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ESTIMATES FOR THE WIDTHS OF CLASSES OF PERIODIC FUNCTIONS OF SEVERAL VARIABLES – I

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Abstract. We establish estimates sharp in order for the Kolmogorov and linear widths of the classes $\mathrm{B}^{s\,m}_{p\,q}(\mathbb{T}^k)$ and $\mathrm{L}^{s\,m}_{p\,q}(\mathbb{T}^k)$ of Nikol'skii-Besov, Lizorkin-Triebel types respectively, in the space $L_r(\mathbb{T}^k)$ for a certain range of the parameters s, p, q, r, and m.

1 Introduction

Let X be a Banach space equipped with the norm $\| \cdot \,|\, X \|$ and F be a centrally symmetric set in X; $N \in \mathbb{N}$. Then the quantity

$$d_N(F, X) = \inf_{\{g_\iota\}_{\iota=1}^N \subset X} \sup_{f \in F} \inf_{\{c_\iota\}_{\iota=1}^N \subset \mathbb{C}} \|f - \sum_{\iota=1}^N c_\iota g_\iota \,|\, X\| \tag{1}$$

is called the Kolmogorov N-width of F in X and the quantity

$$\lambda_N(F, X) = \inf_A \sup_{f \in F} \|f - Af \,|\, X\| \tag{2}$$

(where the inf is taken over all linear operators $A : X \to X$ such that $rank(A) \le N$) is called the linear N-width of F in X. Recall that widths (1) and (2) were introduced by A.N. Kolmogorov [10], by V.M. Tikhomirov [22] respectively. Many papers are devoted to calculating (mainly in the one-dimensional case) and estimating those widths of various function classes in various function spaces. Some of them containing detailed historical comments are cited in books [11], [21]; see also Remark 2 below for more references and comments.

In the present paper we consider widths (1) and (2) with the classes $\mathrm{B}^{s\,m}_{p\,q}(\mathbb{T}^k)$ and $\mathrm{L}^{s\,m}_{p\,q}(\mathbb{T}^k)$ as F and the space $L_r(\mathbb{T}^k)$ as X.

Below we use the following notation. Let $k \in \mathbb{N}$, $\mathrm{z}_k = \{1, \ldots, k\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}_+ = (0, +\infty)$; for a finite set Y we denote by $|Y|$ the number of its elements. For $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in \mathbb{R}^k$, put $xy = x_1y_1 + \ldots + x_ky_k$, $|x| = |x_1| + \ldots + |x_k|$, $\|x\| = \sqrt{xx}$, $|x|_\infty = \max\{\,|x_j| \,:\, j \in \mathrm{z}_k\}$; $x \le y$ $(x < y)$ $\Leftrightarrow$ $x_j \le y_j$ $(x_j < y_j)$ for all $j \in \mathrm{z}_k$.

Let $\widetilde{L}_p = L_p(\mathbb{T}^k)$ $(1 \le p < \infty)$ be the space of functions $f : \mathbb{T}^k \to \mathbb{C}$, whose pth-power is integrable on the k-dimensional torus $\mathbb{T}^k = (\mathbb{R}/\mathbb{Z})^k$, with the norm

$$\| f \,|\, \widetilde{L}_p \,\| = \left(\int_{\mathbb{T}^k} |f(x)|^p dx \right)^{1/p};$$

and let

$$\hat{g}(\xi) = \int_{\mathbb{T}^k} g(x) e^{-2\pi i\, \xi x} dx, \quad \xi \in \mathbb{Z}^k,$$

be the trigonometric Fourier coefficients of a function $g \in \widetilde{L}_1$.

Let $J \neq \varnothing$ be at most countable set and $\ell_q(J)$ $(1 \le q \le \infty)$ be the space of all numerical sequences $(c_j) = (c_j)_{j \in J}$ with the finite norm

$$\| \,(c_j) \,|\, \ell_q(J) \,\| = \left(\sum_{j \in J} |c_j|^q \right)^{1/q}, \quad 1 \le q < \infty,$$

$$\| \,(c_j) \,|\, \ell_\infty(J) \,\| = \sup_{j \in J} |c_j|.$$

Furthermore, we fix $n \in \mathbb{N}$, $n \le k$, and a multi-index $m = (m_1, ..., m_n) \in \mathbb{N}^n$ with $m_1 + \cdots + m_n = k$ ($m = k$ when $n = 1$ and $m = \mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^k$ when $n = k$). We represent $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ in the form $x = (x^1, \ldots, x^n)$, where $x^\nu = (x_{j_{\nu-1}+1}, ..., x_{j_\nu}) \in \mathbb{R}^{m_\nu}$, $j_0 = 0, j_\nu = m_1 + ... + m_\nu$, $\nu \in \mathrm{z}_n$.

For brevity, we put $\ell_q \equiv \ell_q(\mathbb{N}_0^n)$ and $\| \cdot \,|\, \ell_q \,\| \equiv \| \cdot \,|\, \ell_q(\mathbb{N}_0^n) \,\|$ as well as $\ell_q^M \equiv \ell_q(\mathrm{z}_M)$ and $\| \cdot \,|\, \ell_q^M \,\| \equiv \| \cdot \,|\, \ell_q^M(\mathrm{z}_M) \,\|$ for $M \in \mathbb{N}$.

By $\ell_q(\widetilde{L}_p) \equiv \ell_q(L_p(\mathbb{T}^k))$ and $\widetilde{L}_p(\ell_q) \equiv L_p(\mathbb{T}^k; \ell_q)$ we denote the spaces of sequences of functions $(g_\alpha(x)) = (g_\alpha(x))_{\alpha \in \mathbb{N}_0^n}$, $x \in \mathbb{T}^k$, with the finite norms

$$\| \,(g_\alpha(x)) \,|\, \ell_q(\widetilde{L}_p) \| = \| \,(\| \,g_\alpha \,|\, \widetilde{L}_p \,\|) \,|\, \ell_q \,\|,$$

$$\| \,(g_\alpha(x)) \,|\, \widetilde{L}_p(\ell_q) \| = \| \,\| \,(g_\alpha(\cdot)) \,|\, \ell_q \,\| \,|\, \widetilde{L}_p \,\|$$

respectively.

We consider infinitely differentiable functions $\eta_0^\nu : \mathbb{R}^{m_\nu} \to \mathbb{R}$ $(\nu \in \mathrm{z}_n)$ such that

$$0 \le \eta_0^\nu(\xi^\nu) \le 1, \;\; \xi^\nu \in \mathbb{R}^{m_\nu};$$

$$\eta_0^\nu(\xi^\nu) = 1, \;\; \text{when } |\xi^\nu|_\infty \le 1;$$

$$\eta_0^\nu(\xi^\nu) = 0, \;\; \text{when } |\xi^\nu|_\infty \ge 3/2;$$

and put

$$\eta^\nu(\xi^\nu) = \eta_0^\nu(2^{-1}\xi^\nu) - \eta_0^\nu(\xi^\nu),$$

$$\eta_j^\nu(\xi^\nu) = \eta^\nu(2^{-j+1}\xi^\nu), \;\; j \in \mathbb{N}.$$

Then the system

$$\eta = \{\eta_\alpha(\xi) = \prod_{\nu \in \mathrm{z}_n} \eta_{\alpha_\nu}^\nu(\xi^\nu), \;\; \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n\}$$

forms a smooth diadic (product-type) resolution of unity. We introduce the operators $\widetilde{\Delta}^{\eta}_{\alpha} : \widetilde{L}_1 \to \widetilde{L}_1$, $\alpha \in \mathbb{N}^n_0$, as follows: for a given $g \in \widetilde{L}_1$,

$$\widetilde{\Delta}^{\eta}_{\alpha}(g, x) = \sum_{\xi \in \mathbb{Z}^k} \eta_\alpha(\xi) \hat{g}(\xi) e^{2\pi i \xi x}.$$

Definition 1. *Let $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_+$, $1 \le p < \infty$, $1 \le q \le \infty$. Then*

(B) *the space $\widetilde{B}^{s\,m}_{p\,q} = B^{s\,m}_{p\,q}(\mathbb{T}^k)$ of Nikol'skii-Besov type consists of all functions $f \in \widetilde{L}_p$ for which the norm*

$$\| f \,|\, \widetilde{B} \| \;=\; \| \big(2^{\alpha s}\, \widetilde{\Delta}^{\eta}_{\alpha}(f, x) \big) \,|\, \ell_q(\widetilde{L}_p) \, \|$$

is finite;

(L) *the space $\widetilde{L}^{s\,m}_{p\,q} = L^{s\,m}_{p\,q}(\mathbb{T}^k)$ of Lizorkin-Triebel type consists of all functions $f \in \widetilde{L}_p$ for which the norm*

$$\| f \,|\, \widetilde{L} \| \;=\; \| \big(2^{\alpha s}\, \widetilde{\Delta}^{\eta}_{\alpha}(f, x) \big) \,|\, \widetilde{L}_p(\ell_q) \, \|$$

is finite.

