ISSN 2077–9879

Eurasian Mathematical Journal

2013, Volume 4, Number 1

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia
the University of Padua

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

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5 Munaitpasov Street
010008 Astana
Kazakhstan

Proceedings of the international workshop

OPERATORS IN MORREY-TYPE SPACES AND APPLICATIONS

(OMTSA 2011)

dedicated to 70th birthday of Professor V.I. Burenkov

held through May 20-27, 2011

at the Ahi Evran University, Kirşehir, Turkey

Volume 2

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 4, Number 1 (2013), 135 – 144

VOLTERRA OPERATOR FROM BERGMAN SPACES TO MORREY SPACES

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Communicated by V. Guliyev

Key words: Volterra operator, Bergman spaces, Morrey spaces, Carleson measures.

AMS Mathematics Subject Classification: Primary 47B38; Secondary 32A35.

Abstract. We characterize the boundedness of Volterra operators from Bergman spaces to Morrey spaces. Tools in the holomorphic function spaces, properties of Carleson measures and the atomic decomposition for functions in Bergman spaces are heavily employed.

1 Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane, and $\partial\mathbb{D}$ be the boundary of the unit disk $\mathbb{D}$. Denote the set of all holomorphic functions on $\mathbb{D}$ by $\mathcal{H}(\mathbb{D})$. For φ and f in $\mathcal{H}(\mathbb{D})$, the Volterra operator V_φ acting on f is defined by

$$V_\varphi(f)(z) = \int_0^z f(w)\varphi'(w)dw\,, \quad z \in \mathbb{D}\,.$$

A natural and interesting question about the Volterra operator is to characterize the symbol function φ so that V_φ is bounded from one holomorphic space to another.

On Hardy and Bergman spaces the boundedness of V_φ in terms of φ are well understood. For example, it is proved in [1] (see also the references therein), that for $p > q > 0$, V_φ maps $\mathcal{H}^p$ into $\mathcal{H}^q$ if and only if $\varphi \in \mathcal{H}^r$ with $1/r = 1/q - 1/p$; V_φ maps $\mathcal{H}^p$ into itself if and only if $\varphi \in$ BMOA; for $0 < p < q$ and $1/p - 1/q \leqslant 1$, V_φ maps $\mathcal{H}^p$ into $\mathcal{H}^q$ if and only if $\varphi \in \mathcal{B}^{1-\frac{1}{p}+\frac{1}{q}}$; and for $0 < p < q$ and $1/p - 1/q > 1$, V_φ maps $\mathcal{H}^p$ into $\mathcal{H}^q$ if and only if φ is constant. On Bergman spaces, or from Hardy spaces to Bergman spaces, the characterization of the boundedness of V_φ can be reduced to the characterization of φ so that the embedding inequality

$$\int_{\mathbb{D}} |f(z)|^q |\varphi'(z)|^q (1-|z|)^\beta dA(z) \leqslant C\,\|f\|^q$$

holds on Bergman or Hardy spaces. These embedding inequalities have been well studied and understood on Bergman and Hardy spaces, even for the most general forms $\int_{\mathbb{D}} |f^{(n)}(z)|^q d\mu(z) \leqslant C\,\|f\|^q$ (see, for example, [2, 4, 6, 7, 8, 15]). The boundedness of V_φ from Bergman spaces to Hardy spaces, and from Hardy spaces to Morrey spaces have been studied recently in [11, 12]. For example, it is proved in [12] that for $2 \leqslant p \leqslant \infty$, V_φ maps $\mathcal{H}^p$ into $\mathcal{L}^{2,1-\frac{2}{p}}$ bounded if and only if $\varphi \in \mathrm{BMOA}$.

In this paper, we investigate the boundedness of the Volterra operator from Bergman space $\mathcal{A}^p$ to Morrey space $\mathcal{L}^{2,q}$ (see definitions later). Leaving the notation and definitions later, we state our main results first.

Main Theorem. *Suppose $2 \leqslant p \leqslant \infty$ and $0 < q \leqslant 1$.*

(1) For $q > 1 - \frac{2}{p}$, $V_\varphi : \mathcal{A}^p \to \mathcal{L}^{2,q}$ is bounded if and only if $\varphi \in \mathcal{B}^{\frac{3-q}{2}-\frac{2}{p}}$.

