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WEAK AND STRONG TYPE ESTIMATES FOR FRACTIONAL INTEGRAL OPERATORS ON MORREY SPACES OVER METRIC MEASURE SPACES

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Communicated by T.V. Tararykova

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Abstract. We discuss here a weak and strong type estimate for fractional integral operators on Morrey spaces over metric measure spaces, where the underlying measure does not always satisfy the doubling condition.

1 Introduction

The aim of this paper is to propose a framework of Morrey spaces and fractional integral operators on a metric measure space (X, d, μ), where μ is a Radon measure.

We recall that the Riesz potential I_α on $\mathbb{R}^d$ is given by

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\, dy.$$

According to the Hardy-Littlewood-Sobolev theorem [2, 3, 10], I_α is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ as long as $p, q \in (1, \infty)$ satisfy $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. Morrey spaces, named after C. Morrey, can also be used to describe the boundedness property of I_α. Here we adopt the following notation to denote Morrey spaces. Let $1 \leqslant q \leqslant p < \infty$. For a measurable function f on $\mathbb{R}^d$, we define

$$\|f\|_{\mathcal{M}^p_q} := \sup \left\{ |B|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(B)} \,:\, B \text{ is a ball} \right\}.$$

The space $\mathcal{M}^p_q(\mathbb{R}^d)$ denotes the set of all measurable functions f for which the norm $\|f\|_{\mathcal{M}^p_q}$ is finite. According to Adams [1], I_α is bounded from $\mathcal{M}^p_q(\mathbb{R}^d)$ to $\mathcal{M}^s_t(\mathbb{R}^d)$, provided that $p, q, s, t \in (1, \infty)$ satisfy $\frac{p}{q} = \frac{t}{s}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$.

In this paper, we aim to show that this theorem is independent from the geometric structure of $\mathbb{R}^d$ by extending it to metric measure spaces, where all we have are the distance function d and the Radon measure μ.

Let (X, d, μ) be a metric measure space with a distance function d and a Borel measure μ. Recall that the measure μ is a doubling measure if it satisfies the so-called *doubling condition*, that is, there exists a constant $C > 0$ such that

$$\mu(B(a, 2r)) \leqslant C\mu(B(a, r)) \tag{1.1}$$

for every ball $B(a,r)$ with center $a \in X$ and radius $r > 0$. The doubling condition was a key property in classical harmonic analysis but around a decade ago, it turned out to be unnecessary. The point is that we modify the related definitions. Indeed, in the present paper, we propose to redefine the fractional integral operator by

$$I_\alpha f(x) := \int_X \frac{f(y)}{\mu(B(x, 2d(x,y)))^{1-\alpha}}\, d\mu(y). \tag{1.2}$$

Note that the definition is independent of any notion of dimensions. The same can be said for Morrey spaces, which we define now. For $k > 0, 1 \leqslant p < \infty$ and $f \in L^1_{\mathrm{loc}}(\mu)$, the norm is given by

$$\begin{aligned}&\|f\|_{\mathcal{M}^p_1(k,\mu)}\\ &:= \sup\left\{\mu(B(x,kr))^{1/p-1}\|\chi_{B(x,r)}f\|_{L^1(\mu)} \,:\, x \in X,\, r > 0,\, \mu(B(x,r)) > 0\right\},\end{aligned}$$

where $\chi_{B(x,r)}$ denotes the characteristic function of the ball $B(x,r)$.

We will prove here that I_α satisfies weak and strong type estimates on Morrey spaces. Our main results are:

Theorem 1.1. *If $1 < p < \infty$, $1 < s < \infty$, $0 < \alpha < \frac{1}{p}$ and $\frac{1}{s} = \frac{1}{p} - \alpha$, then there exists $C > 0$ such that*

$$\mu\{x \in B(a,r) : I_\alpha f(x) > \gamma\} \leqslant C\mu(B(a,6r))^{1-1/p}\left(\frac{\|f\|_{\mathcal{M}^p_1(2,\mu)}}{\gamma}\right)^{s/p}$$

for all positive μ-measurable functions f.

Theorem 1.2. *If $1 < q \leqslant p < \infty$, $1 < s < \infty$, $0 < \alpha < \frac{1}{p}$, $\frac{q}{p} = \frac{t}{s}$ and $\frac{1}{s} = \frac{1}{p} - \alpha$, then there exists $C > 0$ such that*

$$\|I_\alpha f\|_{\mathcal{M}^s_t(6,\mu)} \leqslant C\|f\|_{\mathcal{M}^p_q(2,\mu)}$$

for all positive μ-measurable functions f.

