\infty \frac{1}{j!}\Big(j\frac{p}{p-1}\Big)^j \frac{1}{2^j \mathrm{e}^{j\frac{p}{p-1}}} \leqslant \sum_{j=1}^\infty \frac{1}{2^j}\Big(\mathrm{e}^{1-\frac{p}{p-1}} \frac{p}{p-1}\Big)^j < 1. \qquad (14)\end{aligned}$$

Thus, for $\Phi(t) = t^\eta - 1$ with $\eta = \frac{p}{p-1}$, there is continuous embedding $W_p^l(\Omega) \hookrightarrow C^{k,\Phi}(\Omega)$ with the norm satisfying necessary condition.

Take any $\eta \in [1, \frac{p}{p-1})$. Then by Hölder inequality

$$\int_\Omega \frac{1}{j!}\Big(\frac{|g(x)|}{\rho}\Big)^{\eta j} dx \leqslant \Big(\int_\Omega \frac{1}{j!}\Big(\frac{|g(x)|}{\rho}\Big)^{j\frac{p}{p-1}} dx\Big)^{\eta\frac{p-1}{p}} \Big(\frac{|\Omega|}{j!}\Big)^{1-\eta\frac{p-1}{p}}.$$

Therefore, for any $\eta < \frac{p}{p-1}$ there will be also continuous embedding.

Now we show the compactness of the embedding for $\eta < \frac{p}{p-1}$. Fix a bounded sequence $\{f_i\}_{i=1}^\infty$ in $W_p^l(\Omega)$. In view of item (5) we have compact embedding $W_p^l(\Omega) \hookrightarrow C^{k-1}(\Omega)$. Hence, there is subsequence $\{f_{i_s}\}$ converging in the C^{k-1}-norm. Denote this subsequence again by $\{f_i\}$.

Take any $\varepsilon > 0$. The sequence $\{f_i\}$ is bounded in $C^{k,\Phi}(\Omega)$ for $\Phi(t) = \exp(t^{\frac{p}{p-1}}) - 1$. Let ρ be the supremum of the norms of $\{f_i\}$: $\rho = \sup_i \|f_i\|_{C^{k,\Phi}(\Omega)}$. Denote $X^{I_h} f_i = g_i$ with $d(I_h) = k$. Then

$$\|g_i\|_{j\frac{p}{p-1},\Omega} \leqslant \rho(j!)^{\frac{p-1}{pj}} \quad \text{for all } i$$

and

$$\|g_i\|_{j\eta,\Omega} \leqslant \rho(j!)^{\frac{p-1}{pj}} |\Omega|^{\frac{p-\eta(p-1)}{\eta j p}}.$$

It is easy to verify that the series

$$\sum_{j=1}^\infty \int_\Omega \frac{|g_{i_1}(x) - g_{i_2}(x)|^{\eta j}}{\varepsilon^{\eta j} j!} dx \leqslant |\Omega|^{1-\frac{\eta(p-1)}{p}} \sum_{j=1}^\infty \Big(\frac{2\rho}{\varepsilon}\Big)^{\eta j} \frac{1}{j!}$$

converges for arbitrary i_1 and i_2. Thus, there is a number $M_1 > 0$ such that

$$\sum_{j=M_1}^\infty \int_\Omega \frac{|g_{i_1}(x) - g_{i_2}(x)|^{\eta j}}{\varepsilon^{\eta j} j!} dx \leqslant |\Omega|^{1-\frac{\eta(p-1)}{p}} \sum_{j=M_1}^\infty \Big(\frac{2\rho}{\varepsilon}\Big)^{\eta j} \frac{1}{j!} < 2 - \mathrm{e}^{\frac{1}{2}}$$

for all i_1 and i_2.