(2) For $q \leqslant 1 - \frac{2}{p}$, $V_\varphi : \mathcal{A}^p \to \mathcal{L}^{2,q}$ is bounded if and only if the measure

$$|\varphi'(z)|^{\frac{2p}{p-2}} (1-|z|)^{\frac{p}{p-2}} dA(z)$$

is a $\frac{pq}{p-2}$-Carleson measure.

One can establish the corresponding results for the compactness of the operator $V_\varphi : \mathcal{A}^p \to \mathcal{L}^{2,q}$ by modifying the proofs provided for the main theorem.

2 Notation, definitions and preliminaries

Let $|d\zeta|$ be the Lebesgue measure on $\partial\mathbb{D}$, $dA(z) = \frac{1}{\pi} dxdy$ be the normalized Lebesgue measure on $\mathbb{D}$. For $0 < p < \infty$, the Hardy space $\mathcal{H}^p$ and the Bergman space $\mathcal{A}^p$ consist of, respectively, all holomorphic functions $f \in \mathcal{H}(\mathbb{D})$ satisfying, respectively

$$\|f\|_{\mathcal{H}^p}^p = \sup_{0<r<1} \int_{\partial\mathbb{D}} |f(r\zeta)|^p \frac{|d\zeta|}{2\pi} < \infty \quad \text{and} \quad \|f\|_{\mathcal{A}^p}^p = \int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty.$$

For $p = \infty$, we say that $f \in \mathcal{H}^\infty = \mathcal{A}^\infty$ if f is a bounded holomorphic function on $\mathbb{D}$.

For $\alpha \geqslant 0$, the Bloch type space $\mathcal{B}^\alpha$ is the set of all analytic functions φ on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} |\varphi'(z)|(1-|z|)^\alpha < \infty \, .$$

Define $\mathcal{B}^\alpha = \{\text{constants}\}$ if $\alpha < 0$.

Let $f \in \mathcal{H}^2$. It is well known that f has the boundary value $f(\zeta) = \lim_{r\to 1_-} f(r\zeta)$ for all $\zeta \in \partial\mathbb{D}$. For $s \in \mathbb{R}$, define the Morrey space $\mathcal{L}^{2,s}$ to be the sets of all $f \in \mathcal{H}^2$ such that

$$\sup_{\text{arc } I\subset\partial\mathbb{D}} \frac{1}{|I|^s} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi} < \infty \, .$$

Here $|I| = \int_I |d\zeta|/(2\pi)$ is the normalized length of arc I, $f_I = \frac{1}{|I|} \int_I f(\zeta) \frac{|d\zeta|}{2\pi}$. It is clear that $\mathcal{L}^{2,s} = \mathcal{H}^2$ if $s \leqslant 0$, $\mathcal{L}^{2,1} = \mathrm{BMOA}$, and $\mathcal{L}^{2,s}$ is Bloch type space if $s > 1$. Therefore $0 < s \leqslant 1$ is the most interesting range for Morrey space $\mathcal{L}^{2,s}$.

The Carleson square based on an arc I in $\mathbb{D}$ is defined by

$$S(I) = \{z \in \mathbb{D} : |z| > 1 - |I| \ \text{ and } z/|z| \in I\} \, .$$

For fixed $s > 0$, a non-negative measure μ on $\mathbb{D}$ is called an $s-$Carleson measure if there exists a $C > 0$ such that

$$\mu\left(S(I)\right) \leqslant C\,|I|^{s} \quad \text{for all arc} \quad I \subset \partial\mathbb{D}\,.$$

It is well known that $1-$Carleson measures are related to BMO space (see, for example, page 240 in [5]), and for $s < 1$, the $s-$Carleson measures are associated with Morrey spaces and Q spaces (see, for example, [13] and [14]). The following theorem is due to Carleson [2] if $p = q$, and Duren [4] if $q > p$.