It hardly looks likely to replace $2d(x,y)$ with $d(x,y)$ in the definition of fractional integral operators and have the similar results according to the example in [9, Section 2]. The proof is a future work.

2 Main Results

We define, for $k > 0$, the centered maximal operator

$$M_k f(x) := \sup_{r>0} \frac{1}{\mu(B(x,kr))}\int_{B(x,r)} |f(y)|\, d\mu(y) \quad (x \in \mathrm{supp}(\mu)).$$

For the maximal operator M_2, we prove the following boundedness property on Morrey spaces.

Theorem 2.1. *For any $\gamma > 0$, any positive μ-measurable function k and any ball $B(a,r)$,*

$$\mu\{x \in B(a,r) : M_2 f(x) > \gamma\} \leqslant 4\frac{\mu(B(a,6r))^{1-1/p}}{\gamma}\|f\|_{\mathcal{M}^p_1(2,\mu)}.$$

Proof. We actually prove the estimate

$$\mu\{x \in B(a,r) : M_2 f(x) > 2\gamma\} \leqslant 2\frac{\mu(B(a,6r))^{1-1/p}}{\gamma}\|f\|_{\mathcal{M}^p_1(2,\mu)}. \tag{2.1}$$

Once we prove that

$$\mu\{x \in B(a,r) : M_2[\chi_{B(a,3r)}f](x) > \gamma\} \leqslant \frac{\mu(B(a,6r))^{1-1/p}}{\gamma}\|f\|_{\mathcal{M}^p_1(2,\mu)} \tag{2.2}$$

and

$$\mu\{x \in B(a,r) : M_2[\chi_{X\setminus B(a,r)}f](x) > \gamma\} \leqslant \frac{\mu(B(a,6r))^{1-1/p}}{\gamma}\|f\|_{\mathcal{M}^p_1(2,\mu)}, \tag{2.3}$$

then estimate (2.1) follows automatically. Estimate (2.2) follows from the weak-$L^1(\mu)$ boundedness of M_2 (see [8, 11]).

Denote by $\mathcal{B}(\mu)$ the set of all balls with positive μ-measure. A geometric observation shows that

$$M_2[\chi_{X\setminus B(a,3r)}f](x) \leqslant \sup_{\substack{B\in\mathcal{B}(\mu),\, B\cap B(a,r)\neq\varnothing,\\ B\cap(X\setminus B(a,3r))\neq\varnothing}} \frac{1}{\mu(2B)}\int_B |f(y)|\,d\mu(y).$$

Let B be a ball which intersects both $B(a,r)$ and $X\setminus B(a,3r)$. The ball B engulfs $B(a,r)$ if we double the radius of B. Thus,

$$\begin{aligned}
&\mu(B(a,6r))^{1/p-1}\mu\{x \in B(a,r) : M_2[\chi_{B(a,3r)}f](x) > \gamma\}\\
&\leqslant \mu(B(a,r))^{1/p-1}\sup_{\substack{B\in\mathcal{B}(\mu),\, B\cap B(a,r)\neq\varnothing,\\ B\cap(X\setminus B(a,3r))\neq\varnothing}} \frac{1}{\mu(2B)}\int_B |f(y)|\,d\mu(y)\\
&\leqslant \sup_{\substack{B\in\mathcal{B}(\mu),\, B\cap B(a,r)\neq\varnothing,\\ B\cap(X\setminus B(a,3r))\neq\varnothing}} \frac{\mu(2B)^{1/p}}{\mu(2B)}\int_B |f(y)|\,d\mu(y)\\
&\leqslant \|f\|_{\mathcal{M}^p_1(2,\mu)}.
\end{aligned}$$

Thus, (2.3) follows. □

Analogously, the following inequality holds:

Theorem 2.2. *Let $1 < q \leqslant p < \infty$. Then there exists $C > 0$ such that*

$$\|M_2 f\|_{\mathcal{M}^p_q(6,\mu)} \leqslant C\|f\|_{\mathcal{M}^p_q(2,\mu)}$$

for all positive μ-measurable functions f.

The proof of Theorem 2.2 being similar to that of Theorem 2.1, we skip the proof, which is based on the $L^q(\mu)$-boundedness of M_2 established in [8].

Next, we prove a Hedberg type estimate [4].

