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AN INVERSE COEFFICIENT PROBLEM FOR NONLINEAR BIHARMONIC EQUATION WITH SURFACE MEASURED DATA[1]

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Abstract. An inverse coefficient problem for the nonlinear biharmonic equation $Au := (g(\xi^2(u))(u_{x_1x_1} + (1/2)u_{x_2x_2})_{x_1x_1} + (g(\xi^2(u))u_{x_1x_2})_{x_1x_2} + (g(\xi^2(u))(u_{x_2x_2} + (1/2)u_{x_1x_1}))_{x_2x_2} = F(x)$, in $\Omega \subset R^2$, is considered. This problem arises in computational material science as a problem of identification of unknown properties of inelastic isotropic homogeneous incompressible bending plate using surface measured data. Within J_2-deformation theory of plasticity these properties are described by the coefficient $g(\xi^2(u))$ which depends on the effective value of the plate curvature: $\xi^2(u) = (u_{x_1x_1})^2 + (u_{x_2x_2})^2 + (u_{x_1x_2})^2 + u_{x_1x_1}u_{x_2x_2}$. The surface measured output data is assumed to be the deflections w_i, $i = \overline{1, M}$, at some points of the surface of a plate and obtained during the quasistatic process of bending. For a given coefficient $g(\xi^2(u))$ mathematical modeling of the bending problem leads to a nonlinear boundary value problem for the biharmonic equation with Dirichlet or mixed types of boundary conditions. Existence of the weak solution in the Sobolev space $H^2(\Omega)$ is proved by using the theory of monotone potential operators. A monotone iteration scheme for the linearized equation is proposed. Convergence in H^2-norm of the sequence of solutions of the linearized problem to the solution of the nonlinear problem is proved, and the rate of convergence is estimated. The obtained continuity property of the solution $u \in H^2(\Omega)$ of the direct problem, and compactness of the set of admissible coefficients $\mathcal{G}$, permit one to prove the existence of a quasi-solution of the considered inverse problem.

1 Introduction

Determing of unknown properties of materials based on boundary/surface measured data is one of central and actual problems of computational material sciences (see, [6 - 7, 9], [13] and references therein). Mathematical modeling of these problems leads

[1]The results have partially been announced at the Satellite Conference of International Congress of Mathematicians, 14-17 August, 2010, Delhi, India

to inverse coefficient problems for nonlinear PDEs of various types ([3], [5], [8 - 10]). It is known from inverse problems theory that inverse coefficient problems are most difficult in comparison with all other types of inverse problems [12]. Moreover, these problems are severely ill-posed problems even when the governing equations are linear ones ([9], [13]) which means that very close measured output data may correspond to quite different materials (i.e. coefficients).

In this paper we study the following *inverse problem of identifying the unknown coefficient* $g(\xi^2(u))$ in the nonlinear bending equation

$$\begin{cases} Au \equiv \frac{\partial^2}{\partial x_1^2}\left[g(\xi^2(u))\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{1}{2}\frac{\partial^2 u}{\partial x_2^2}\right)\right]+\frac{\partial^2}{\partial x_1 \partial x_2}\left[g(\xi^2(u))\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)\right] \\ +\frac{\partial^2}{\partial x_2^2}\left[g(\xi^2(u))\left(\frac{\partial^2 u}{\partial x_2^2}+\frac{1}{2}\frac{\partial^2 u}{\partial x_1^2}\right)\right]=F(x)\ , \quad x\in\Omega\subset R^2, \\ u(x)=0, \ \ \frac{\partial u}{\partial n}(x)=0, \quad x\in\partial\Omega, \end{cases} \tag{1}$$

from the deflections

$$w_i[\tau_k] := w(\lambda_i;\tau_k), \quad i=\overline{1,M}, \quad k=\overline{1,K}, \tag{2}$$

measured at some points $\lambda_i=(x_1^{(i)},x_2^{(i)})$ of a plate and corresponding to the given values of the external normal load $q(x)$ (Figure 1(a)). Here $F(x)=3q(x;\mathcal{T})/h^3$, $q(x;\mathcal{T})$ is the intensity (per unit area) of the load normal to the middle surface of a plate with the thickness $h>0$, and n is the unit outward normal to the boundary $\partial\Omega$ of a plate. The coordinate plane Ox_1x_2 is assumed to be the middle surface of the plate occupying the square $\Omega=\{(x_1,x_2)\in R^2: -l<x_1,\ x_2<l,\ l>0\}$.

According to J_2-deformation theory of plasticity [8], the deflection $u=u(x)$ of a point $x\in\Omega$ on the middle surface of a plate, in equilibrium under the action of normal loads, satisfies the nonlinear biharmonic equation (1). Besides of the above rigid clamped boundary conditions, other (mixed) types of boundary conditions can also be considered depending on experiment. The process of bending is assumed to be quasistatic, generating by the increasing values $0<\tau_1<\tau_2<...<\tau_K$ of the loading parameter $\mathcal{T}$.