Theorem A. *Let $0 < p \leqslant q \leqslant \infty$ and μ be a non-negative measure on $\mathbb{D}$. Then the inequality*

$$\int_{\mathbb{D}} |f(z)|^{q}\, d\mu(z) \leqslant C\,\|f\|^{q}_{\mathcal{H}^p}\,, \qquad \forall f \in \mathcal{H}^p$$

holds if and only if μ is a $\frac{q}{p}-$Carleson measure.

Part (2) of the following theorem (for $0 < s < 1$) was proved in [13], the other parts are either easy or well known.

Theorem B. *Let $s \in \mathbb{R}$ and φ be analytic on $\mathbb{D}$. Let*

$$d\mu_{\varphi}(z) = \left|\varphi'(z)\right|^{2}(1 - |z|)dA(z).$$

(1) If $s \leqslant 0$, then μ_φ is an s–Carleson measure if and only if $\varphi \in \mathcal{H}^2$;

(2) If $0 < s \leqslant 1$, then μ_φ is an s–Carleson measure if and only if $\varphi \in \mathcal{L}^{2,s}$;

(3) If $1 < s \leqslant 3$, then μ_φ is an s–Carleson measure if and only if $\varphi \in \mathcal{B}^{\frac{3-s}{2}}$;

(4) If $s > 3$, then μ_φ is an s–Carleson measure if and only if $\varphi \equiv$ constant.

Throughout this paper, C and c denote positive constants that may change from one step to the next. We say that two positive functions a and b are equivalent, denoted by $a \asymp b$, if there are two positive constants c and C such that $ca \leqslant b \leqslant Ca$.

3 Proof of Main Theorem

Let φ_a be a Möbius transformation of $\mathbb{D}$ such that $\varphi_a(0) = a$ and $\varphi_a(a) = 0$. The Bergman distance of z and w in $\mathbb{D}$ is defined by

$$d(z,w) = \frac{1}{2}\log\left(\frac{1 + |\varphi_w(z)|}{1 - |\varphi_w(z)|}\right),$$

which is Möbius invariant. For fixed $a \in \mathbb{D}$ and $t > 0$, the Bergman disk of center a and radius t is defined by

$$D(a,t) = \{\, w \in \mathbb{D} \,:\, d(w,a) < t\}\,,$$

which is also a Euclidean disk of center $\frac{1-r^2}{1-|a|^2r^2}a$ and radius $r\frac{1-|a|^2}{1-|a|^2r^2}$, where $r = \frac{e^{2t}-1}{e^{2t}+1}$.

Geometries associated with the Bergman distance are useful in the study of holomorphic spaces and the operator theory. It is standard that for fixed $a \in \mathbb{D}$

$$|1-\bar{a}z| \asymp (1-|z|) \asymp (1-|a|)\,, \qquad \forall z \in D(a,t)\,. \tag{3.1}$$

If $|D(a,t)|$ is the the area of $D(a,t)$ with respect to $dA(z)$, then the we have clearly

$$|D(a,t)| = \int_{D(a,t)} dA(z) \asymp (1-|a|)^2\,, \qquad \forall a \in \mathbb{D}. \tag{3.2}$$

For $f \in \mathcal{H}(\mathbb{D})$ and $r,t>0$, it is standard that the following mean value inequality holds:

$$|f(z)|^r \leqslant \frac{C}{|D(z,t)|}\int_{D(z,t)} |f(w)|^r\, dA(w), \qquad \forall z \in \mathbb{D}\,. \tag{3.3}$$

Proof of Main Theorem, part (1). Suppose $q > 1-\frac{2}{p}$ and $\varphi \in \mathcal{B}^{\frac{3-q}{2}-\frac{2}{p}}$. By Theorem B, we need to show that the measure

$$d\mu_{V_\varphi(f)}(z) = |V_\varphi(f)'(z)|^2\,(1-|z|)dA(z) = |f(z)|^2\,|\varphi'(z)|^2\,(1-|z|)dA(z)$$

is a q-Carleson measure for any $f \in \mathcal{A}^p$.