The unit balls

$$\widetilde{\mathrm{B}}^{s\,m}_{p\,q} = \mathrm{B}^{s\,m}_{p\,q}(\mathbb{T}^k) = \{\, f \in \widetilde{B}^{s\,m}_{p\,q} \quad | \quad \| f \,|\, \widetilde{B} \| \le 1 \,\}$$

and

$$\widetilde{\mathrm{L}}^{s\,m}_{p\,q} = \mathrm{L}^{s\,m}_{p\,q}(\mathbb{T}^k) = \{\, f \in \widetilde{L}^{s\,m}_{p\,q} \quad | \quad \| f \,|\, \widetilde{L} \| \le 1 \,\}$$

of these spaces will be called the Nikol'skii-Besov classes, Lizorkin-Triebel classes respectively.

Remark 1. *For $n = 1$, the spaces $\widetilde{B}^{s}_{p\,q} \equiv \widetilde{B}^{s\,k}_{p\,q}$ and $\widetilde{L}^{s}_{p\,q} \equiv \widetilde{L}^{s\,k}_{p\,q}$ coincide with classical (periodic) Nikol'skii-Besov spaces, Lizorkin-Triebel spaces respectively [13], [19, Ch.3]; for $n = k$, the spaces $M\widetilde{B}^{s}_{p\,q} \equiv \widetilde{B}^{s\,\mathbf{1}}_{p\,q}$ and $M\widetilde{L}^{s}_{p\,q} \equiv \widetilde{L}^{s\,\mathbf{1}}_{p\,q}$ are the spaces with mixed smoothness (proper) (see [13], [1], [20], [19, Ch.2]). Next, $\widetilde{L}^{s}_{p\,2}$ is the Sobolev space $\widetilde{W}^{s}_{p}$, and $M\widetilde{L}^{s}_{p\,2}$ is the corresponding space $M\widetilde{W}^{s}_{p}$ of functions with dominating mixed derivative belonging to $\widetilde{L}_p$ when $1 < p < \infty$; the spaces $\widetilde{L}^{s}_{1\,2}$ and $M\widetilde{L}^{s}_{1\,2}$ are somewhat narrower than the spaces $\widetilde{W}^{s}_{1}$, $M\widetilde{W}^{s}_{1}$ respectively; $\widetilde{H}^{s}_{p} \equiv \widetilde{B}^{s}_{p\,\infty}$ is the Nikol'skii space, and $M\widetilde{H}^{s}_{p} \equiv M\widetilde{B}^{s}_{p\,\infty}$ is the corresponding space of functions with dominating mixed difference belonging to $\widetilde{L}_p$.*

The study of the spaces $\widetilde{B}^{s\,m}_{p\,q}$ and $\widetilde{L}^{s\,m}_{p\,q}$ in more general context ($1 \le n \le k$) and of some other similar function spaces started in the 1980s (see [19, Ch.2]).

The current state of a number of aspects in the theory of these spaces is described in survey [18]. In particular, these spaces do not depend on the choice of the system η, and the norms defined by different such systems (and even more general) are equivalent.

2 Main results

In this section we state and discuss estimates sharp in order for the Kolmogorov and linear widths of the Nikol'skii-Besov class $\widetilde{\mathrm{B}}^{s\,m}_{p\,q}$ and the Lizorkin-Triebel class $\widetilde{\mathrm{L}}^{s\,m}_{p\,q}$ in space $\widetilde{L}_r$ for a certain range of the parameters of these classes and of the space.

For $s \in \mathbb{R}^n_+$, $m \in \mathbb{N}^n$, $a \in \mathbb{R}$, and $1 < p < \infty$, we define the following numbers: $\sigma_\nu = s_\nu/m_\nu$, $\nu \in \mathrm{z}_n$; $p_* = \min\{p, 2\}$, $p^* = \max\{p, 2\}$, $p' = p/(p-1)$; $a_+ = \max\{a, 0\}$. Without loss of generality, we assume that for some $\omega \in \mathrm{z}_n$, $\sigma \equiv \min\{\sigma_\nu : \nu \in \mathrm{z}_n\} = \sigma_1 = ... = \sigma_\omega < \sigma_\nu$, $\nu \in \mathrm{z}_n \setminus \mathrm{z}_\omega$. Below, $\log \equiv \log_2$, and for functions $F : \mathbb{R}_+ \to \mathbb{R}_+$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ we write $F(u) \ll H(u)$ as $u \to \infty$, if there exists a constant $C = C(F, H) > 0$ such that $F(u) \le CH(u)$ for $u \ge u_0 > 0$, and $F(u) \asymp H(u)$ if $F(u) \ll H(u)$ and $H(u) \ll F(u)$ simultaneously.

The main results are as follows.

Theorem 1. I. *Let* $1 < r \le p < \infty$; $1 \le q \le \infty$; $s \in \mathbb{R}^n_+$. *Then*

$$d_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q}, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma} (\log^{\omega-1} N)^{(\frac{1}{p_*}-\frac{1}{q})_+};$$

$$d_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q}, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma} (\log^{\omega-1} N)^{(\frac{1}{2}-\frac{1}{q})_+}.$$

II. *Let* $1 < p < r \le 2$; $1 \le q \le \infty$; $s \in \mathbb{R}^n_+$ *and* $\sigma > \frac{1}{p} - \frac{1}{r}$. *Then*

$$d_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q}, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{p}+\frac{1}{r}} (\log^{\omega-1} N)^{(\frac{1}{r}-\frac{1}{q})_+};$$

$$d_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q}, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{p}+\frac{1}{r}}.$$

III ∪IV. *Let* $1 < p \le 2 < r < \infty$; $1 \le q \le \infty$; $s \in \mathbb{R}^n_+$ *and* $\sigma > \frac{1}{p}$. *Then*

$$d_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q}, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{p}+\frac{1}{2}} (\log^{\omega-1} N)^{(\frac{1}{2}-\frac{1}{q})_+};$$

$$d_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q}, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{p}+\frac{1}{2}}.$$

V. *Let* $2 \le p \le r < \infty$; $1 \le q \le \infty$; $s \in \mathbb{R}^n_+$ *and* $\sigma > \sigma(p, r) \equiv (\frac{1}{p} - \frac{1}{r})/(1 - \frac{2}{r})$. *Then*

$$d_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q}, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma} (\log^{\omega-1} N)^{(\frac{1}{2}-\frac{1}{q})_+};$$

$$d_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q}, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma} (\log^{\omega-1} N)^{(\frac{1}{2}-\frac{1}{q})_+}.$$