For any $q_j = j\eta$, $\eta < \frac{p}{p-1}$, there is $p_j < p$ such that $q_j < \frac{\nu p_j}{\nu-(l-k)p_j}$. Hence, the embedding $W_{p_j}^l(\Omega) \hookrightarrow W_{q_j}^k(\Omega)$ is compact by item (1). In view of Cantor's diagonal method there is a subsequence $\{f_{i_s}\}$ which is the Cauchy sequence in $W_{q_j}^k$-norm for all j. Denote it again by $\{f_i\}$. Then, there is an integer $M_2 > 0$ such that

$$\|g_{i_1} - g_{i_2}\|_{q_j,\Omega} \leqslant \frac{\varepsilon}{2^{1/\eta}}$$

for all $i_1, i_2 > M_2$ and for $j < M_1$. It follows

$$\sum_{j=1}^\infty \int_\Omega \frac{|g_{i_1}(x) - g_{i_2}(x)|^{\eta j}}{\varepsilon^{\eta j} j!} dx \leqslant \sum_{j=1}^{M_1-1} \frac{1}{j! 2^j} + 2 - \mathrm{e}^{\frac{1}{2}} < \mathrm{e}^{\frac{1}{2}} - 1 + 2 - \mathrm{e}^{\frac{1}{2}} = 1.$$

That is, the sequence $\{f_j\}$ is a Cauchy sequence in $C^{k,\Phi}$-norm for $\eta < \frac{p}{p-1}$.

(3) $p = 1$ **and** $\nu = l - k$**.** The proof of the embedding into $C^k(\Omega)$ is obvious, it follows the proof of the item (4).

Obviously, $C^k(\Omega) \hookrightarrow C^{k,\Phi}(\Omega)$ for all $\eta \in (0,\infty)$. Show the compactness of this embedding. Fix $\eta \in (0,\infty)$. Take any $\eta_0 > \eta$. Then $W_p^l(\Omega)$ is continuously embedded in $C^{k,\eta_0}(\Omega)$. As in the item (2) we can derive from here that $W_p^l(\Omega)$ is compactly embedded in $C^{k,\eta}(\Omega)$. □

Lemma 9. *Let $P \in \mathcal{P}_{m+1}$ be a polynomial on $\mathbb{G}$. Then there is a constant $C = C(m) \geqslant 0$ such that*

$$|P(xh) + P(xh^{-1}) - 2P(x)| \leqslant Cr^2 \|\nabla_{\mathcal{L}}^2 P\|_{\infty,B(x,r)}$$

for any two points $x, h \in \mathbb{G}$, $r = d_{cc}(h,e)$.

Proof. Write down the polynomial $P \in \mathcal{P}_{m+1}$:

$$P(y) = \sum_{d(I)=0}^{m} a_I (x^{-1}y)^I.$$

Let $\mathcal{N}(P)$ be the following norm of the polynomial P:

$$\mathcal{N}(P) = \sup_{d(I)\leqslant m} |a_I| r^{d(I)}.$$

One can verify that the norms $\|\cdot\|_{\infty,B(x,r)}$ and $\mathcal{N}$ are equivalent.

We have

$$|P(xh) + P(xh^{-1}) - 2P(x)| = 2\Bigg| \sum_{\substack{d(I)=2 \\ d(I) \text{ is even}}}^{m} a_I h^I \Bigg| \leqslant Cr^2 \sup_{2\leqslant d(I)\leqslant m} |a_I| r^{d(I)-2}.$$

Recall $\nabla_{\mathcal{L}}^2 P = \{X^{I_h}P\}_{d(I_h)=2}$ and $X^{I_h}P(y) = \sum_{d(I)=2}^{m} a_I X_y^{I_h}(x^{-1}y)^I$ where $X_y^{I_h}(x^{-1}y)^I$ is either zero or polynomial of degree $d(I)-2$. Therefore

$$\sup_{2\leqslant d(I)\leqslant m} |a_I| r^{d(I)-2} \leqslant C \sup_{d(I_h)=2} \mathcal{N}(X^{I_h}P) \leqslant C\|\nabla_{\mathcal{L}}^2 P\|_{\infty,B(x,r)}.$$

