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A STRONG CONVERGENCE THEOREM FOR TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN A STRICTLY CONVEX BANACH SPACE

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Abstract. An iterative scheme containing two asymptotically nonexpansive mappings has been used in this paper to approximate common fixed points in a strictly convex Banach space. This improves and extends some known recent results to the case of two mappings.

1 Introduction and preliminaries

The class of asymptotically nonexpansive mappings, which is a natural generalization of the important class of nonexpansive mappings, was introduced by Goebel and Kirk [2], where it was shown that, if C is a nonempty bounded closed convex subset of a uniformly convex Banach space X and $T : C \to C$ is asymptotically nonexpansive, then T has a fixed point. Moreover, the set $F(T)$ of all fixed points of T, is closed and convex. Recall that a mapping $T : C \to C$ is called asymptotically nonexpansive if there is a sequence $\{k_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ and $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and for all $n \in \mathbb{N}$. T is called $L-$Lipschitzian if $\|Tx - Ty\| \leq L \|x - y\|$ for all $x, y \in C$ and some $L > 0$. Note that every asymptotically nonexpansive mapping is $L-$Lipschitzian. The problem of existence and approximation of fixed points of such mappings have attracted many mathematicians, see [2], [4], [5], [6] to mention only a few. Many authors starting from Das and Debata [1] and including Takahashi and Tamura [4] have studied the two mappings case of iterative schemes for different types of mappings. That approximating common fixed points has its own significance is clear from the fact that the two mappings case has a direct link with the minimization problem, see for example [7].

In this manuscript, we intend to prove, in the context of strictly convex Banach spaces, a strong convergence theorem to approximate common fixed points of two asymptotically nonexpansive mappings under certain control conditions. Before we go further, we recall the definition of a strictly convex Banach space. A Banach space X is said to be strictly convex if for two elements $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$, we have $\left\|\frac{x+y}{2}\right\| < 1$.

We will need the following results in the proof of our main theorem. The lemma below, which can be found in [3], is used to prove the existence of a certain limit in the proof of our main theorem.

Lemma 1 ([3]). *Let $\{r_n\}, \{s_n\}$ and $\{t_n\}$ be nonnegative sequences satisfying*

$$r_{n+1} \leq (1+s_n)r_n + t_n$$

for all $n \geq 1$. If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.

The following definition can be found, for example, in [9].

Definition. *Let d denote (throughout the paper) the diameter of a nonempty bounded convex subset of a Banach space X. The modulus of convexity of C is denoted by $\delta(C,\varepsilon)$ with $0 \leq \varepsilon \leq 1$ and is defined as*

$$\delta(C,\varepsilon) = \frac{1}{d} \inf \left\{ \max\left(\|x-z\|, \|y-z\|\right) - \left\| z - \frac{x+y}{2} \right\| : x,y,z \in C, \|x-y\| \geq d\varepsilon \right\}.$$

Next two lemmas were proved in [9].

Lemma 2. *Let C be a nonempty compact convex subset of a Banach space X with $d > 0$. Let $x,y,z \in C$ and suppose that $\|x-y\| \geq d\varepsilon$ for some ε with $0 \leq \varepsilon \leq 1$. Then, for all λ with $0 \leq \lambda \leq 1$, we have*

$$\|\lambda(x-z) + (1-\lambda)(y-z)\| \leq \max\left(\|x-z\|, \|y-z\|\right) - 2\lambda(1-\lambda)d\delta(C,\varepsilon).$$

Lemma 3. *Let C be a nonempty compact convex subset of a strictly convex Banach space X with $d > 0$. If $\lim_{n\to\infty} \delta(C,\varepsilon_n) = 0$, then $\lim_{n\to\infty} \varepsilon_n = 0$.*

2 Main theorem

We are now all set to prove our main result.

Theorem. *Let X be a strictly convex Banach space and C its compact convex subset. Let S and T be self mappings of C satisfying*

$$\|S^n x - S^n y\| \leq k_n \|x-y\| \quad \text{and} \quad \|T^n x - T^n y\| \leq k_n \|x-y\|$$

for all $n \in \mathbb{N}$ where $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\gamma \in (0, \frac{1}{2})$. Let $x_1 = x \in C$ be the starting point of the iterative scheme $\{x_n\}$ defined as

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)\,x_n + a_n S^n y_n, \\ y_n = (1-b_n)\,x_n + b_n T^n x_n, \quad n \in \mathbb{N} \end{cases} \tag{1}$$

with $\{a_n\}, \{b_n\}$ in $[\gamma, 1-\gamma]$ for all $n \in \mathbb{N}$.