Theorem 2. I. *Let* $1 < r \le p < \infty$; $1 \le q \le \infty$*;* $s \in \mathbb{R}^n_+$. *Then*

$$\lambda_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma} (\log^{\omega-1} N)^{(\frac{1}{p_*}-\frac{1}{q})_+};$$

$$\lambda_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma} (\log^{\omega-1} N)^{(\frac{1}{2}-\frac{1}{q})_+}.$$

II. *Let* $1 < p < r \le 2$; $1 \le q \le \infty$*;* $s \in \mathbb{R}^n_+$ *and* $\sigma > \frac{1}{p} - \frac{1}{r}$. *Then*

$$\lambda_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{p}+\frac{1}{r}} (\log^{\omega-1} N)^{(\frac{1}{r}-\frac{1}{q})_+};$$

$$\lambda_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{p}+\frac{1}{r}}.$$

III. *Let* $2 \le r < \infty$, $1 \le \frac{1}{p} + \frac{1}{r}$; $1 \le q \le \infty$; $s \in \mathbb{R}^n_+$ *and* $\sigma > \frac{1}{p}$. *Then*

$$\lambda_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{p}+\frac{1}{2}} (\log^{\omega-1} N)^{(\frac{1}{2}-\frac{1}{q})_+};$$

$$\lambda_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{p}+\frac{1}{2}}.$$

IV. *Let* $1 < p \le 2$, $\frac{1}{p} + \frac{1}{r} < 1$; $1 \le q \le r$; $s \in \mathbb{R}^n_+$ *and* $\sigma > 1 - \frac{1}{r}$. *Then*

$$\lambda_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{2}+\frac{1}{r}} (\log^{\omega-1} N)^{(\frac{1}{r}-\frac{1}{q})_+};$$

$$\lambda_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{2}+\frac{1}{r}} (\log^{\omega-1} N)^{(\frac{1}{r}-\frac{1}{q})_+}.$$

V. *Let* $2 \le p < r < \infty$, $1 \le q \le \infty$; $s \in \mathbb{R}^n_+$ *and* $\sigma > \frac{1}{p} - \frac{1}{r}$. *Then*

$$\lambda_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{p}+\frac{1}{r}} (\log^{\omega-1} N)^{(\frac{1}{r}-\frac{1}{q})_+};$$

$$\lambda_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \asymp \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{p}+\frac{1}{r}}.$$

Remark 2. *Here we mention previous results which are directly related to Theorems 1 and 2; moreover, we point out only those of them which are related to the case* $1 < p, r < \infty$.

A) First we concentrate on results on the Kolmogorov widths:

a) note that estimates sharp in order for the Kolmogorov widths of the classes $\widetilde{\mathrm{W}}^s_p \equiv \widetilde{\mathrm{L}}^{s\,k}_{p\,2}$, $\widetilde{\mathrm{H}}^s_p \equiv \widetilde{\mathrm{B}}^{s\,k}_{p\,\infty}$, $\mathrm{M}\widetilde{\mathrm{W}}^s_p \equiv \widetilde{\mathrm{L}}^{s\,\mathbf{1}}_{p\,2}$ *for all* $1 < p, r < \infty$, *as well as of class* $\mathrm{M}\widetilde{\mathrm{H}}^s_p \equiv \widetilde{\mathrm{B}}^{s\,\mathbf{1}}_{p\,\infty}$ *when* $r \geq p^*$ *or* $p \geq r^*$ *are well known: their complete exposition including the history of the question is given in books [20, Ch.III], [21, Ch.II, §4; Ch.III, §4];*

b) estimates sharp in order for $d_N(\widetilde{\mathrm{B}}^{s\,\mathbf{1}}_{p\,q}, \widetilde{L}_r)$ *were obtained by A.S. Romanyuk for the case* `II` *in [14], for the cases* `III` $\cup$ `IV` *and* `V` *in [15], by E.M. Galeev [6] and A.S. Romanyuk [17] for the cases* $2 \leq r \leq p$ $(1 \leq q \leq \infty)$ *and* $r < 2 \leq p$ $(2 \leq q \leq \infty)$*;*

c) finally, note that estimates in Theorem 1 related to the class $\widetilde{\mathrm{B}}^{s\,m}_{p\,q}$ *were partly announced in [3].*

B) Now we turn our attention to previous results on the linear widths:

a) estimates sharp in order for the linear widths of the classes $\widetilde{\mathrm{W}}^s_p$ *and* $\widetilde{\mathrm{H}}^s_p$ *for all* p, r *are also well known; for detailed exposition see [21, Ch. II, §4];*

b) E.M. Galeev found estimates sharp in order for $\lambda_N(\mathrm{M}\widetilde{\mathrm{W}}^s_p, \widetilde{L}_r)$ *for all* $1 < p$, $r < \infty$ *and for* $\lambda_N(\mathrm{M}\widetilde{\mathrm{H}}^s_p, L_r)$ *in the cases* `II`, `III`*, as well as when* $p \geq r^*$*; see [5] for details;*

c) A.S.Romanyuk [16, 17] established estimates sharp in order for $\lambda_N(\widetilde{\mathrm{B}}^{s\,\mathbf{1}}_{p\,q}, \widetilde{L}_r)$ *in the following cases:* $2 \leq r < p$ $(1 \leq q \leq \infty)$*;* $r < 2 \leq p$ $(2 \leq q \leq \infty)$*;* $p \leq r \leq 2$ $(1 \leq q \leq \infty)$*;* $p \leq 2 \leq r < p'$ $(1 \leq q \leq \infty)$*;* $p \leq 2$, $p' < r$ $(2 \leq q \leq r)$*;* $2 \leq p < r$ $(2 \leq q \leq r)$*;*

d) finally, note that estimates in Theorem 2 related to the class $\widetilde{\mathrm{B}}^{s\,m}_{p\,q}$ *were partly announced in [2].*

3 Wavelet characterization of the spaces $\widetilde{B}^{s\,m}_{p\,q}$ and $\widetilde{L}^{s\,m}_{p\,q}$

Proofs of the main results in next sections will be based on embeddings and wavelet characterization for the spaces $\widetilde{B}^{s\,m}_{p\,q}$ and $\widetilde{L}^{s\,m}_{p\,q}$ and related theorem of Littlewood-Paley type.

In this section we list those results with relevant background.

Let $v = w^0 : \mathbb{R} \to \mathbb{R}$ and $w^1 : \mathbb{R} \to \mathbb{R}$ be a Meyer scaling function, a wavelet respectively [12, Ch.2, §12, Ch.3, §2]. They are defined as follows. Let $\vartheta(\tau)$ be an odd infinitely differentiable function equal to $\pi/4$ for $\tau > \pi/3$ and monotonic on $(-\pi/3, \pi/3)$. Next, let $\psi(\tau)$ be the even function defined by

$$\psi(\tau) = \begin{cases} \pi/4 + \vartheta(\tau - \pi), & \text{if } \tau \in [2\pi/3, 4\pi/3]; \\ \pi/4 - \vartheta(\frac{\tau}{2} - \pi), & \text{if } \tau \in [4\pi/3, 8\pi/3]; \\ 0, & \text{if } \tau \in [0, 2\pi/3) \cup (8\pi/3, \infty). \end{cases} \tag{3}$$

Then

$$w^0(t) = \frac{1}{\pi} \int_0^{4\pi/3} \cos(t\tau) \cos(\psi(\tau)) d\tau \tag{4}$$

and

$$w^1(t) = \frac{1}{\pi} \int_{2\pi/3}^{8\pi/3} \cos((t - 1/2)\tau) \sin(\psi(\tau)) d\tau. \tag{5}$$