□

6.3 Proof of Theorem 6

It's evident that, $W_p^l(\Omega) \subset \widetilde{W}_p^l(\Omega)$ and this embedded operator is bounded. In view of Lemma 1, $C^\infty(\Omega)$-functions are dense in $\widetilde{W}_p^l(\Omega)$ and the integral representation formulas in Theorems 2 and 3 hold for $\widetilde{W}_p^l(\Omega)$. It follows from here Theorem 4 for $f \in \widetilde{W}_p^l(\Omega)$. Hence, Theorem 4 implies that an arbitrary function $f \in \widetilde{W}_p^l(\Omega)$ belongs also to $W_p^l(\Omega)$. It means, that $W_p^l(\Omega) = \widetilde{W}_p^l(\Omega)$. By well-known Banach theorem the inverse embedded operator is also bounded.

6.3 Extension operator (proof of Theorem 7)

Proof. Using the well-known technique of Jones [15] we can construct an extension operator

$$\mathrm{ext}'\colon W_p^k(\Omega) \to W_p^k(\mathbb{G}).$$

The norm of ext' depends only on $\varepsilon, \delta, k, p$ and the radius of the domain Ω (see, e. g. [19, 43]).

Let $\mathrm{ext}\, u = \widetilde{u} = \Pi u + \mathrm{ext}'(u - \Pi u)$. Then

$$\|Q\widetilde{u}\|_{p,\mathbb{G}} = \|Q(\mathrm{ext}'(u - \Pi u))\|_{p,\mathbb{G}} \leqslant C\|\mathrm{ext}'(u - \Pi u)\|_{W_p^k(\mathbb{G})} \leqslant C\|u - \Pi u\|_{W_p^k(\Omega)}.$$

By Theorem 1, there is a projector P on kernel of Q such that $\|u - Pu\|_{W_p^k(\Omega)} \leqslant C\|Qu\|_{p,\Omega}$. From here

$$\|u - \Pi u\|_{W_p^k(\Omega)} \leqslant \|u - Pu\|_{W_p^k(\Omega)} + \|\Pi(u - Pu)\|_{W_p^k(\Omega)} \leqslant C\|Qu\|_{p,\Omega}.$$

Since $\Pi\widetilde{u} = \Pi u$ the theorem follows. □

Acknowledgements. The research was partially supported by the Russian Foundation for Basic Research (Grant 08–01–00531), the Federal Program "Research and educational resources of innovative Russia in 2009–2013" (contract No. P2224), and the State Maintenance Program for Young Russian Scientists and the Leading Scientific Schools of the Russian Federation (Grant NSh-6613.2010.1).

References

[1] N. Aronszajn, *On coercive integro-differential quadratic forms*, Conference on Partial Differential Equations. Univ. of Kansas, Report, 14 (1954), 94 – 106.

[2] O.V. Besov, *Coerciveness in a nonisotropic S. L. Sobolev space.* Mat. Sb. (N.S.), 73, no. 115 (1967), 585 – 599 (in Russian).

[3] O.V. Besov, V.P. Il'in, S.M. Nikol'skii, *Integral representations of functions and imbedding theorems. Volumes. I & II.* Scripta Series in Mathematics. V.H. Winston & Sons, Washington, D.C.; John Wiley & Sons, New York etc., Vol. I, 1978; Vol. II, 1979.

[4] S. Buckley, P. Koskela, G. Lu, *Boman equals John.* In: Proc. XVI Rolf Nevanlinna colloq., 1996, 91 – 99.

[5] S.M. Buckley, *Strong doubling condition.* Math. Ineq. Appl., 1, no. 4 (1998), 533 – 542.

[6] B. Franchi, C.E. Gutiérrez, R.L. Wheeden, *Weighted Sobolev-Poincaré inequalities for Grushin type operators.* Commun. Partial Differ. Equations, 19, no. 3–4 (1994), 523 – 604.

[7] L. Capogna, N. Garofalo, *Non tangentially accessible domains for Carnot-Carateódory metrics and a Fatou type theorem.* C. R. Acad. Sci., Paris, Sér. I, 321, no.12 (1995), 1565 – 1570.