If $F(S) \cap F(T) \neq \emptyset$ then $\{x_n\}$ converges strongly to a common fixed point of S and T.

Proof of Theorem. Let $w \in F(S) \cap F(T)$. Then from (1),

$$\begin{aligned} \|S^n y_n - w\| &= \|S^n y_n - S^n w\| \\ &\le k_n \|y_n - w\| \\ &= k_n \|b_n(T^n x_n - w) + (1-b_n)(x_n - w)\| \qquad (2) \\ &\le b_n k_n^2 \|x_n - w\| + k_n(1-b_n)\|x_n - w\| \\ &\le b_n k_n^2 \|x_n - w\| + k_n^2(1-b_n)\|x_n - w\| \\ &= k_n^2 \|x_n - w\|. \qquad (3) \end{aligned}$$

Using (2), we have

$$\begin{aligned} \|x_{n+1} - w\| &= \|a_n S^n y_n + (1-a_n)x_n - w\| \\ &= \|a_n(S^n y_n - w) + (1-a_n)(x_n - w)\| \\ &\le a_n \|S^n y_n - w\| + (1-a_n)\|x_n - w\| \\ &\le a_n k_n \|b_n(T^n x_n - w) + (1-b_n)(x_n - w)\| + (1-a_n)\|x_n - w\| \\ &\le a_n b_n k_n^2 \|x_n - w\| + a_n(1-b_n)k_n \|x_n - w\| + (1-a_n)\|x_n - w\| \\ &= [1 + a_n b_n k_n(k_n - 1) + a_n(k_n - 1)]\|x_n - w\| \\ &= [1 + (a_n b_n k_n + a_n)(k_n - 1)]\|x_n - w\| \\ &\le \big[1 + \{(1-\gamma)^2 k + (1-\gamma)\}(k_n - 1)\big]\|x_n - w\| \end{aligned}$$

where $k = \sup\limits_{n \in \mathbb{N}} k_n$.

Using Lemma 1 with $s_n = \{(1-\gamma)^2 k + (1-\gamma)\}(k_n - 1)$, $t_n = 0$ and $r_n = \|x_n - w\|$, we conclude that $\lim\limits_{n\to\infty} \|x_n - w\|$ exists.

Denote $\varepsilon_n = \|S^n y_n - x_n\| d^{-1}$. Then $0 \le \varepsilon_n \le 1$ because $0 \le \|S^n y_n - x_n\| \le d < \infty$. Applying Lemma 2 and then using (3), we get

$$\begin{aligned} \|x_{n+1} - w\| &= \|a_n(S^n y_n - w) + (1-a_n)(x_n - w)\| \\ &\le \max(\|S^n y_n - w\|, \|x_n - w\|) - 2a_n(1-a_n)d\delta(C, \varepsilon_n) \\ &\le k_n^2 \|x_n - w\| - 2a_n(1-a_n)d\delta(C, \varepsilon_n). \end{aligned}$$

This gives

$$\begin{aligned} 2a_n(1-a_n)d\delta(C,\varepsilon_n) &\le (k_n^2 - 1)\|x_n - w\| + \|x_n - w\| - \|x_{n+1} - w\| \\ &\le d(k_n^2 - 1) + \|x_n - w\| - \|x_{n+1} - w\|. \end{aligned}$$

Since $a_n \in [\gamma, 1-\gamma]$, we have $a_n(1-a_n) \ge \gamma^2$ for all $\gamma \in (0, \frac{1}{2})$. Let $m \ge n$, then the above inequality can be written as

$$\begin{aligned} 2d\gamma^2 \sum_{n=1}^{m} \delta(C, \varepsilon_n) &\le (k+1)d \sum_{n=1}^{m}(k_n - 1) + \|x_1 - w\| - \|x_{m+1} - w\| \\ &\le (k+1)d \sum_{n=1}^{m}(k_n - 1) + \|x_1 - w\|. \end{aligned}$$