Next, put

$$\mathrm{E}^k = \mathrm{E}^k(0) = \{0, 1\}^k, \quad \mathrm{E}^k(1) = \mathrm{E}^k \setminus \{(0, \ldots, 0)\};$$

$$\Lambda(k,j) = \mathbb{Z}^k \cap [0, 2^j - 1]^k,\ j \in \mathbb{N}_0,$$

and define the functions $w^\iota : \mathbb{R}^k \to \mathbb{R}$ $(\iota = (\iota_1, \ldots, \iota_k) \in \mathrm{E}^k)$ as follows:

$$w^\iota(x) = w^{\iota_1}(x_1) \times \cdots \times w^{\iota_k}(x_k).$$

Furthermore, let

$$w^\iota_j(x) = w^\iota(2^j x) \quad (j \in \mathbb{N}_0).$$

Finally, we define the functions $\tilde{w}^\iota_{j\lambda} : \mathbb{T}^k \to \mathbb{R}$ as follows:

$$\tilde{w}^\iota_{j\lambda}(x) = 2^{jk/2}\widetilde{w}^\iota_j(x - 2^{-j}\lambda) \quad (\lambda \in \Lambda(k,j),\ j \in \mathbb{N}_0,\ \iota \in \mathrm{E}^k),$$

where $\widetilde{h} : \mathbb{T}^k \to \mathbb{C}$ is the periodization of a function $h : \mathbb{R}^k \to \mathbb{C}$:

$$\widetilde{h}(x) = \sum_{\xi \in \mathbb{Z}^k} h(\,x - \xi).$$

It is well known [12, ch.3] that the system of the Meyer wavelets

$$\widetilde{\mathcal{W}}_k = \{\tilde{w}^\iota_{j\lambda} \ \mid\ \lambda \in \Lambda(k,j),\ \iota \in \mathrm{E}^k(\mathrm{sign} j),\ j \in \mathbb{N}_0\}$$

forms a complete orthonormal system in $L_2(\mathbb{T}^k)$. Finally, we introduce the (m-multiple) system of wavelets

$$\widetilde{\mathcal{W}}_m \equiv \widetilde{\mathcal{W}}_{m_1} \otimes \cdots \otimes \widetilde{\mathcal{W}}_{m_n} \equiv \tag{6}$$

$$\{\tilde{w}^\iota_{\alpha\lambda}(x) = \tilde{w}^{\iota^1}_{\alpha_1\,\lambda^1}(x^1) \times \cdots \times \tilde{w}^{\iota^n}_{\alpha_n\,\lambda^n}(x^n) \ \mid\ \lambda \in \Lambda(m,\alpha),\ \iota \in \mathrm{E}^m(\alpha),\ \alpha \in \mathbb{N}_0^n\};$$

here $x \in \mathbb{T}^k$, $\mathrm{E}^m(\alpha) = \{\iota \in \mathrm{E}^k : \iota^\nu \in \mathrm{E}^{m_\nu}(\mathrm{sign}\,\alpha_\nu),\ \nu \in \mathrm{z}_n\}$, and $\Lambda(m,\alpha) = \{\lambda \in \mathbb{Z}^k \mid \lambda^\nu \in \Lambda(m_\nu, \alpha_\nu),\ \nu \in \mathrm{z}_n\}$.

We also introduce the operators $\widetilde{\Delta}^w_\alpha$ $(\alpha \in \mathbb{N}_0^n)$ by

$$\widetilde{\Delta}^w_\alpha(f,x) = \sum_{\iota \in \mathrm{E}^m(\alpha)} \sum_{\lambda \in \Lambda(m,\alpha)} \langle f, \tilde{w}^\iota_{\alpha\lambda}\rangle \tilde{w}^\iota_{\alpha\lambda}(x), \text{ where } \langle f, \tilde{w}^\iota_{\alpha\lambda}\rangle = \int_{\mathbb{I}^k} f(x)\, \tilde{w}^\iota_{\alpha\lambda}(x)\, dx.$$

We need the following spaces $\widetilde{\mathtt{B}}^{s\,m}_{p\,q}$ and $\widetilde{\mathtt{L}}^{s\,m}_{p\,q}$ of numerical sequences that are closely related to $\widetilde{B}^{s\,m}_{p\,q}$, $\widetilde{L}^{s\,m}_{p\,q}$ respectively. Let $\chi_{\alpha\lambda} = \chi_{P(m,\alpha,\lambda)}$ be the characteristic function of the parallelepiped

$$P(m,\alpha,\lambda) = Q(m_1, \alpha_1, \lambda^1) \times \cdots \times Q(m_n, \alpha_n, \lambda^n),$$

$$Q(m_\nu, \alpha_\nu, \lambda^\nu) = \{\, x^\nu \in \mathbb{R}^\nu \ :\ 2^{\alpha_\nu} x^\nu - \lambda^\nu \in [\,0, 1\,]^{m_\nu}\,\} \quad (\alpha \in \mathbb{N}_0^n, \lambda \in \mathbb{Z}^k).$$

Definition 2. *Let $1 \le p,\, q \le \infty$ and $s \in \mathbb{R}^n_+$. For a numerical sequence $(a^\iota_{\alpha\lambda}) = (a^\iota_{\alpha\lambda} \,|\, \alpha \in \mathbb{N}_0^n, \iota \in \mathrm{E}^m(\alpha), \lambda \in \Lambda(m,\alpha)\,)$, we introduce the following two norms:*

$$\|\,(a^\iota_{\alpha\lambda}) \,|\, \widetilde{\mathtt{B}}\,\| = \Bigg(\sum_{\alpha \in \mathbb{N}_0^n} \sum_{\iota \in \mathrm{E}^m(\alpha)} 2^{\alpha s\, q} \| \sum_{\lambda \in \Lambda(m,\alpha)} a^\iota_{\alpha\lambda} \chi_{\alpha\lambda}(\cdot) \,|\, L_p\,\|^q \Bigg)^{1/q} =$$

$$= \Big(\sum_{\alpha\in\mathbb{N}_0^n} \sum_{\iota\in\mathrm{E}^m(\alpha)} 2^{\alpha s\, q} 2^{-\alpha m q/p} \Big(\sum_{\lambda\in\Lambda(m,\alpha)} |\, a^{\iota}_{\alpha\lambda}|^p \Big)^{q/p} \Big)^{1/q}$$

and

$$\|\, (a^{\iota}_{\alpha\lambda}) \,|\, \widetilde{\mathtt{L}}\, \| = \| \Big(\sum_{\alpha\in\mathbb{N}_0^n} \sum_{\iota\in\mathrm{E}^m(\alpha)} 2^{\alpha s\, q} | \sum_{\lambda\in\Lambda(m,\alpha)} a^{\iota}_{\alpha\lambda}\chi_{\alpha\lambda}(\cdot)|^q \Big)^{1/q} \,|\, L_p \,\| =$$

$$= \| \Big(\sum_{\alpha\in\mathbb{N}_0^n} \sum_{\iota\in\mathrm{E}^m(\alpha)} 2^{\alpha s\, q} \sum_{\lambda\in\Lambda(m,\alpha)} |\, a^{\iota}_{\alpha\lambda}|^q \chi_{\alpha\lambda}(\cdot) \Big)^{1/q} \,|\, L_p \,\|$$

(with the natural modification for $p = \infty$ or $q = \infty$).

Then

$$\widetilde{\mathtt{B}}^{s\,m}_{p\,q} \equiv \{\, (a^{\iota}_{\alpha\lambda}) \,:\, \|\, (a^{\iota}_{\alpha\lambda}) \,|\, \widetilde{\mathtt{B}}\, \| < \infty \,\},$$

and

$$\widetilde{\mathtt{L}}^{s\,m}_{p\,q} \equiv \{\, (a^{\iota}_{\alpha\lambda}) \,:\, \|\, (a^{\iota}_{\alpha\lambda}) \,|\, \widetilde{\mathtt{L}}\, \| < \infty \,\}.$$

Now we are in position to formulate a theorem on characterization and representation of functions in the spaces $\widetilde{B}^{s\,m}_{p\,q}$ and $\widetilde{L}^{s\,m}_{p\,q}$ using the system $\widetilde{\mathcal{W}}_m$.