The coefficient $g(\xi^2(u))$ of equation (1) depends on the effective value of the plate curvature

$$\xi^2(u)=\left(\frac{\partial^2 u}{\partial x_1^2}\right)^2+\left(\frac{\partial^2 u}{\partial x_2^2}\right)^2+\left(\frac{\partial^2 u}{\partial x_1\partial x_2}\right)^2+\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2}, \tag{3}$$

which in its turn, depends on the deflection function $u(x)$, the solution of problem (1). This coefficient describes the elastoplasticitic properties of an increasingly hardening plate, and is usually said to be the plasticity function. Within J_2-deformation theory, this function is assumed to be piecewise differentiable and to satisfy the following conditions [7]:

(C1) $c_0\le g(\xi^2)\le c_1,\ \ \xi\in(0,\xi^*)$;
(C2) $g(\xi^2)+2g'(\xi^2)\xi^2\ge c_2$;
(C3) $g'(\xi^2)\le 0,\ \ \forall\xi\in[0,\xi_M]$;

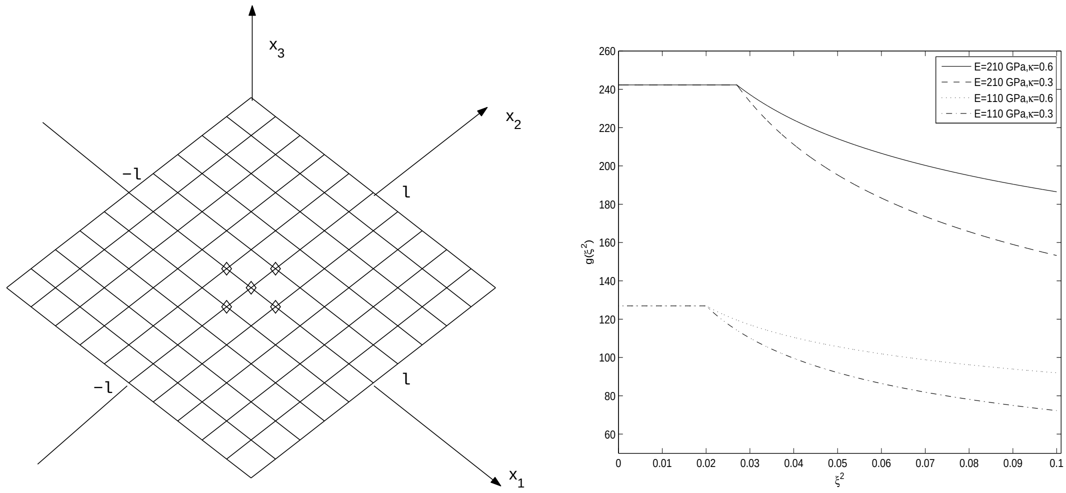

Figure 1: (a) Geometry of the bending plate (left figure); (b) plasticity functions corresponding to rigid and soft materials (right figure)

(C4) $\exists \xi_0 \in (0, \xi_M), \quad g(\xi^2) = G, \quad \forall \xi \in [0, \xi_0],$
where c_i are positive constants. For pure elastic bending when $\xi \in [0, \xi_0]$, this function represents the modulus of rigidity: $g(\xi^2) = G$, $G = E/(2(1+\nu))$. Here $E > 0$ is the Young modulus and $\nu > 0$ is the Poisson ratio.

The plasticity functions $g(\xi^2)$ corresponding to some rigid and soft engineering materials are given in Figure 1(b).

In the pure elastic bending case the nonlinear equation (1) becomes the well-known biharmonic equation

$$\frac{\partial^4 u}{\partial x_1^4} + 2\frac{\partial^4 u}{\partial x_1^1 \partial x_2^2} + \frac{\partial^4 u}{\partial x_2^4} = F(x;\lambda)/G \ , \quad x \in \Omega \subset R^2.$$

This article presents mathematical analysis of the inverse coefficient problem (1) – (2), and also of the nonlinear boundary value problem (1), defined to be the *direct problem*, according to inverse problems terminology. For the direct problem a variational approach is proposed by using the monotone potential operator theory [4, 15]. For the inverse problem the quasisolution method [11] is applied to obtain the existence of a quasisolution.

Note that nonlinear boundary value problems with monotone potential operators arise in various areas of mechanics and physics (see, [3 - 4], [12], [15]).

The paper is organized as follows. In Section 2 solvability of the weak solution of the direct problem (1) is proved in the Sobolev space $H^2(\Omega)$ by using the Browder-Minty theorem. In Section 3 the linearization scheme is proposed for the nonlinear boundary value problem (1). Then monotonicity of the sequence of potentials of the linearized problem and convergence of the sequence of solutions of the linearized problem to the solution of the nonlinear problem (1) in H^2-norm is proved. In the final Section 4 the existence of a quasisolution of the inverse coefficient problem (1) – (2) is obtained in the natural class of coefficients given by the above physical model.

2 Monotonicity of the nonlinear biharmonic operator A and solvability of problem (1)

Let $H^2(\Omega)$ be the Sobolev space of functions [1] defined on the domain Ω with piecewise smooth boundary $\partial\Omega$ and

$$\overset{0}{H}{}^2(\Omega) = \left\{ v \in H^2(\Omega) : \; u(x) = \frac{\partial u(x)}{\partial n} = 0, \; x \in \partial\Omega \right\}.$$

Multiplying the both sides of equation (1) by $v \in \overset{0}{H}{}^2(\Omega)$, integrating on Ω and using the Dirichlet boundary conditions (1) we obtain the following integral identity

$$\int_\Omega g(\xi^2(u))H(u,v)dx = \int_\Omega F(x)v(x)dx, \quad \forall v \in \overset{0}{H}{}^2(\Omega), \tag{4}$$

where

$$H(u,v) = \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}\right) \tag{5}$$

is a bilinear differential form defined on $H^2(\Omega) \times H^2(\Omega)$. It follows from (3) that $H(v,v) = \xi^2(v)$.

Definition 1. *A function $u \in \overset{0}{H}{}^2(\Omega)$ satifying the integral identity (4) for all $v \in \overset{0}{H}{}^2(\Omega)$ is said to be a weak solution of the nonlinear problem (1) – (2).*

Recalling equation (1) and definition (5) we may introduce the nonlinear functional (form)

$$\langle Au, v\rangle = \int_\Omega g(\xi^2(u))H(u,v)dx, \quad \forall v \in \overset{0}{H}{}^2(\Omega). \tag{6}$$

corresponding to the nonlinear biharmonic operator A, and rewrite the *variational problem* (4) as follows

$$\langle Au, v\rangle = l(v), \quad \forall v \in \overset{0}{H}{}^2(\Omega), \tag{7}$$

where the linear functional

$$l(v) = \int_\Omega F(x)v(x)dx$$

is defined by the source function $F \in H^2(\Omega)$.