Theorem 2.3. *If $1 < p < \infty$ and $0 < \alpha < \frac{1}{p}$, then there exists $C > 0$ such that*

$$|I_\alpha f(x)| \leqslant C M_2 f(x)^{1-p\alpha} \|f\|^{p\alpha}_{\mathcal{M}^p_1(2,\mu)} \quad (x \in X)$$

for all positive μ-measurable functions.

Proof. Let $x \in X$ be fixed. We define

$$R_k(x) := \inf\left(\{R > 0 : \mu(B(x, 2R)) > 2^k\} \cup \{\infty\}\right).$$

Then, we have

$$\begin{aligned}
&|I_\alpha f(x)| \\
&\leqslant \sum_{k=-\infty}^{\infty} \lim_{\varepsilon \downarrow 0} \int_{B(x,R_k(x))\setminus B(x,R_{k-1}(x))} \frac{|f(y)|}{\mu(B(x, 2d(x,y)+\varepsilon))^{1-\alpha}}\, d\mu(y) \\
&\leqslant \sum_{k=-\infty}^{\infty} \lim_{\varepsilon \downarrow 0} \int_{B(x,R_k(x))\setminus B(x,R_{k-1}(x))} \frac{|f(y)|}{\mu(B(x, 2R_{k-1}(x)+\varepsilon))^{1-\alpha}}\, d\mu(y) \\
&= \sum_{k=-\infty}^{\infty} \lim_{\varepsilon \downarrow 0} \frac{1}{\mu(B(x, 2R_{k-1}(x)+\varepsilon))^{1-\alpha}} \int_{B(x,R_k(x))\setminus B(x,R_{k-1}(x))} |f(y)|\, d\mu(y) \\
&\leqslant \sum_{k\in\mathbb{Z}; R_{k-1}(x) < R_k(x)} \lim_{\varepsilon \downarrow 0} \frac{1}{\mu(B(x, 2R_{k-1}(x)+\varepsilon))^{1-\alpha}} \int_{B(x,R_k(x))} |f(y)|\, d\mu(y).
\end{aligned}$$

The condition $R_{k-1}(x) < R_k(x)$ means that

$$2^{k-1} < \mu(B(x, 2R_{k-1}(x)+\varepsilon)) \leqslant 2^k$$

for each $\varepsilon \in (0, R_k(x) - R_{k-1}(x))$. Therefore

$$\begin{aligned}
|I_\alpha f(x)| &\leqslant C \sum_{k=-\infty}^{\infty} 2^{k\alpha} \min\left(M_2 f(x), 2^{-k/p} \|f\|_{\mathcal{M}^p_1(2,\mu)}\right) \\
&\leqslant C M f(x)^{1-p\alpha} \|f\|^{p\alpha}_{\mathcal{M}^p_1(2,\mu)}.
\end{aligned}$$

Thus, the estimate is proved. □

Now we prove Theorem 1.1.

Proof. For $|I_\alpha f(x)| > \gamma$, Theorem 2.3 gives us

$$M_2 f(x) > \left(\frac{\gamma}{C\|f\|^{p\alpha}_{\mathcal{M}^p_1(2,\mu)}}\right)^{1/(1-p\alpha)}.$$

Hence, by applying Theorem 2.1, we obtain

$$\begin{aligned}
&\mu\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\} \\
&\leqslant \mu\left\{x \in B(a,r) : M_2 f(x) > \left(\frac{\gamma}{C\|f\|^{p\alpha}_{\mathcal{M}^p_1(2,\mu)}}\right)^{1/(1-p\alpha)}\right\} \\
&\leqslant C\mu(B(a,6r))^{1-1/p}\|f\|_{\mathcal{M}^p_1(2,\mu)}\left(\frac{\|f\|^{p\alpha}_{\mathcal{M}^p_1(2,\mu)}}{\gamma}\right)^{1/(1-p\alpha)} \\
&\leqslant C\mu(B(a,6r))^{1-1/p}\frac{\|f\|^{1+\alpha s}_{\mathcal{M}^p_1(2,\mu)}}{\gamma^{s/p}} \\
&\leqslant C\mu(B(a,6r))^{1-1/p}\left(\frac{\|f\|_{\mathcal{M}^p_1(2,\mu)}}{\gamma}\right)^{s/p}.
\end{aligned}$$

Thus, the proof is complete. □

Theorem 1.2 can be proved in a similar way by using Theorem 2.3.

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