Since $|\varphi'(z)|\,(1-|z|)^{\frac{3-q}{2}-\frac{2}{p}} \leqslant C$ holds for all $z \in \mathbb{D}$, we have

$$|\varphi'(z)|^2\,(1-|z|) \leqslant C(1-|z|)^{q-2(1-\frac{2}{p})}, \quad \forall z \in \mathbb{D}.$$

Hence for any arc $I \subset \partial\mathbb{D}$, we have

$$\begin{aligned}
\mu_{V_\varphi(f)}(S(I)) &= \int_{S(I)} |f(z)|^2\,|\varphi'(z)|^2\,(1-|z|)dA(z)\\
&\leqslant C\int_{S(I)} |f(z)|^2\,(1-|z|)^{q-2(1-\frac{2}{p})}dA(z)\\
&\leqslant \left(\int_{S(I)} |f(z)|^p dA(z)\right)^{\frac{2}{p}} \left(\int_{S(I)} (1-|z|)^{\frac{pq}{p-2}-2} dA(z)\right)^{\frac{p-2}{p}}.
\end{aligned}$$

Since $q > 1-\frac{2}{p}$ implies $\frac{pq}{p-2} > 1$, we have clearly

$$\int_{S(I)} (1-|z|)^{\frac{pq}{p-2}-2} dA(z) = C|I|^{\frac{pq}{p-2}}.$$

Therefore

$$\mu_{V_\varphi(f)}(S(I)) \leqslant C\,\|f\|^2_{\mathcal{A}^p}\,|I|^q.$$

Suppose now $q > 1-\frac{2}{p}$ and $V_\varphi : \mathcal{A}^p \to \mathcal{L}^{2,q}$ is bounded. We need to show that $\varphi \in \mathcal{B}^{\frac{3-q}{2}-\frac{2}{p}}$. By Theorem B, we know that the measure $|f(z)|^2\,|\varphi'(z)|^2\,(1-|z|)dA(z)$ is a q-Carleson measure for any $f \in \mathcal{A}^p$.

For $a \in \mathbb{D}$, let $f_a(z) = \frac{1-|a|}{(1-\bar{a}z)^{1+\frac{2}{p}}}$. It is easy to check that

$$\|f_a\|_{\mathcal{A}^p} \asymp 1, \quad \forall a \in \mathbb{D}.$$

Let $I(a)$ be the arc of center $a/|a|$ and length $1-|a|$. We have, for small enough $t>0$

$$\begin{aligned}\int_{D(a,t)} &|f_a(z)|^2 |\varphi'(z)|^2 (1-|z|)dA(z)\\ &\leqslant \int_{S(I(a))} |f_a(z)|^2 |\varphi'(z)|^2 (1-|z|)dA(z)\\ &\leqslant C\|f_a\|^2_{\mathcal{A}^p} |I(a)|^q\\ &\leqslant C(1-|a|)^q.\end{aligned}$$

By (3,1), (3.2) and (3.3), we can conclude that

$$|\varphi'(a)|^2 (1-|a|)^{3-\frac{4}{p}} \leqslant C(1-|a|)^q, \quad \text{for all } \ a\in\mathbb{D}.$$

This is enough to obtain the desired result. □

We need atomic decomposition theorem for Bergman spaces in order to prove part (2) of the main theorem.

It is well-known (see for example [3]) that for every $\delta>0$, there exists a distinct sequence $\{z_j\}$ in $\mathbb{D}$, called a $\delta-$lattice, such that $d(z_j,z_k)>\delta/5$ if $j\neq k$, and

$$\bigcup_j D(z_j,\delta)=\mathbb{D} \qquad \text{and} \qquad \sum_j \chi_{D(z_j,5\delta)}(z)\leqslant L\,, \quad \forall z\in\mathbb{D}\,. \tag{3.4}$$

Here, $L>0$ is a uniform constant and χ_E is the characteristic function of a set E. The following theorem is due to Rochberg [9].