Theorem A. *Let $1 \le p, q \le \infty$ and $s \in \mathbb{R}^n_+$.*

`B`. *For a function $f \in \widetilde{L}_p$ the following conditions are equivalent:*

`(i)` *the function f belongs to the space $\widetilde{B}^{s\,m}_{p\,q}$;*

`(ii)` *the sequence of functions $(\, 2^{\alpha s}\widetilde{\Delta}^w_\alpha(f, x)\,)$ belongs to the space $\ell_q(\widetilde{L}_p)$; i.e., the quantity*

$$\|\, (2^{\alpha s}\, \widetilde{\Delta}^w_\alpha(f, x)) \;|\; \ell_q(\widetilde{L}_p)\, \|; \tag{7}$$

is finite;

`(iii)` *the numerical sequence $(\, 2^{\alpha m/2}\langle\, f, \widetilde{w}^{\iota}_{\alpha\lambda}\,\rangle\,)$ belongs to the space $\widetilde{\mathtt{B}}^{s\,m}_{p\,q}$; i.e., the quantity*

$$\|\, (2^{\alpha m/2}\langle\, f, \widetilde{w}^{\iota}_{\alpha\lambda}\,\rangle) \,|\, \widetilde{\mathtt{B}}\, \|. \tag{8}$$

is finite.

The functionals (33) and (34) are norms in the space $\widetilde{B}^{s\,m}_{p\,q}$ which are equivalent to the original norm $\|\, \cdot \;|\; \widetilde{B}\, \|$.

`L`. *For a function $f \in \widetilde{L}_p\,, p < \infty$ the following conditions are equivalent:*

`(i)` *the function f belongs to the space $\widetilde{L}^{s\,m}_{p\,q}$;*

`(ii)` *the sequence of functions $(\, 2^{\alpha s}\widetilde{\Delta}^w_\alpha(f, x)\,)$ belongs to the space $\widetilde{L}_p(\ell_q)$; i.e., the quantity*

$$\|\, (2^{\alpha s}\, \widetilde{\Delta}^w_\alpha(f, x)) \;|\; \widetilde{L}_p(\ell_q)\, \|; \tag{9}$$

is finite;

`(iii)` *the numerical sequence $(\, 2^{\alpha m/2}\langle\, f, \widetilde{w}^{\iota}_{\alpha\lambda}\,\rangle\,)$ belongs to the space $\widetilde{\mathtt{L}}^{s\,m}_{p\,q}$; i.e., the quantity*

$$\|\, (2^{\alpha m/2}\langle\, f, \widetilde{w}^{\iota}_{\alpha\lambda}\,\rangle) \,|\, \widetilde{\mathtt{L}}\, \|. \tag{10}$$

is finite.

The functionals (35) and (36) are norms in the space $\widetilde{L}^{s\,m}_{p\,q}$ which are equivalent to the original norm $\|\, \cdot \;|\; \widetilde{L}\, \|$.

We also need the following theorem of Littlewood-Paley type for wavelet decompositions with respect to the system $\widetilde{\mathcal{W}}_m$ and its corollary.

Theorem B. *Let $1 < p < \infty$. Then there exists a constant $C = C(v, m, p) > 0$ such that*

$$C^{-1} \| f \,|\, \widetilde{L}_p \| \le \| \, (\widetilde{\Delta}^w_\alpha(f, x)) \,|\, \widetilde{L}_p(\ell_2) \, \| \le C \, \| f \,|\, \widetilde{L}_p \|,$$

$$C^{-1} \| f \,|\, \widetilde{L}_p \| \le \| \sum_\alpha \sum_\iota \sum_\lambda |\langle f, \widetilde{w}^\iota_{\alpha\lambda} \rangle \, \widetilde{w}^\iota_{\alpha\lambda}|^2 \,|\, \widetilde{L}_{p/2} \, \|^{1/2} \le C \, \| f \,|\, \widetilde{L}_p \|.$$

for all functions $f \in \widetilde{L}_p$.

Corollary A. *Let $1 < p < \infty$. Then, for any function $f \in \widetilde{L}_p$*

$$\| f \,|\, \widetilde{L}_p \| \le C(v, m, p) \, \| \, (\widetilde{\Delta}^w_\alpha(f, x)) \,|\, \ell_{p_*}(\widetilde{L}_p) \, \|.$$

For proofs of Theorems A and B and of Corollary A and detailed comments on related results see [4].

Here we formulate embedding (of different metrics) theorem for spaces $\widetilde{B}^{s\,m}_{p\,q}$ and $\widetilde{L}^{s\,m}_{p\,q}$.

Theorem C. *Let $s = (s_1, ..., s_n)$, $\tau = (\tau_1, ..., \tau_n) \in \mathbb{R}^n_+$, $1 \le p < r \le \infty$, $1 \le q, u \le \infty$ and*

$$s_\nu - \tau_\nu = m_\nu \left(\frac{1}{p} - \frac{1}{r} \right), \quad \nu \in \mathrm{z}_n.$$

Then
(i) *the embedding*

$$B^{s\,m}_{p\,q}(\mathbb{T}^k) \hookrightarrow B^{\tau\,m}_{r\,u}(\mathbb{T}^k) \tag{11}$$

holds if and only if $q \le u$;
(ii) *the embedding*

$$L^{s\,m}_{p\,q}(\mathbb{T}^k) \hookrightarrow B^{\tau\,m}_{r\,u}(\mathbb{T}^k) \tag{12}$$

holds if and only if $p \le u$.

Moreover, let $r < \infty$. Then
(iii) *the embedding*

$$B^{s\,m}_{p\,q}(\mathbb{T}^k) \hookrightarrow L^{\tau\,m}_{r\,u}(\mathbb{T}^k) \tag{13}$$

holds if and only if $q \le r$;
(iv) *the embedding*

$$L^{s\,m}_{p\,q}(\mathbb{T}^k) \hookrightarrow L^{\tau\,m}_{r\,u}(\mathbb{T}^k) \tag{14}$$

holds for all $1 \le q, u \le \infty$.

In the cases of $n = 1$ and $n = k$, Theorem C and its counterpart for the spaces $B^{s\,m}_{p\,q}(\mathbb{R}^k)$ and $L^{s\,m}_{p\,q}(\mathbb{R}^k)$ are well known (see [13] - [1], [18]). In the case of $1 < n < k$, the nonperiodic counterpart of Theorem C is proved in [8]. Concerning Theorem C in case of $1 < n < k$, see Remark 5.1 in [4].

When deriving estimates of both theorems and estimating dimensions of certain finite-dimensional linear spans we will apply the following lemma, which is a modification of Lemmas B, C, and D from [20] for our case; the proof can be carried out in a similar way.

Lemma A. *Let $\beta, \gamma \in \mathbb{R}^n_+$ be such that $\beta_\nu = \gamma_\nu$ for $\nu \in \mathrm{z}_\omega$ and $\beta_\nu > \gamma_\nu$ $\nu \in \mathrm{z}_n \setminus \mathrm{z}_\omega$; and let $L > 0$. Then the following relations hold:*

$$\mathcal{I}_L^{\beta,\gamma}(u) \equiv \sum_{\alpha \in \mathbb{N}_0^n : \alpha\gamma > u} 2^{-L\alpha\beta} \asymp 2^{-Lu} u^{\omega-1} \qquad \textit{as} \quad u \to +\infty; \tag{15}$$

$$\mathcal{J}_L^{\gamma,\beta}(u) \equiv \sum_{\alpha \in \mathbb{N}_0^n : \alpha\beta \le u} 2^{L\alpha\gamma} \asymp 2^{Lu} u^{\omega-1} \qquad \textit{as} \quad u \to +\infty. \tag{16}$$