Let us introduce now the following nonlinear functional

$$J(u) = \frac{1}{2}\int_\Omega \left\{ \int_0^{\xi^2(u)} g(\tau)d\tau \right\} dx, \quad u \in \overset{0}{H}{}^2(\Omega). \tag{8}$$

Lemma 1. *The nonlinear operator A, defined by (1), is a potential operator with the potential $J(u)$ defined by (8).*

Proof. Calculating the first Gateaux derivative

$$\langle J'(u), v\rangle := \frac{d}{dt}J(u+tv)|_{t=0}, \quad u, v \in \overset{0}{H}{}^2(\Omega)$$

of the functional $J(u)$ at $u \in \overset{0}{H}{}^2(\Omega)$ and on $v \in \overset{0}{H}{}^2(\Omega)$, we obtain

$$\begin{aligned}\langle J'(u), v\rangle &= \frac{1}{2}\int_\Omega \left\{ \frac{d}{dt}\left[\int_0^{\xi^2(u+tv)} g(\tau)d\tau\right]_{t=0}\right\} dx \\ &= \frac{1}{2}\int_\Omega \left\{ g(\xi^2(u+tv))\frac{d}{dt}\xi^2(u+tv)\right\}_{t=0} dx.\end{aligned}$$

On the other hand, due to definition (3) of the function $\xi^2(u)$, we have

$$\frac{d}{dt}\left[\xi^2(u+tv)\right]_{t=0} = \\ 2\left[\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}\right)\right].$$

This and (5) imply

$$\frac{d}{dt}\left[\xi^2(u+tv)\right]_{t=0} = 2H(u,v).$$

Thus

$$\langle J'(u), v\rangle = \int_\Omega g(\xi^2(u))H(u,v)dx.$$

By definition (6) we obtain the statement. □

Following this lemma let us introduce the nonlinear functional

$$\Pi(u) := J(u) - l(u) = \frac{1}{2}\int_\Omega \left\{\int_0^{\xi^2(u)} g(\tau)d\tau - 2F(x)u(x)\right\} dx, \quad v \in \overset{0}{H}{}^2(\Omega), \tag{9}$$

defined to be the *potential of the nonlinear problem (1) – (2)*, and consider the following minimization problem.

Find such a function $u \in \overset{0}{H}{}^2(\Omega)$ that

$$\Pi(u) = \min \Pi(v), \quad v \in \overset{0}{H}{}^2(\Omega). \tag{10}$$

Using Lemma 1 and the standard results of nonlinear analysis we can prove the following

Lemma 2. *Any solution $u \in C^4(\Omega) \cup C^2(\overline{\Omega})$ of the nonlinear problem (1) – (2) is also solution of the variational problem (7), which is equivalent to the minimization problem (10). Further, if $u \in \overset{0}{H}{}^2(\Omega)$ is a solution of the variational problem (7) and, in addition, belongs to $C^4(\Omega) \cup C^2(\overline{\Omega})$, then this function is also a solution of the nonlinear problem (1) – (2).*

To derive a solvability result for the variational problem (7), let us analyse monotonicity of the nonlinear biharmonic operator A. For this aim we define the energy norm $\|v\|_E$ and the seminorm $|v|_2$ in $H^2(\Omega)$, respectively

$$\|v\|_E := \left\{ \int_\Omega H(v,v)dx \right\}^{1/2}, \quad |v|_2 := \left\{ \int_\Omega \left[\left(\frac{\partial^2 v}{\partial x_1^2} \right)^2 + \left(\frac{\partial^2 v}{\partial x_2^2} \right)^2 + \left(\frac{\partial^2 v}{\partial x_1 \partial x_2} \right)^2 \right] dx \right\}^{1/2}.$$

The following lemma shows that these norms are equivalent.

Lemma 3. *If $v \in \overset{0}{H}{}^2(\Omega)$, then $|v|_2^2 \le \|v\|_E^2 \le 2|v|_2^2$.*

Proof. It is known that for all $v \in \overset{0}{H}{}^2(\Omega)$ the following formula holds (see, [14])

$$\int_\Omega \left[\left(\frac{\partial^2 v}{\partial x_1 \partial x_2} \right)^2 - \frac{\partial^2 v}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} \right] dx = 0.$$

Then taking into account (5) we get

$$\begin{aligned} \|v\|_E^2 = \int_\Omega H(v,v)dx &= \int_\Omega \left[\left(\frac{\partial^2 v}{\partial x_1^2} \right)^2 + \left(\frac{\partial^2 v}{\partial x_2^2} \right)^2 + \left(\frac{\partial^2 v}{\partial x_1 \partial x_2} \right)^2 + \frac{\partial^2 v}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} \right] dx \\ &= \int_\Omega \left[\left(\frac{\partial^2 v}{\partial x_1^2} \right)^2 + \left(\frac{\partial^2 v}{\partial x_2^2} \right)^2 + 2\left(\frac{\partial^2 v}{\partial x_1 \partial x_2} \right)^2 \right] dx \\ &\ge \int_\Omega \left[\left(\frac{\partial^2 v}{\partial x_1^2} \right)^2 + \left(\frac{\partial^2 v}{\partial x_2^2} \right)^2 + \left(\frac{\partial^2 v}{\partial x_1 \partial x_2} \right)^2 \right] dx \\ &= |v|_2^2. \end{aligned}$$