Theorem C. *Let $p>0$. There exists $\delta_0>0$ such that for any $\delta\in(0,\delta_0)$, any $\delta-$lattice $\{z_j\}$ in $\mathbb{D}$, and any $m>\max\{-1,2(1-1/p)\}$, the following statements hold.*

(B.1) If $f\in\mathcal{A}^p$, then there exists $\{a_j\}\in\ell^p$ with $\|\{a_j\}\|_{\ell^p}\leqslant C\|f\|_{\mathcal{A}^p}$ such that

$$f(z)=\sum_j a_j\frac{(1-|z_j|)^m}{(1-\overline{z_j}z)^{m+\frac{2}{p}}}\,.$$

(B.2) If $\{a_j\}\in\ell^p$, then the function f defined by the above series converges in $\mathcal{A}^p$ and $\|f\|_{\mathcal{A}^p}\leqslant C\|\{a_j\}\|_{\ell^p}$.

Proof of Main Theorem, part (2). Suppose $0<q\leqslant 1-\frac{2}{p}$ and the measure

$$|\varphi'(z)|^{\frac{2p}{p-2}} (1-|z|)^{\frac{p}{p-2}} dA(z)$$

is a $\frac{pq}{p-2}$-Carleson measure. By Theorem B, we need to show that the measure

$$d\mu_{V_\varphi(f)}(z)=|V_\varphi(f)'(z)|^2(1-|z|)dA(z)=|f(z)|^2|\varphi'(z)|^2(1-|z|)dA(z)$$

is a q-Carleson measure for any $f\in\mathcal{A}^p$.

For an arc $I \subset \partial\mathbb{D}$, we have

$$\begin{aligned}
\mu_{V_\varphi(f)}(S(I)) &= \int_{S(I)} |f(z)|^2 |\varphi'(z)|^2 (1-|z|)dA(z) \\
&\leqslant \left(\int_{S(I)} |f(z)|^p dA(z)\right)^{\frac{2}{p}} \\
&\quad \times \left(\int_{S(I)} |\varphi'(z)|^{\frac{2p}{p-2}} (1-|z|)^{\frac{p}{p-2}} dA(z)\right)^{\frac{p-2}{p}} \\
&\leqslant C \|f\|^2_{\mathcal{A}^p} \left(|I|^{\frac{pq}{p-2}}\right)^{\frac{p-2}{p}} \\
&= C \|f\|^2_{\mathcal{A}^p} |I|^q.
\end{aligned}$$

This is the desired result.

To prove the “only if" part, let I be a arc on $\partial\mathbb{D}$. By Theorem B, we know that the measure

$$d\mu_{V_\varphi(f)}(z) = |f(z)|^2 |\varphi'(z)|^2 (1-|z|)dA(z)$$

is a q-Carleson measure for any $f \in \mathcal{A}^p$.

We start with the estimate

$$\int_{S(I)} |f(z)|^2 |\varphi'(z)|^2 (1-|z|)dA(z) \leqslant C |I|^q \|f\|^2_{\mathcal{A}^p}, \quad \forall f \in \mathcal{A}^p. \tag{3.5}$$

If $p = \infty$, then (3.5) implies clearly the desired result. Therefore we consider $p < \infty$.

In the following, we will employ Theorem C and Khinchine's inequality to refine the above estimate. This idea was first used by Luecking in [8] (see also [10, 12]).

For $x \in [0,1)$, let $r_j(x) = r_0(2^j x)$, $j = 1, 2, \cdots$ with

$$r_0(y) = \begin{cases} 1, & 0 \leqslant y - [y] < 1/2; \\ -1, & 1/2 \leqslant y - [y] < 1. \end{cases}$$

Khinchine's inequality says: for any $0 < p < \infty$ and integer $N > 0$, there exists $c_p > 0$ such that

$$c_p \left(\sum_{j=0}^N |c_j|^2\right)^{p/2} \leqslant \int_0^1 \left|\sum_{j=0}^N c_j r_j(x)\right|^p dx \leqslant \frac{1}{c_p} \left(\sum_{j=0}^N |c_j|^2\right)^{p/2}.$$

Let $\{z_j\}$ be a $\delta-$lattice in $\mathbb{D}$, $m > \max\{-1, 2(1-1/p)\}$ (> 0), and

$$f_j(z) = \frac{(1-|z_j|)^m}{(1-\overline{z_j}z)^{m+\frac{2}{p}}}, \quad j = 1, 2, \cdots.$$