4 Upper bounds in Theorems 1 and 2: simple cases

Together with $s = (s_1, ..., s_n)$, $m = (m_1, ..., m_n)$ and $\sigma_\nu = \frac{s_\nu}{m_\nu}$, $\nu \in \mathrm{z}_n$, we consider following vectors and numbers:

$$\bar{s} = s - m\left(\frac{1}{p} - \frac{1}{r}\right)_+; \quad \bar{\sigma}_\nu = \frac{\bar{s}_\nu}{m_\nu} = \sigma_\nu - \left(\frac{1}{p} - \frac{1}{r}\right)_+, \quad \nu \in \mathrm{z}_n;$$

$$\gamma = \frac{1}{\sigma} s = (m_1, ..., m_\omega, m_{\omega+1}\frac{\sigma_{\omega+1}}{\sigma}, ..., m_n\frac{\sigma_n}{\sigma}),$$
$$\bar{\gamma} = \frac{1}{\bar{\sigma}} \bar{s} = (m_1, ..., m_\omega, m_{\omega+1}\frac{\bar{\sigma}_\nu}{\bar{\sigma}}, ..., m_n\frac{\bar{\sigma}_n}{\bar{\sigma}}).$$

It is clear that

$$\bar{\sigma} \equiv \min\{\bar{\sigma}_\nu : \quad \nu \in \mathrm{z}_n\} = \sigma - \left(\frac{1}{p} - \frac{1}{r}\right)_+ = \bar{\sigma}_1 = .. = \bar{\sigma}_\omega < \bar{\sigma}_\nu, \quad \nu > \omega,$$

and

$$\bar{\gamma} \ge \gamma \ge m.$$

For vector $\bar{\gamma}$ we choose numbers σ'_ν, $\nu \in \mathrm{z}_n$, such that:

$$\sigma'_1 = ... = \sigma'_\omega = \bar{\sigma} < \sigma'_\nu < \bar{\sigma}_\nu, \qquad \nu > \omega,$$

then the vector

$$\gamma' \equiv (m_1, ..., m_\omega, m_{\omega+1}\frac{\sigma'_{\omega+1}}{\bar{\sigma}}, ..., m_n\frac{\sigma'_n}{\bar{\sigma}})$$

satisfies the inequality $\bar{\gamma} \ge \gamma' \ge m$.

Next by Lemma A, formula(16) with

$$\beta = \gamma', \quad \gamma = m, \quad L = 1, \tag{17}$$

we obtain the following relation

$$\sum_{\alpha\gamma' \le u} 2^{\alpha m} \asymp 2^u u^{\omega-1} \qquad \text{as} \quad u \to +\infty. \tag{18}$$

We begin with upper estimates in Theorem 1 for cases `I, II` and in Theorem 2 for cases `I, II, V`.

Cases I and II: for both classes $\widetilde{\mathrm{B}}^{s\,m}_{p\,q}$ and $\widetilde{\mathrm{L}}^{s\,m}_{p\,q}$ the required estimates follow by the estimates of approximation of these classes by wavelets of the system $\widetilde{\mathcal{W}}_m$ obtained before in [4]. For a function $f \in \widetilde{L}_r$ consider its "hyperbolic" partial Fourier sum with respect to $\widetilde{\mathcal{W}}_m$:

$$\widetilde{\mathcal{S}}^{w,\beta}_u(f,x) = \sum_{\alpha\beta \le u} \widetilde{\Delta}^w_\alpha(f,x)$$

(here $u > 0$ and $\beta \in \mathbb{R}^n_+$).

Then by Theorem 4.1 in [4] we have for $1 < r \le p < \infty$

$$\sup\{\| f - \widetilde{\mathcal{S}}^{w,\gamma'}_u(f) \,|\, \widetilde{L}_r\| \,|\, f \in \widetilde{\mathrm{B}}^{s\,m}_{p\,q}\} \asymp 2^{-\sigma u} u^{(\omega-1)(\frac{1}{p_*}-\frac{1}{q})}, \tag{19}$$

$$\sup\{\| f - \widetilde{\mathcal{S}}^{w,\gamma'}_u(f) \,|\, \widetilde{L}_r\| \,|\, f \in \widetilde{\mathrm{L}}^{s\,m}_{p\,q}\} \asymp 2^{-\sigma u} u^{(\omega-1)(\frac{1}{2}-\frac{1}{q})_+}, \tag{20}$$

and for $1 < p < r < \infty$

$$\sup\{\| f - \widetilde{\mathcal{S}}^{w,\gamma'}_u(f) \,|\, \widetilde{L}_r\| \,|\, f \in \widetilde{\mathrm{B}}^{s\,m}_{p\,q}\} \asymp 2^{-(\sigma-\frac{1}{p}+\frac{1}{r})u} u^{(\omega-1)(\frac{1}{r}-\frac{1}{q})_+}; \tag{21}$$

$$\sup\{\| f - \widetilde{\mathcal{S}}^{w,\gamma'}_u(f) \,|\, \widetilde{L}_r\| \,|\, f \in \widetilde{\mathrm{L}}^{s\,m}_{p\,q}\} \asymp 2^{-(\sigma-\frac{1}{p}+\frac{1}{r})u}. \tag{22}$$

The dimension $\delta(\gamma', u)$ of the linear span of $\{\widetilde{w}^{\iota}_{\alpha\lambda} :\ \lambda \in \Lambda(m,\alpha), \iota \in \mathrm{E}^m(\alpha), \alpha\gamma' \le u\}$ is of order $2^u u^{\omega-1}$ (see Remark 5.2. in [4]). Choose $u > 0$ such that $\delta(\gamma', u) \le N$ and $N \asymp 2^u u^{\omega-1}$. Therefore, from (19), (20) and definition (1) of the Kolmogorov width and (2) of the linear width it follows that in case I we obtain

$$d_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \le \lambda_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \ll \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma} (\log^{\omega-1} N)^{(\frac{1}{p_*}-\frac{1}{q})_+},$$

$$d_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \le \lambda_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \ll \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma} (\log^{\omega-1} N)^{(\frac{1}{2}-\frac{1}{q})_+},$$

and by (21), (22) and (1), (2) it follows that in case II we have

$$d_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \le \lambda_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \ll \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{p}+\frac{1}{r}} (\log^{\omega-1} N)^{(\frac{1}{r}-\frac{1}{q})_+},$$

$$d_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \le \lambda_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \ll \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{p}+\frac{1}{r}}.$$

Similarly by (21), (22) and (2) it follows that in case V we have the following upper bounds

$$\lambda_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \ll \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{p}+\frac{1}{r}} (\log^{\omega-1} N)^{(\frac{1}{r}-\frac{1}{q})_+};$$

$$\lambda_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \ll \left(\frac{\log^{\omega-1} N}{N}\right)^{\sigma-\frac{1}{p}+\frac{1}{r}}.$$

Thus, the required upper estimates in Theorem 1 for cases I and II and in Theorem 2 for cases I, II and V are proved.

Let us consider case $\mathtt{III}\cup\mathtt{IV}$ of Theorem 1. In virtue of Theorem C we have

$$\widetilde{B}^{s\,m}_{p\,q} \hookrightarrow \widetilde{B}^{\bar{s}\,m}_{2\,q},$$

$$\widetilde{L}^{s\,m}_{p\,q} \hookrightarrow \widetilde{L}^{\bar{s}\,m}_{2\,1},$$

(here $\bar{s} = s - m\big(\frac{1}{p} - \frac{1}{2}\big)$; note that according to the assumptions of Theorem 1 in case $\mathtt{III}\cup\mathtt{IV}$ $\bar{s} > 0$; therefore, these embeddings hold true). Hence, upper bounds in this case is reduced to corresponding upper estimates for case $\mathtt{V}$:

$$d_N(\widetilde{\mathrm{B}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \ll d_N(\widetilde{\mathrm{B}}^{\bar{s}\,m}_{2\,q},\, \widetilde{L}_r),$$

$$d_N(\widetilde{\mathrm{L}}^{s\,m}_{p\,q},\, \widetilde{L}_r) \ll d_N(\widetilde{\mathrm{L}}^{\bar{s}\,m}_{2\,1},\, \widetilde{L}_r).$$

Note that the assumption $\sigma > \frac{1}{p}$ for case $\mathtt{III}\cup\mathtt{IV}$ is equivalent to the assumption of case $\mathtt{V}$ with $\bar{s}$ and 2 replacing s, p respectively.