On the other hand,

$$\int_\Omega H(v,v)dx \le 2 \int_\Omega \left[\left(\frac{\partial^2 v}{\partial x_1^2} \right)^2 + \left(\frac{\partial^2 v}{\partial x_2^2} \right)^2 + \left(\frac{\partial^2 v}{\partial x_1 \partial x_2} \right)^2 \right] dx = |v|_2^2.$$

These two estimates imply the proof. □

By using the equivalence of the norm $\|\cdot\|_2$ and the seminorm $|\cdot|_2$ in $\overset{0}{H}{}^2(\Omega)$ [2], we obtain the following

Corollary 1. *If $v \in \overset{0}{H}{}^2(\Omega)$, then*

$$\exists \alpha_1, \alpha_2 > 0, \quad \alpha_1 \|v\|_2 \le \|v\|_E \le \alpha_2 \|v\|_2, \tag{11}$$

i.e. the H^2-norm and the energy norm are equivalent.

The lemma permits one to obtain also the following upper estimate.

Corollary 2. *If* $u, v \in \overset{0}{H}{}^2(\Omega)$, *then*

$$\int_\Omega |H(u,v)|dx \le \alpha_2^2 \|u\|_2 \|v\|_2. \tag{12}$$

Proof. We use the Schwartz inequality for the bilinear form $H(u,v)$

$$(H(u,v))^2 \le H(u,u) \cdot H(v,v).$$

This together with (11) implies

$$\begin{aligned} \int_\Omega |H(u,v)|dx &\le \int_\Omega (H(u,u))^{1/2} (H(v,v))^{1/2} \\ &\le \left(\int_\Omega |H(u,u)|dx \right)^{1/2} \left(\int_\Omega |H(v,v)|dx \right)^{1/2} \\ &\le \alpha_2^2 \|u\|_2 \|v\|_2, \end{aligned}$$

and we obtain the statement. □

By using these auxiliary results we can prove strong convexity of the potential $J(u)$ of the nonlinear operator A in $\overset{0}{H}{}^2(\Omega)$.

Lemma 4. *Let the coefficient* $g = g(\xi^2)$ *of equation (1) satisfy conditions* **(C1)**–**(C4)**. *Then the potential operator* A *is strongly convex in* $\overset{0}{H}{}^2(\Omega)$, *i.e.*

$$\forall u, v \in \overset{0}{H}{}^2(\Omega), \quad \langle J''(u), v, v \rangle \ge \gamma_1 \|v\|_2^2, \quad \gamma_1 > 0. \tag{13}$$

Proof. Calculating the second Gateaux derivative of the functional $J(u)$, defined by (8), we have

$$\begin{aligned} \langle J''(u), v, w \rangle &:= \frac{d}{dt} \langle J'(u+tw), v \rangle|_{t=0} = \frac{d}{dt} \left\{ \int_\Omega g(\xi^2(u+tw)) H(u+tw, v) dx \right\}_{t=0} \\ &= \left\{ \int_\Omega \left[2g'(\xi^2(u+tw)) H(u,w) H(u+tw, v) + g(\xi^2(u+tw)) H(w,v) \right] dx \right\}_{t=0} \\ &= \int_\Omega \left[2g'(\xi^2(u)) H(u,w) H(u,v) + g(\xi^2(u)) H(w,v) \right] dx. \end{aligned}$$

Substituting here $w = v$ and using $H(v,v) = \xi^2(v)$ we find

$$\langle J''(u), v, v \rangle := \int_\Omega \left[g(\xi^2(u)) \xi^2(v) + 2g'(\xi^2(u)) H^2(u,v) \right] dx, \quad u, v \in \overset{0}{H}{}^2(\Omega).$$

Condition **(C3)** and the inequality $H(u,v) \le H(u,u) H(v,v)$ together with the relation $H(v,v) = \xi^2(v)$ between the differential form $H(v,v)$ and the effective curvature $\xi^2(v)$ imply that $g'(\xi^2(u)) H^2(u,v) \ge g'(\xi^2(u)) \xi^2(u) \xi^2(v)$. Hence

$$\begin{aligned} \langle J''(u), v, v \rangle &\ge \int_\Omega \left[g(\xi^2(u)) \xi^2(v) + 2g'(\xi^2(u)) \xi^2(u) \xi^2(v) \right] dx \\ &= \int_\Omega [g(\xi^2(u)) + 2g'(\xi^2(u)) \xi^2(u)] \xi^2(v) dx. \end{aligned}$$

By using condition **(C2)** on the right hand side and applying Corollary 1 finally we get

$$\langle J''(u), v, v\rangle \geq c_2 \int_\Omega \xi^2(v)dx = c_2 \int_\Omega H(v,v)dx \geq \gamma_1 \|v\|_2^2, \quad \gamma_1 = \alpha_1 c_2 \alpha_1 c_2 > 0.$$

This implies the statement. □

Corollary 3. *Let the coefficient* $g = g(\xi^2)$ *of equation (1) satisfy conditions* **(C1)**–**(C4)**. *Then the potential operator* A *is strongly monotone in* $\overset{0}{H}{}^2(\Omega)$, *i.e.*

$$\forall u, v \in \overset{0}{H}{}^2(\Omega), \quad \langle Au - Av, u - v\rangle \geq \gamma_1 \|u - v\|_2^2, \quad \gamma_1 > 0. \tag{14}$$

Corollary 4. *Since* $A\Theta = \Theta$, *where* $\Theta \in \overset{0}{H}{}^2(\Omega)$ *is zero element, monotonicity (14) of the nonlinear operator* A *also means its coercivity*

$$\langle Av, v\rangle \geq \gamma_1 \|v\|_2^2, \quad \gamma_1 > 0.$$

Lemma 5. *The nonlinear operator* A *is radially continuous (hemicontinuous), i.e. the real valued function* $t \to \langle A(u + tv), v\rangle$, *for fixed* $u, v \in \overset{0}{H}{}^2(\Omega)$, *is continuous.*