By Theorem C, we know that for sufficient small $\delta > 0$, and any $\Lambda = \{\lambda_j\} \in \ell^p$, the function

$$f_\Lambda(z) = \sum_j \lambda_j f_j(z)$$

is a function in $\mathcal{A}^p$ and

$$\|f_\Lambda\|_{\mathcal{A}^p} \leqslant C\,\|\Lambda\|_{\ell^p}\,.$$

For every $x \in [0,1)$, let $\Lambda(x) = \{\lambda_j r_j(x)\}$. It is clear that

$$\|\Lambda(x)\|_{\ell^p} = \|\Lambda\|_{\ell^p}\,, \quad \forall x \in [0,1).$$

Replacing $f(z)$ by $f_{\Lambda(x)}(z)$ in the estimate (3.5), we have

$$\int_{S(I)} \left|f_{\Lambda(x)}(z)\right|^2 |\varphi'(z)|^2 (1-|z|)dA(z) \leqslant C\,|I|^q \left\|f_{\Lambda(x)}\right\|^2_{\mathcal{A}^p} \leqslant C\,|I|^q\,\|\Lambda\|^2_{\ell^p}\,.$$

Integrating both sides of the above inequality over $[0,1)$ with respect to x, and then using Khinchine's inequality, we obtain

$$\int_{S(I)} \sum_j |\lambda_j|^2\,|f_j(z)|^2\,|\varphi'(z)|^2 (1-|z|)dA(z) \leqslant C\,|I|^q\,\|\Lambda\|^2_{\ell^p}\,.$$

For convenience, write $D_j = D(z_j,\delta)$. By the estimate (3.1), we know

$$\int_{D_j} |f_j(z)|^2\,|\varphi'(z)|^2 (1-|z|)dA(z) \asymp (1-|z_j|)^{-\frac{4}{p}} \int_{D_j} |\varphi'(z)|^2 (1-|z|)dA(z).$$

Hence the above estimate implies

$$\begin{aligned}
&\sum_{j:D_j\subset S(I)} \frac{|\lambda_j|^2}{(1-|z_j|)^{\frac{4}{p}}} \int_{D_j} |\varphi'(z)|^2 (1-|z|)dA(z) \\
&\asymp \sum_{j:D_j\subset S(I)} |\lambda_j|^2 \int_{D_j} |f_j(z)|^2\,|\varphi'(z)|^2 (1-|z|)dA(z) \\
&\leqslant C \sum_{j:D_j\subset S(I)} \int_{D_j} \sum_k |\lambda_k|^2\,|f_k(z)|^2\,|\varphi'(z)|^2 (1-|z|)dA(z) \\
\text{(by (3.4))}\quad &\leqslant CL \int_{S(I)} \sum_j |\lambda_j|^2\,|f_j(z)|^2\,|\varphi'(z)|^2 (1-|z|)dA(z) \\
&\leqslant C\,|I|^q\,\|\Lambda\|^2_{\ell^p}\,.
\end{aligned}$$

This further implies, by the duality relation of $\ell^{\frac{p}{2}}$ and $\ell^{\frac{p}{p-2}}$, that

$$\left(\sum_{j:D_j\subset S(I)} \left(\frac{1}{(1-|z_j|)^{\frac{4}{p}}} \int_{D_j} |\varphi'(z)|^2 (1-|z|)dA(z)\right)^{\frac{p}{p-2}}\right)^{1-\frac{2}{p}} \leqslant C\,|I|^q\,.$$

By (3.1), (3,2) and (3.3), we have for each $j = 1, 2, \cdots$ and $z \in D_j$

$$|\varphi'(z)|^2 (1-|z|) \leqslant C(1-|z_j|)^{-2} \int_{D_j} |\varphi'(w)|^2 (1-|w|)dA(w).$$