Concerning upper estimates in both Theorems 1 and 2 it remains to consider case $\mathtt{V}$ of Theorem 1 and cases $\mathtt{III}$ and $\mathtt{IV}$ of Theorem 2.

Proofs of the rest of upper bounds as well as proofs of all lower bounds in both Theorems 1 and 2 will be given in part II of this paper.

We conclude part I by constructing approximation operators $G^{N,w}$ and $H^{N,w}$ which will be useful when we shall estimate from above the widths of the classes $\widetilde{\mathrm{B}}^{s\,m}_{p\,q}$ and $\widetilde{\mathrm{L}}^{s\,m}_{p\,q}$ in the remaining cases.

First we recall the following well-known estimates for the widths of finite-dimensional sets which are due to B.S. Kashin [9] and E.D. Gluskin [7].

Let $\mathfrak{b}^M_p$ be the unit ball of ℓ^M_p, $1 \le N < M$.

$\mathtt{I}$. Let $2 \le p < r < \infty$, then

$$d_N(\mathfrak{b}^M_p, \ell^M_r) \asymp \min\{1, M^{2\sigma(p,r)/r} N^{-\sigma(p,r)}\}.$$

$\mathtt{II}$. Let $1 < p < 2 < r < \infty$ and $\frac{1}{r} + \frac{1}{p} \ge 1$, then

$$\lambda_N(\mathfrak{b}^M_p, \ell^M_r) \asymp \max\Big\{M^{\frac{1}{r}-\frac{1}{p}}, \min\{1, M^{\frac{1}{r}} N^{-\frac{1}{2}}\}\sqrt{1-\frac{N}{M}}\Big\}.$$

Hence we have the following facts:

$\mathtt{I}$. let $2 \le p < r < \infty$, then there exist an N–dimensional subspace $L^N \subset \mathbb{R}^M$ and a map $G^N_M : \mathbb{R}^M \to L^N$ such that for any $\mathrm{x} \in \mathbb{R}^M$

$$\|\,\mathrm{x} - G^N_M\mathrm{x}\,|\,\ell^M_r\| \ll d_N(\mathfrak{b}^M_p, \ell^M_r)\,\|\,\mathrm{x}\,|\,\ell^M_p\| \ll \|\,\mathrm{x}\,|\,\ell^M_p\|\, M^{2\sigma(p,r)/r} N^{-\sigma(p,r)}; \tag{23}$$

$\mathtt{II}$. let $1 < p < 2 < r < \infty$ and $\frac{1}{r} + \frac{1}{p} \ge 1$, then there exists a linear operator $H^N_M : \mathbb{R}^M \to \mathbb{R}^M$ such that the dimension of the subspace $H^N_M(\mathbb{R}^M)$ does not exceed N and for any $\mathrm{x} \in \mathbb{R}^M$

$$\|\,\mathrm{x} - H^N_M\mathrm{x}\,|\,\ell^M_r\| \ll \lambda_N(\mathfrak{b}^M_p, \ell^M_r)\,\|\,\mathrm{x}\,|\,\ell^M_p\| \ll$$

$$\ll \|\,\mathrm{x}\,|\,\ell^M_p\|\, \max\Big\{M^{\frac{1}{r}-\frac{1}{p}}, \min\{1, M^{\frac{1}{r}} N^{-\frac{1}{2}}\}\sqrt{1-\frac{N}{M}}\Big\}. \tag{24}$$

Let also $\mathrm{Id}_M : \mathbb{R}^M \to \mathbb{R}^M$ be the identity map.

Now let us construct operators $G^{N,w} : \widetilde{L}_r \to \widetilde{L}_r$ and $H^{N,w} : \widetilde{L}_r \to \widetilde{L}_r$ as follows.

Let $\varepsilon > 0$ be sufficiently small; define numbers $M_\alpha, N_\alpha \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n$ as follows:

$$M_\alpha = 2^{\alpha m}, \quad N_\alpha = \begin{cases} 2^{\alpha m}, & \text{if} \quad \alpha\gamma' \le u, \\ \lfloor 2^{u(1+\varepsilon\bar{\sigma})-\varepsilon\alpha\bar{s}} \rfloor, & \text{if} \quad \alpha\gamma' > u \end{cases} \tag{25}$$

(here $\lfloor a \rfloor$ is the integer part of a real number a).

Then taking into account (17) and (18) by Lemma A (formula (15) with $\beta = \bar{\gamma},\ \gamma = \gamma',\ L = \varepsilon\bar{\sigma}$) we find that

$$N(u) = \sum_\alpha \sum_\iota N_\alpha \asymp \sum_\alpha N_\alpha = \sum_{\alpha\gamma' \le u} 2^{\alpha m} + \sum_{\alpha\gamma' > u} \lfloor 2^{u(1+\varepsilon\bar{\sigma})-\varepsilon\alpha\bar{s}} \rfloor \asymp 2^u u^{\omega-1} +$$

$$+ 2^{u(1+\varepsilon\bar{\sigma})} \sum_{\alpha\beta' > u} 2^{-\varepsilon\bar{\sigma}\alpha\bar{\gamma}} \asymp 2^u u^{\omega-1} + 2^{u(1+\varepsilon\bar{\sigma})} 2^{-\varepsilon\bar{\sigma}u} u^{\omega-1} \asymp 2^u u^{\omega-1}. \tag{26}$$

We choose $u > 0$ such that $N(u) \le N$ and $2^u u^{\omega-1} \asymp N$.

Let $f \in \widetilde{L}_r$. Then we put

$$(g^\iota_{\alpha\lambda})_\lambda = G^{N_\alpha}_{M_\alpha}((f^\iota_{\alpha\lambda})_\lambda),\ (h^\iota_{\alpha\lambda})_\lambda = H^{N_\alpha}_{M_\alpha}((f^\iota_{\alpha\lambda})_\lambda), \quad \text{if} \ \ \iota \in \mathrm{E}^m(\alpha), \alpha \in \mathbb{N}_0^n :\ 1 \le N_\alpha < M_\alpha.$$

Here $G^{N_\alpha}_{M_\alpha} : \mathbb{R}^{M_\alpha} \to \mathbb{R}^{M_\alpha}$ is operator from (23) and $H^{N_\alpha}_{M_\alpha} : \mathbb{R}^{M_\alpha} \to \mathbb{R}^{M_\alpha}$ is operator from (24) and

$$f^\iota_{\alpha\lambda} = \langle f, \widetilde{w}^\iota_{\alpha\lambda} \rangle,\ \lambda \in \Lambda(m,\alpha),\ \iota \in \mathrm{E}^m(\alpha),\ \alpha \in \mathbb{N}_0^n.$$

Furthermore,

$$(g^\iota_{\alpha\lambda})_\lambda \equiv G^{N_\alpha}_{M_\alpha}((f^\iota_{\alpha\lambda})_\lambda) = (h^\iota_{\alpha\lambda})_\lambda \equiv H^{N_\alpha}_{M_\alpha}((f^\iota_{\alpha\lambda})_\lambda) = 0 \in \mathbb{R}^{M_\alpha},$$

$$\text{if} \ \ \iota \in \mathrm{E}^m(\alpha),\ \alpha \in \mathbb{N}_0^n : N_\alpha = 0;$$

$$(g^\iota_{\alpha\lambda})_\lambda \equiv G^{N_\alpha}_{M_\alpha}((f^\iota_{\alpha\lambda})_\lambda) = (h^\iota_{\alpha\lambda})_\lambda \equiv H^{N_\alpha}_{M_\alpha}((f^\iota_{\alpha\lambda})_\lambda) = \mathrm{Id}_{M_\alpha}((f^\iota_{\alpha\lambda})_\lambda) = (f^\iota_{\alpha\lambda}),$$

$$\text{if} \ \ \iota \in \mathrm{E}^m(\alpha), \alpha \in \mathbb{N}_0^n :\ \alpha\gamma' \le u.$$