Proof. Since both mappings $t \mapsto g(\xi^2(u+tw)$, $t \mapsto H((u+tw), v)$ are continuous, the proof immediately follows from (6). □

Thus, the potential operator A is radially continuous, strongly monotone and coercive. Then by the Browder-Minty theorem [4] we obtain the following

Theorem 1. *Let conditions* **(C1)**–**(C4)** *hold. Then the nonlinear problem (1) has a unique solution in* $\overset{0}{H}{}^2(\Omega)$, *defined by the integral identity (4).*

3 Linearization of the nonlinear problem (1), the monotone iteration scheme and convergence of the approximate solutions

To study the inverse coefficient problem, as well as to apply any numerical method for solution of the direct problem, one needs to perform linearization of the nonlinear problem (1), and then prove the convergence of the sequence of approximate solutions in appropriate Sobolev norm. For this aim we will use the so-called convexity argument for nonlinear monotone potential operators, introduced in [7]. For clarity we will explain here the convexity argument in its abstract form.

Let $A : H \mapsto H^*$ be a strongly monotone potential operator defined on an abstract Hilbert space H, and $a(u; \cdot, \cdot)$ be the corresponding bounded, symmetric continuous and coercive tri-linear form (functional): $a(u; u, v) := \langle Au, v\rangle$, $u, v \in H$, and

$$\begin{cases} \langle Au - Av, u - v\rangle \geq \gamma_1 \|u - v\|_H, \quad \gamma_1 > 0; \\ |a(u; u, v)| \leq \gamma_2 \|u\|_H \|v\|_H, \quad \gamma_2 > 0, \quad \forall u, v \in H. \end{cases} \tag{15}$$

Here $\|\cdot\|_H$ is the norm of the space H.

Assume that the functional $J(u)$, $u \in H$ is the potential of the operator A. Then we have $\langle J'(u), v\rangle = a(u; u, v)$.

Definition 2. *A monotone potential operator $A : H \to H^*$ defined on an abstract Hilbert space H is said to satisfy the convexity argument, if the following inequality holds*

$$\frac{1}{2}a(u; v, v) - \frac{1}{2}a(u; u, u) - J(v) + J(u) \geq 0, \quad \forall u, v \in H. \tag{16}$$

Consider the following variational problem

$$a(u; u, v) = l(v), \quad v \in H, \tag{17}$$

which defines the weak solution $u \in H$ of the abstract operator equation $Au = F$, where $l(v)$ is the linear functional defined by the element $F \in H^*$. Denote by $\Pi(u)$ the potential of the operator equation $Au = F$.

Let us linearize the nonlinear variational problem (17) as follows

$$a(u^{(n-1)}; u^{(n)}, v) = l(v), \quad \forall v \in H, \quad n = 1, 2, 3, \ldots, \tag{18}$$

where $u^{(0)} \in H$ is an initial iteration. The function $u^{(n)} \in H$ is said to be an approximate solution of the abstract variational problem (17). Evidently, at each nth iteration the variational problem (17) is a linear one, since $u^{(n-1)}$ is known from the previous iteration. The iteration scheme (18) is said to be *the abstract iteration scheme* for the monotone potential operator A.

Theorem 2 ([7])**.** *Let $A : H \mapsto H^*$ be a strongly monotone potential operator defined on an abstract Hilbert space H, and $a(u; \cdot, \cdot)$ be the corresponding bounded, symmetric continuous and coercive tri-linear form. If the convexity argument (16) holds, then*
(a1) *the sequence of potentials $\{\Pi(u^{(n)})\} \subset R$, corresponding to the sequence of solutions $\{u^{(n)}\} \subset H$, $n = 1, 2, 3, ..$, of the linearized problem (18), is a monotonically decreasing one;*
(a2) *the sequence of approximate solutions $\{u^{(n)}\} \subset H$ defined by the abstract iteration scheme (18) converges to the solution $u \in H$ of the nonlinear problem (17) in the norm of the space H;*
(a3) *for the rate of convergence the following estimate holds*

$$\|u - u^{(n)}\| \leq \frac{\sqrt{2}\gamma_2}{\gamma_1^{3/2}} \left[\Pi(u^{(n-1)}) - \Pi(u^{(n)})\right]^{1/2}, \tag{19}$$

where $\gamma_1, \gamma_2 > 0$ are the constants defined in (15).

Let us linearize now the nonlinear variational problem (6), according to the linearization scheme (18). The solution $u^{(n)} \in \overset{0}{H}{}^2(\Omega)$ of the linear variational problem

$$\int_\Omega g(\xi^2(u^{(n-1)}))H(u^{(n)}, v)dx = \int_\Omega F(x)v(x)dx, \quad n = 1, 2, 3, ..., \tag{20}$$

is said to be an *appoximate solution of the nonlinear variational problem (6).* Here $u^{(0)} \in \overset{0}{H}{}^2(\Omega)$ is an initial iteration. In contrast to the potential $J(u)$, defined by (8), the potential of the linearized operator, defined to be $J_0(u^{(n)})$, is a quadratic functional, since the right hand side of (20) is a bilinear functional. The potentials of the linearized operator and the linearized problem (20) are defined as follows

$$\begin{cases} J_0(u^{(n)}) = \frac{1}{2}\int_\Omega g(\xi^2(u^{(n-1)}))H(u^{(n)},u^{(n)})dx; \\ \Pi_0(u^{(n)}) = J_0(u^{(n)}) - l(u^{(n)}), \quad u^{(n)} \in \overset{0}{H}{}^2(\Omega). \end{cases} \tag{21}$$

To apply the above theorem we need, first of all, to the analyze fulfilment of the convexity argument (16) for the nonlinear biharmonic operator (1).