[8] G.B. Folland, *Lipschitz classes and Poisson integrals on stratified groups.* Stud. Math., 66 (1979), 37 – 55.

[9] G.B. Folland, E.M. Stein, *Hardy spaces on homogeneous groups.* Mathematical Notes, 28. Princeton, New Jersey: Princeton University Press; University of Tokyo Press, 1982.

[10] B. Franchi, G. Lu, R.L. Wheeden, *Representation formulas and weighted Poincaré inequalities for Hörmander vector fields.* Ann. Inst. Fourier, Grenoble, 45, no. 2 (1992), 577 – 604.

[11] N. Garofalo, D.-M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces.* Commun. Pure Appl. Math., 49, no. 10 (1996), 1081 – 1144.

[12] P. Hajłasz, P. Koskela, *Sobolev met Poincaré.* Memoirs of the AMS, 688 (2000).

[13] D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition.* Duke Math. J., 53 (1986), 503 – 523.

[14] F. John, *Rotation and strain.* Comm. Pure Appl. Math. 14 (1961), 391–413.

[15] P.W. Jones, *Quasiconformal mappings and extendability of functions in Sobolev spaces.* Acta Math. 147 (1981), 71–88.

[16] A.W. Knapp, E.M. Stein, *Intertwining operators for semisimple groups.* Ann. Math., 93, no. 3 (1971), 489 – 578.

[17] O.A. Ladyzhenskaya, T.N. Shilkin, *On coercive estimates for solutions of linear systems of hydrodynamic type.* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 288 (2002), Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts., 32, 104 – 133 (in Russian). English transl. in J. Math. Sci. (N.Y.), 126, no. 6 (2004), 4580 – 4596.

[18] G. Lu, *Local and global interpolation inequalities on the Folland-Stein Sobolev spaces and polynomials on stratified groups.* Math. Res. Lett., 4, no.6 (1997), 777 – 790.

[19] G. Lu, *Polynomials, higher order Sobolev extension theorems and interpolation inequalities on weighted Folland-Stein spaces on stratified groups.* Acta Math. Sin., Engl. Ser., 16, no. 3 (2000), 405 – 444.

[20] V.G. Maz'ya, *Sobolev spaces.* Springer-Verlag, Berlin etc., 1985.

[21] P.P. Mosolov, V.P. Mjasnikov, *Correctness of boundary value problems in the mechanics of continuous media.* Mat. Sb. (N.S.), 88 (1972), 256 – 267.

[22] A. Nagel, E.M. Stein, S. Wainger, *Balls and metrics defined by vector fields. I: Basic properties.* Acta Math., 155 (1985), 103 – 147.

[23] A.I. Parfenov, *Straightenness criteria of Lipschitzian surface by Liezorkin – Triebel. I.* Mat. trudy, 12, no. 1 (2009), 144 – 204 (in Russian).

[24] S.I. Pohozhaev, *On Sobolev embedding theorem in the case $lp = n$.* Proceedings of the Scientific and Technical Conference, Moscow Power Engineering Institute, 1965, 158–170 (in Russian).

[25] E.A. Plotnikova, *Integral representations and the generalized Poincaré inequality on Carnot groups.* Sib. Math. Zh., 49, no. 2 (2008), 421 – 437 (in Russian). English transl. in Sib. Mat. J., 49, no. 2 (2008), 339 – 352.

[26] Yu.G. Reshetnyak, *Some integral representations of differentiable functions.* Sib. Mat. Zh., 12 (1971), 420 – 432 (in Russian).

[27] Yu.G. Reshetnyak, *Integral representations of differentiable functions in domains with nonsmooth boundary.* Sib. Math. J., 21, no. 6 (1980), 833 – 839.

[28] Yu.G. Reshetnyak, *Stability theorems in geometry and analisys.* Mathematics and its Applications (Dordrecht), 304. Kluwer Academic Publishers, Dordrecht, 1994.