Next, we define

$$g_N(x) \equiv G^{N,w}(f,x) \equiv \sum_\alpha \widetilde{\triangle}^w_\alpha (g_N, x) \equiv \sum_\alpha \sum_\iota G^{N_\alpha,w}_\iota (f,x),$$

where

$$G^{N_\alpha,w}_\iota (f,x) = \sum_{\lambda \in \Lambda(m,\alpha)} g^\iota_{\alpha\lambda} \widetilde{w}^\iota_{\alpha\lambda}(x),$$

and

$$h_N(x) \equiv H^{N,w}(f,x) \equiv \sum_\alpha \widetilde{\triangle}^w_\alpha (h_N, x) \equiv \sum_\alpha \sum_\iota H^{N_\alpha,w}_\iota (f,x),$$

where

$$H^{N_\alpha,w}_\iota (f,x) = \sum_{\lambda \in \Lambda(m,\alpha)} h^\iota_{\alpha\lambda} \widetilde{w}^\iota_{\alpha\lambda}(x).$$

It it clear that $H^{N,w}$ is a linear operator and the dimensions of the linear span $\{g_N = G^{N,w}(f) : f \in \widetilde{L}_r\}$ and of the set $\{h_N = H^{N,w}(f) : f \in \widetilde{L}_r\}$ do not exceed $N(u) \leq N$.

Therefore, we obtain that

$$d_N(\widetilde{\mathrm{F}}^{s\,m}_{p\,q}, \widetilde{L}_r) \leq \sup\{\|f - g_N \,|\, \widetilde{L}_r\,\| : f \in \widetilde{\mathrm{F}}^{s\,m}_{p\,q}\}, \tag{27}$$

$$\lambda_N(\widetilde{\mathrm{F}}^{s\,m}_{p\,q}, \widetilde{L}_r) \leq \sup\{\|f - h_N \,|\, \widetilde{L}_r\,\| : f \in \widetilde{\mathrm{F}}^{s\,m}_{p\,q}\}, \tag{28}$$

where $\widetilde{\mathrm{F}}$ is $\widetilde{\mathrm{B}}$ or $\widetilde{\mathrm{L}}$. □

References

[1] T.I. Amanov, *Spaces of differentiable functions with dominating mixed derivative.* Nauka, Alma-Ata, 1976 (in Russian).

[2] D.B. Bazarkhanov, *Estimates for linear widths of unit ball of Nikol'skii-Besov space with generalized mixed smoothness.* Math. J., 8, no. 4 (2008), 44 – 47 (in Russian).

[3] D.B. Bazarkhanov, *Estimates for certain approximation characteristics of Nikol'skii-Besov spaces with generalized mixed smoothness.* Doklady Ross. Akad. Nauk. Mathematika, 426, no. 1 (2009), 11 – 14 (in Russian). English transl. Acad. Sci. Dokl. Math., 79, no. 3 (2009), 305 – 308.

[4] D.B. Bazarkhanov, *Wavelet approximation and Fourier widths of classes of periodic functions of several variables. I.* Tr. Math Inst. im. V.A. Steklova, 269 (2010), 8 – 30 (in Russian). English transl. in Proc. Steklov Inst. Math., 269, no. 1 (2010), 2 – 24

[5] E.M. Galeev, *Linear widths of Hölder-Nikol'skii classes of periodic functions of several variables.* Mat. Zametki, 59, no. 2 (1996), 189 – 199 (in Russian). English transl. in Math. Notes, 59, no. 2 (1996), 133 – 140.

[6] E.M. Galeev, *Widths of the Besov classes* $B^r_{p,\theta}(\mathbb{T}^d)$. Mat. Zametki, 69, no 5 (2001), 656 – 665 (in Russian). English transl. in Math. Notes, 69, no. 5-6 (2001), 605 – 613.

[7] E.D. Gluskin, *Norms of random matrices and widths of finite-dimensional sets.* Matem.sb., 120, no 2 (1983), 180 – 189 (in Russian). English transl. in Mathematics of the USSR-Sbornik, 48, no. 1 (1984), 173 – 182.

[8] M. Hansen, J. Vybiral, *The Jawerth-Franke embedding of spaces with dominating mixed smoothness.* Georgian Math. J., 16, no. 4 (2009), 667 – 682.

[9] B.S. Kashin, *Diameters of some finite-dimensional sets and classes of smooth functions.* Izv. AN SSSR. Ser. mat., 41, no. 2 (1977), 334 – 351 (in Russian). English transl. in Mathematics of the USSR-Izvestiya, 11, no 2 (1977), 317 – 333.

[10] A.N. Kolmogoroff, *Über die beste annäherung von funktionen einer gegebenen funktionenklasse.* Ann. Math., 37, no. 1 (1936), 107 – 111.

[11] N.P. Korneichuk, *Exact constants in approximation theory.* Nauka, Moscow, 1987 (in Russian).

[12] Y. Meyer, *Wavelets and Operators.* Cambridge Studies in Advanced Mathematics, Cambridge Univ. Press, 37, 1992.

[13] S.M.Nikol'skii, *Approximation of functions of several variables and imbedding theorems.* Nauka, Moscow, 1977 (in Russian).

[14] A.S. Romanyuk, *Approximation of Besov classes of periodic functions of many variables in the space* L_q. Ukrain. Mat. Zh., 43, no. 10 (1991), 1398 – 1408 (in Russian). English transl. in Ukrainian Math. J., 43, no. 10 (1991), 1297 – 1306.

[15] A.S. Romanyuk, *The best trigonometric approximations and the Kolmogorov diameters of the Besov classes of functions of many variables.* Ukrain. Mat. Zh., 45, no. 5 (1993), 663 – 675 (in Russian). English transl. in Ukrainian Math. J., 45, no. 5 (1993), 724 – 738.

[16] A.S. Romanyuk, *Linear widths of the Besov classes of periodic functions of many variables. I.* Ukrain. Mat. Zh., 53, no 5 (2001), 647 – 661 (in Russian). English transl. in Ukrainian Math. J., 53, no. 5 (2001), 744 – 761.

[17] A.S. Romanyuk, *Linear widths of the Besov classes of periodic functions of many variables. II.* Ukrain. Mat. Zh., 53, no 6 (2001), 820 – 829 (in Russian). English transl. in Ukrainian Math. J., 53, no. 6 (2001), 965 – 977.

[18] H.-J. Schmeisser, *Recent developments in the theory of function spaces with dominating mixed smoothness.* in Nonlinear Analysis, Function Spaces and Application: Proc. Spring School, Prague, May 30 – June 6, 2006 (Czech Acad. Sci., Math. Inst., Prague), 8 (2007), 145 – 204.

[19] H.-J. Schmeisser, H. Triebel, *Topics in Fourier analysis and function spaces.* J.Wiley and Sons, Chichester, 1987.

[20] V.N. Temlyakov, *Approximation of functions with a bounded mixed derivative.* Tr. Math Inst. im. V.A. Steklova, 178 (1986), 3 – 113 (in Russian). English transl. in Proc. Steklov Inst. Math., 178 (1989), 1 – 121.

[21] V.N. Temlyakov, *Approximation of periodic functions.* Nova Sci. Publ., Commack, New York, 1993.

[22] V.M. Tikhomirov, *Widths of sets in function spaces and the theory of best approximation.* Uspekhi Math. Nauk, 15 (1960), 81 – 120 (in Russian).

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