Lemma 6. *If the coefficient $g = g(\xi^2)$ satisfies condition* **(C3)**, *then the convexity argument holds for the nonlinear biharmonic operator A, defined by (1).*

Proof. Using definitions (8) and (9) we calculate the left hand side of inequality (16)

$$\begin{aligned} &\frac{1}{2}a(u;v,v) - \frac{1}{2}a(u;u,u) - J(v) + J(u) \\ &= \frac{1}{2}\int_\Omega g(\xi^2(u))H(v,v)dx - \frac{1}{2}\int_\Omega g(\xi^2(u))H(u,u)dx \\ &-\frac{1}{2}\int_\Omega \left\{\int_0^{\xi^2(v)} g(\tau)d\tau\right\}dx + \frac{1}{2}\int_\Omega\left\{\int_0^{\xi^2(u)} g(\tau)d\tau\right\}dx \\ &= \frac{1}{2}\int_\Omega\left\{g(\xi^2(u))[\xi^2(v)-\xi^2(u)] - \int_0^{\xi^2(v)} g(\tau)d\tau + \int_0^{\xi^2(u)} g(\tau)d\tau\right\}dx. \end{aligned}$$

Now introduce the function

$$Q(t) = \int_0^t g(\tau)d\tau.$$

Due to condition **(C2)** we conclude $Q''(t) = g'(t) \le 0$, and hence $Q = Q(t)$ is a concave function

$$Q'(t_1)(t_2 - t_1) - Q(t_2) + Q(t_1) \ge 0, \quad \forall t_2 > t_1 > 0.$$

Using this inequality in the right-hand side of the above integral expression, with $\xi^2(u)$ and $\xi^2(v)$ instead of t_1, t_2 respectively, we get

$$g(\xi^2(u))[g(\xi^2(v)) - g(\xi^2(u))] - \int_0^{\xi^2(v)} g(\tau)d\tau + \int_0^{\xi^2(u)} g(\tau)d\tau \ge 0.$$

This implies the statement. □

Evidently the functional

$$a(u;v,w) = \int_\Omega g(\xi^2(u))H(v,w)dx, \quad u,v,w \in \overset{0}{H}{}^2(\Omega). \tag{22}$$

satisfies the boundedness condition (15), due to condition **(C1)** and Corollary 2

$$|a(u;u,v)| \le c_1\int_\Omega |H(u,v)|dx \le c_2\alpha_2^2\|u\|_2\|v\|_2\ .$$

Thus all conditions of Theorem 2 hold, and we may apply it to the nonlinear problem (6).

Theorem 3. *Let* $u \in \overset{0}{H}{}^2(\Omega)$ *and* $u^{(n)} \in \overset{0}{H}{}^2(\Omega)$ *be solutions of the nonlinear problem (6), and linearized problem (20), respectively. If conditions* **(C1)-(C3)** *hold, then*
(b1) *the sequence of potentials* $\{\Pi_0(u^{(n)})\} \subset R$*, defined by (21) and corresponding to the sequence of solutions* $\{u^{(n)}\} \subset \overset{0}{H}{}^2(\Omega)$*,* $n = 1, 2, 3, ..,$ *of the linearized problem (20), is a monotonically decreasing one*

$$\Pi(u^{(n)}) \leq \Pi(u^{(n-1)}), \quad \forall u^{(n-1)}, u^{(n)} \in \overset{0}{H}{}^2(\Omega);$$

(b2) *the sequence of approximate solutions* $\{u^{(n)}\} \subset \overset{0}{H}{}^2(\Omega)$ *defined by the iteration scheme (20) converges to the solution* $u \in \overset{0}{H}{}^2(\Omega)$ *of the nonlinear problem (6) in the norm of the Sobolev space* $\overset{0}{H}{}^2(\Omega)$*;*
(b3) *for the rate of convergence the following estimate holds*

$$\|u - u^{(n)}\| \leq \frac{\sqrt{2}\gamma_2}{\gamma_1^{3/2}} \left[\Pi_0(u^{(n-1)}) - \Pi_0(u^{(n)})\right]^{1/2}. \tag{23}$$

4 Existence of a quasisolution of the inverse coefficient problem

Let us denote by $\mathcal{G}$ be the set of admissible coefficients $g(\xi^2)$, satisfying conditions **(C1)**–**(C3)**. Denote by $u(x; g)$ the corresponding solution of nonlinear problem (1). Then for each step of the quasistatic prosess of bending, given by the parameter τ_k, $k = \overline{1, K}$, the inverse coefficient problem can be reformulated as the following nonlinear functional equation

$$u(x; g) = w_i[\tau_k], \quad g \in \mathcal{G}. \tag{24}$$

The left-hand side operator $\Phi : \mathcal{G} \mapsto w_i[\tau_k]$, is defined to be the *input-output mapping.* In practice an exact equality in (24) is not possible due to measurements errors. For this reason we will introduce the auxiliary functional

$$I(g) = \sum_{k=1}^{K} \sum_{i=1}^{M} (u(x; g) - w_i[\tau_k])^2, \quad g \in \mathcal{G}, \tag{25}$$

and consider the following minimization problem:

$$J(g_*) = \inf_{g \in \mathcal{G}} J(g). \tag{26}$$

A quasisolution of this minimization problem will be defined to be as a *quasisolution of the inverse problem (1) – (2).*

To obtain the existence of a quasisolution one needs to show compactness of the set of admissible coefficients $\mathcal{G}$, and continuity of the functional $I(g)$ in the appropriate norm.