Therefore

$$\begin{aligned}
&\int_{D_j} |\varphi'(z)|^{\frac{2p}{p-2}} (1-|z|)^{\frac{p}{p-2}} dA(z) \\
&\qquad \leqslant \; C\,|D_j|\,(1-|z_j|)^{\frac{-2p}{p-2}} \left(\int_{D_j} |\varphi'(z)|^2 (1-|z|) dA(z) \right)^{\frac{p}{p-2}} \\
&\qquad \asymp \; C \left(\frac{1}{(1-|z_j|)^{\frac{4}{p}}} \int_{D_j} |\varphi'(z)|^2 (1-|z|) dA(z) \right)^{\frac{p}{p-2}} .
\end{aligned}$$

or equivalently

$$\sum_{j:D_j\subset S(I)} \int_{D_j} |\varphi'(z)|^{\frac{2p}{p-2}} (1-|z|)^{\frac{p}{p-2}} dA(z) \leqslant C\,|I|^{\frac{pq}{p-2}} . \tag{3.6}$$

For an arc I, let $J_I = \{j : D_j \cap S(I) \neq \varnothing\}$. Clearly

$$S(I) \subset \bigcup_{j\in J_I} D_j.$$

Let $\tilde{I}$ be the smallest arc on $\partial\mathbb{D}$ such that

$$\bigcup_{j\in J_I} D_j \subset S(\tilde{I}).$$

It is not hard to see that $I \subset \tilde{I}$ and $|\tilde{I}| \asymp |I|$. Since estimate (3.6) holds for any arc I, we have

$$\begin{aligned}
&\int_{S(I)} |\varphi'(z)|^{\frac{2p}{p-2}} (1-|z|)^{\frac{p}{p-2}} dA(z) \\
&\qquad \leqslant \; \sum_{j\in J_I} \int_{D_j} |\varphi'(z)|^{\frac{2p}{p-2}} (1-|z|)^{\frac{p}{p-2}} dA(z) \\
&\qquad \leqslant \; \sum_{j:D_j\subset S(\tilde{I})} \int_{D_j} |\varphi'(z)|^{\frac{2p}{p-2}} (1-|z|)^{\frac{p}{p-2}} dA(z) \\
&\qquad \leqslant \; C|\tilde{I}|^{\frac{pq}{p-2}} \\
&\qquad \leqslant \; C\,|I|^{\frac{pq}{p-2}} .
\end{aligned}$$

The above is enough to conclude the desired result. □

For $\alpha \geqslant 0$, the little Bloch type space $\mathcal{B}_0^\alpha$ is the set of all analytic functions $\varphi \in \mathcal{B}_0^\alpha$ such that

$$\lim_{|z|\to 1_-} |\varphi'(z)|(1-|z|)^\alpha = 0 .$$

For fixed $s > 0$, a non-negative measure μ on $\mathbb{D}$ is called a compact $s-$Carleson measure if it is an s-Carleson measure and satisfies

$$\lim_{|I|\to 0} \frac{\mu\,(S(I))}{|I|^s} = 0 .$$

To end this paper, we state the following theorem, which can be proved similarly.

Theorem 3.1. *Suppose* $2 \leqslant p \leqslant \infty$ *and* $0 < q \leqslant 1$.

(1) For $q > 1 - \frac{2}{p}$, $V_\varphi : \mathcal{A}^p \to \mathcal{L}^{2,q}$ *is compact if and only if* $\varphi \in \mathcal{B}_0^{\frac{3-q}{2} - \frac{2}{p}}$.

(2) For $q \leqslant 1 - \frac{2}{p}$, $V_\varphi : \mathcal{A}^p \to \mathcal{L}^{2,q}$ *is compact if and only if the measure*

$$|\varphi'(z)|^{\frac{2p}{p-2}} (1 - |z|)^{\frac{p}{p-2}} dA(z)$$

is a compact $\frac{pq}{p-2}$*-Carleson measure.*

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Received: 24.08.2012

EURASIAN MATHEMATICAL JOURNAL

2013 – Том 4, № 1 – Астана: ЕНУ. – 145 с.

Подписано в печать 23.05.2013 г. Тираж – 150 экз.

Адрес редакции: 010008, Астана, ул. Мирзояна 2,
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Дизайн: К. Булан

Отпечатано в типографии ЕНУ имени Л.Н.Гумилева

Свидетельство о постановке на учет печатного издания
Министерства культуры и информации Республики Казахстан
№ 10330 – Ж от 25.09.2009 г.