[29] N.N. Romanovskii, *Integral representations and embedding theorems for functions defined on the Heisenberg groups $\mathbb{H}^n$.* Algebra Anal., 16, no. 2 (2004), 82 – 119 (in Russian). English transl. in St. Petersbg. Math. J., 16, no. 2 (2005), 349 – 375.

[30] N.N. Romanovskii, *Mikhlin's problem on Carnot groups.* Sib. Mat. Zh., 49, no. 1 (2008), 193 – 206 (in Russian). English transl. in Sib. Math. J. 49, no. 1 (2008), 155 – 165.

[31] V.S. Rychkov, *On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains.* J. Lond. Math. Soc., II. Ser., 60, no.1 (1999), 237 – 257.

[32] V.S. Rychkov, *Littlewood-Paley theory and function spaces with A_p^{loc} weights.* Math. Nachr., 224 (2001), 145 – 180.

[33] S.L. Sobolev, *Sur quelques evaluations concernant les familles de fonctions ayant des derivees a carre integrable.* Doklady Akad. Nauk, 1(10):7(84) (1936), 267 – 270 (in Russian). French transl. in C. R. (Dokl.) Acad. Sci. URSS, New Ser., 1 (1936), 279,–,282.

[34] S.L. Sobolev, *On one boundary problem for polyharmonic equations.* Rec. Math. Moscou, New Ser., 2 (1937), 465 – 499 (in Russian).

[35] S.L. Sobolev, *Sur un theoreme de l'analyse fonctionnelle.* Doklady Akad. Nauk, 20:1 (1938), 5 – 10 (in Russian). French transl. in C. R. (Dokl.) Acad. Sci. URSS, New Ser., 20 (1938), 5 – 9.

[36] S.L. Sobolev, *On one theorem of functional analisys.* Rec. Math. Moscou, New Ser. ,4 (1938), 471 – 497 (in Russian).

[37] S.L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics. 3rd ed.* Translations of Mathematical Translations of Mathematical Monographs, 90. AMS, Providence, RI, 1991.

[38] E.M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Princeton University Press, Princeton, 1993.

[39] B.V. Trushin, *Embedding of Sobolev space in Orlicz space for a domain with irregular boundary.* Mat. Zametki, 79, no. 5 (2006), 767 – 778 (in Russian). English transl. in Math. Notes, 79, no. 5 (2006), 707 – 718.

[40] S.K. Vodopyanov, *Equivalent normalizations of Sobolev and Nikol'skii spaces in domains. Boundary values and extension.* In: Function Spaces and Applications. (Proceedings of the US – Swedish Seminar, Lund, Sweden, June 1986). Springer – Verlag, Berlin a.o., 1988, 397–409. (Lecture notes in Mathematics, 1302.)

[41] S.K. Vodopyanov, *Weighted L_p-potential theory on homogeneous groups.* Sib. Mat. Zh., 33, no. 2 (1992), 29–48 (in Russian). English transl. in Sib. Math. J., 33, no. 2 (1992), 201 – 218.

[42] S.K. Vodop'yanov, A.V. Greshnov, *On extension of functions of bounded mean oscillation from domains in a space of homogeneous type with intrinsic metric.* Sib. Mat. Zh., 36, no. 5 (1995), 1015 – 1048 (in Russian). English transl. in Sib. Math. J. 36, no. 5 (1995), 873 – 901.

[43] S.K. Vodop'yanov, I.M.Pupyshev, *Whitney–type theorems on extension of functions on Carnot groups.* Sib. Mat. Zh., 47, no. 4 (2006), 731 – 752 (in Russian). English transl. in Sib. Math. J., 47, no. 4 (2006), 601 – 620.

Daria Isangulova, Sergey Vodopyanov
Sobolev Institute of Mathematics
Siberian Branch of the Russian Academy of Sciences
av. Akademika Koptyuga, 4
630090 Novosibirsk, Russia

Mechanics and Mathematics Department
Novosibirsk State University
2 Pirogova str.
630090 Novosibirsk, Russia
E-mail: dasha@math.nsc.ru, vodopis@math.nsc.ru

Received: 15.06.2010