Conditions **(C1)**–**(C3)** allow to conclude that the admissible coefficients $\mathcal{G}$ are in the Sobolev space $H^1(0, \xi^*)$. Further, conditions **(C1)** and **(C3)** imply that the admissible coefficients $g(\xi^2)$ are uniformly bounded and monotonically increasing. According to Tikhonov's lemma [16] this class of functions is compact in $L_2(0,\xi^*) \equiv H^0(0,\xi^*)$. Based on this result, now we assume that in addition to conditions **(C1)–(C3)**, the coefficients satisfiy the condition

(C4) *$g'(\xi^2)$ is a monotonically decreasing function in (ξ_0,ξ^*).*

Then we can conclude that the class of functions $\mathcal{G}_0 \subset \mathcal{G}$ satisfying the conditions **(C1)-(C4)** is compact in $H^1(0,\xi^*)$ [6]. The existence of a quasisolution of the inverse problem (1)-(2) will be derived in the set of admissible coefficients $\mathcal{G}_0 \subset H^1(0,\xi^*)$. For this aim let us first analyze continuity of the solution $u(x;g) \in \overset{0}{H}{}^2(\Omega)$ of the nonlinear variational problem (6)-(7) with respect to the coefficient $g \in \mathcal{G}_0$.

Lemma 7. *Let conditions* **(C1)–(C4)** *hold, and $F \in H^0(\Omega)$. Denote by $u_m \in \overset{0}{H}{}^2(\Omega)$, $u_m = u(x;g_m)$, the sequence of solutions of the variational problem (6) – (7) corresponding to the sequence of coefficients $\{g_m\} \subset \mathcal{G}_0$. Assume that the sequence $\{g_m\}$ of coefficients converges to the function $g \in \mathcal{G}_0$ in H^1-norm, as $m \to \infty$. Then the sequence of the solutions $\{u(x;g_m)\}$ converges to the solution $u(x;g)$ of the variational problem*

$$a(u;u,v) := \int_\Omega g(\xi^2(u))H(u,v)dx = \int_\Omega F(x)v(x)dx, \quad \forall v \in \overset{0}{H}{}^2(\Omega) \tag{27}$$

weakly in $H^2(\Omega)$.

Proof. Let us denote by $u_m^{(n)} \in \overset{0}{H}{}^2(\Omega)$, $u_m^{(n)} = u^{(n)}(x;g_m)$, the sequence solutions of the linearized problem (20) corresponding to the sequence of coefficients $\{g_m\} \subset \mathcal{G}_0$

$$a_m(u^{(n-1)};u_m^{(n)},v) := \int_\Omega g_m(\xi^2(u^{(n-1)}))H(u_m^{(n)},v)dx = \int_\Omega F(x)v(x)dx, \forall v \in \overset{0}{H}{}^2(\Omega). \tag{28}$$

Note that the index m in the above bilinear form $a_m(u_m^{(n-1)};\cdot,\cdot)$ means that in the right-hand side integral there is the function $g_m(\xi^2(u^{(n-1)}))$, instead of $g(\xi^2(u^{(n-1)}))$.

Substituting $v = u_m^{(n)}$ in (28) we get

$$|a_m(u_m^{(n-1)};u_m^{(n)},u_m^{(n)})| \le \|F\|_0\|u_m^{(n)}\|_0 \le \|F\|_0\|u_m^{(n)}\|_2, \quad u_m^{(n)} \in \overset{0}{H}{}^2(\Omega).$$

On the other hand, due to coercitiveness of the bilinear form $a_m(u_m^{(n-1)};\cdot,\cdot)$ we conclude

$$|a_m(u_m^{(n-1)};u_m^{(n)},u_m^{(n)})| \ge \gamma_1\|u_m^{(n)}\|_2^2, \quad \gamma_1 > 0.$$

These two inequalities imply the uniform boundedness of the sequence $\{u_m^{(n)}\}$

$$\|u_m^{(n)}\|_2 \le \|F\|_0/\gamma_1, \quad \gamma_1 > 0$$

in H^2-norm. This implies the weak convergence of the sequence $\{u_m^{(n)}\}$ in $H^2(\Omega)$. Hence there exists such an element $\tilde{u}^{(n)} \in \overset{0}{H}{}^2(\Omega)$, that $u_m^{(n)} \rightharpoonup \tilde{u}^{(n)}$ weakly in $H^2(\Omega)$. We need to prove that $\tilde{u}^{(n)} = u^{(n)}(x;g)$, where $g \in \mathcal{G}_0$ is the limit of the sequence $\{g_m\} \subset \mathcal{G}_0$. For this aim let us estimate the difference $|a(u^{(n-1)};\tilde{u}^{(n)},v) - a_m(u^{(n-1)};u_m^{(n)},v)|$

$$\begin{aligned}
&|a(u^{(n-1)};\tilde{u}^{(n)},v) - a_m(u^{(n-1)};u_m^{(n)},v)| \\
&= \left|\int_\Omega [g(\xi^2(u^{(n-1)}))H(\tilde{u}^{(n)},v) - g_m(\xi^2(u^{(n-1)}))H(u_m^{(n)},v)]dx\right| \\
&\le \left|\int_\Omega [g(\xi^2(u^{(n-1)}))H(\tilde{u}^{(n)},v) - g_m(\xi^2(u^{(n-1)}))H(\tilde{u}^{(n)},v)]dx\right| \\
&+ \left|\int_\Omega [g_m(\xi^2(u^{(n-1)}))H(u^{(n)},v) - g_m(\xi^2(u^{(n-1)}))H(u_m^{(n)},v)]dx\right| \\
&\le \max_{[\xi_*,\xi^*]} |g(\xi^2(u^{(n-1)})) - g_m(\xi^2(u^{(n-1)}))| \int_\Omega H(\tilde{u}^{(n)},v)dx + c_1 \int_\Omega H(\tilde{u}^{(n)} - u_m^{(n)},v)dx.
\end{aligned}$$

The first right-hand side term tends to zero, $\max_{[\xi_*,\xi^*]} \left|g(\xi^2(u^{(n-1)})) - g_m(\xi^2(u^{(n-1)}))\right| \to 0$, since $g_m \to g \in \mathcal{G}_0$ in H^1-norm, as $m \to \infty$. Further, by the weak convergence $u_m^{(n)} \rightharpoonup \tilde{u}^{(n)}$, as $m \to \infty$, in $H^2(\Omega)$, we conclude that the second right-hand side term also tends to zero. Thus, passing to the limit in (28), as $m \to \infty$, we obtain

$$a(u^{(n-1)};\tilde{u}^{(n)},v) := \int_\Omega g_m(\xi^2(u^{(n-1)}))H(\tilde{u}^{(n)},v)dx = \int_\Omega F(x)v(x)dx, \quad \forall v \in \overset{0}{H}{}^2(\Omega),$$

i.e. the limit function $\tilde{u}^{(n)}$ is the solution of the linerized variational problem (20). Due to the uniqueness of the solution of this problem we conclude $\tilde{u}^{(n)} = u^{(n)}(x;g)$.

Thus the convergence $g_m \to g$ of the sequence of coefficients $\{g_m\} \subset \mathcal{G}_0$ in H^1-norm, implies the weak convergence $u_m^{(n)} := u^{(n)}(x;g_m) \rightharpoonup u^{(n)} := u^{(n)}(x;g)$, $m \to \infty$, in $H^2(\Omega)$ of the approximate solutions $u^{(n)} \in \overset{0}{H}{}^2(\Omega)$, defined by (20), of the nonlinear variational problem (27)

$$|a(u^{(n-1)};u^{(n)},v) - a_m(u^{(n-1)};u_m^{(n)},v)| \to 0, \quad n \to \infty. \tag{29}$$

The above results permit to conclude that $|a(u;u,v) - a_m(u^{(n-1)};u_m^{(n)},v)| \to 0$, as $m,n \to \infty$. Indeed,

$$\begin{aligned}
&|a(u;u,v) - a_m(u^{(n-1)};u_m^{(n)},v)| \\
&\le |a(u;u,v) - a(u^{(n-1)};u^{(n)},v)| + |a(u^{(n-1)};u^{(n)},v) - a_m(u^{(n-1)};u_m^{(n)},v)|.
\end{aligned}$$

The first and second right-hand side terms tend to zero, due to Theorem 3 and (29), accordingly. This completes the proof. □

Taking into account the compactness of the embedding $H^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$, $\Omega \subset R^2$, we conclude that the sequence of the solutions $\{u(x;g_m)\} \subset \overset{0}{H}{}^2(\Omega)$ converges to the solution $u(x;g) \in \overset{0}{H}{}^2(\Omega)$ of the variational problem (27) in $C^0(\overline{\Omega})$. This means the continuity of the minimizing functional (25).

Theorem 4. *Let conditions of Lemma 7 hold. Then the inverse problem (1) – (2) has at least one solution $g_* \in \mathcal{G}_0$ defined as a solution of the minimization problem (26).*

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References

[1] R.A. Adams, *Sobolev spaces.* Academic Press, New York, 1975.

[2] P. Ciarlet, *Finite-element method for elliptic problems.* Amsterdam, North-Holland, 1978.

[3] P. DuChateau, R. Thelwell, G. Butters, *Analysis of an adjoint problem approach to the identification of an unknown diffusion coefficient.* Inverse Problems, 20 (2004), 601 – 625.

[4] H. Gajewski, K. Greger, K. Zacharias, *Nichtlineare operator gleichungen und operator differential gleichungen.* Akademic-Verlag, Berlin, 1974.

[5] A. Hasanov, A. Mamedov, *An inverse problem related to the determination of elastoplastic properties of a plate.* Inverse Problems, 10 (1994), 601 – 615.

[6] A. Hasanov, *An Inverse coefficient problem for an elasto-plastic medium.* SIAM J. Appl. Math., 55 (1995), 1736 – 1752.

[7] A. Hasanov, *Convexity argument for monotone potential operators.* Nonlinear Analysis: TMA, 47 (2000), 906 – 918.

[8] A. Hasanov, *Variational approach to non-linear boundary value problems for elasto-plastic incompressible bending plate.* Int. J. Non-Linear Mech., 42 (2007), 711 – 721.

[9] A. Hasanov, *An inversion method for identification of elastoplastic properties for engineering materials from limited spherical indentation measurements.* Inverse Problems in Science and Engineering, 15, no. 6 (2007), 601 – 627.

[10] A. Hasanov, *An introduction to inverse source and coefficient problems for PDEs based on boundary measured data.* Mathematics in Science and Technology (Editors A.H. Siddigi, R.C. Singh, P. Manchanda), Abstracts of the Satellite Conference of International Congress of Mathematicians, Delhi (2010).

[11] V.K. Ivanov, V.V. Vasin, V.P. Tanana, *Theory of linear Ill-posed problems and its applications.* Nauka, Moscow, 1978.

[12] A. Kufner, S. Fučik, *Nonlinear differential equations studies in applied mathematics: 2.* Elsevier Scientific Publ. Comp., Amsterdam, 1980.

[13] L. Liu, N. Ogasawara, N. Chiba, X. Chen, *Can indentation test measure unique elastoplastic properties.* J. Material Research, 24, no. 3 (2009), 784 – 800.

[14] S.G. Mikhlin, *Variational methods in mathematical physics.* Pergamon, New York, 1964.

[15] A.E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations.* Mathematical Surveys and Monographs, 49, Amer. Math. Soc., Providence, 1996.

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