Let $m \to \infty$. Then $\sum\limits_{n=1}^{\infty} \delta(C, \varepsilon_n) < \infty$ and hence $\lim\limits_{n\to\infty} \delta(C, \varepsilon_n) = 0$. Now, applying Lemma 3, we obtain that $\lim\limits_{n\to\infty} \varepsilon_n = 0$. Hence

$$\lim_{n\to\infty} \|S^n y_n - x_n\| = 0. \tag{4}$$

Next suppose that $\zeta_n = \|T^n x_n - x_n\| d^{-1}$. Then using (2), Lemma 2 and the fact that $\|T^n x_n - w\| \le k_n \|x_n - w\|$, we obtain

$$\begin{aligned}
\|x_{n+1} - w\| &= \|a_n S^n y_n + (1-a_n)x_n - w\| \\
&\le a_n \|S^n y_n - w\| + (1-a_n)\|x_n - w\| \\
&\le a_n k_n [\|b_n(T^n x_n - w) + (1-b_n)(x_n - w)\|] + (1-a_n)\|x_n - w\| \\
&\le a_n k_n [\max(\|T^n x_n - w\|, \|x_n - w\|) - 2b_n(1-b_n)d\delta(C,\zeta_n)] \\
&+ (1-a_n)\|x_n - w\| \\
&\le a_n k_n^2 \|x_n - w\| - 2a_n b_n k_n (1-b_n) d\delta(C,\zeta_n) + (1-a_n)\|x_n - w\| \\
&= \|x_n - w\| + a_n(k_n^2 - 1)\|x_n - w\| - 2a_n b_n k_n(1-b_n)d\delta(C,\zeta_n) \\
&\le \|x_n - w\| + a_n(k+1)(k_n - 1)d - 2a_n b_n k_n (1-b_n)d\delta(C,\zeta_n).
\end{aligned}$$

Since $a_n, b_n \in [\gamma, 1-\gamma]$, it follows that $a_n b_n(1-b_n) \ge \gamma^3$. Let $m \ge n$, then the above inequality becomes

$$2\gamma^3 d \sum_{n=1}^{m} k_n \delta(C,\zeta_n) \le \|x_1 - w\| - \|x_{m+1} - w\| + (1-\gamma)(k+1)d \sum_{n=1}^{m} (k_n - 1)$$

By letting $m \to \infty$, we obtain $\sum\limits_{n=1}^{\infty} k_n \delta(C,\zeta_n) < \infty$. This gives $\lim\limits_{n\to\infty} k_n \delta(C,\zeta_n) = 0$. Since $\lim\limits_{n\to\infty} k_n = 1$, we have $\lim\limits_{n\to\infty} \delta(C,\zeta_n) = 0$. Again applying Lemma 3, we get $\lim\limits_{n\to\infty} \zeta_n = 0$. That is

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{5}$$

Further observe that

$$\begin{aligned}
\|S^n x_n - x_n\| &\le \|S^n x_n - S^n y_n\| + \|S^n y_n - x_n\| \\
&\le k_n \|y_n - x_n\| + \|S^n y_n - x_n\| \\
&= b_n k_n \|T^n x_n - x_n\| + \|S^n y_n - x_n\|.
\end{aligned}$$

Since $\{b_n\}$ and $\{k_n\}$ are both bounded, there exists a positive integer M such that $b_n k_n \le M$ for all $n \in \mathbb{N}$. Thus

$$\|S^n x_n - x_n\| \le M \|T^n x_n - x_n\| + \|S^n y_n - x_n\|.$$

Now, using (4) and (5), we obtain

$$\lim_{n\to\infty} \|S^n x_n - x_n\| = 0. \tag{6}$$

Next $\|x_n - x_{n+1}\| \leq a_n \|x_n - S^n y_n\|$ implies by (4) that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{7}$$

Since every asymptotically nonexpansive mapping is $L-$Lipschitzian, it follows

$$\begin{aligned}
\|x_{n+1} - Sx_{n+1}\| &\leq \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| + \left\|S^{n+1}x_{n+1} - Sx_{n+1}\right\| \\
&\leq \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| + L\left\|S^n x_{n+1} - x_{n+1}\right\| \\
&\leq \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| \\
&\quad + L\left(\|x_n - x_{n+1}\| + \|S^n x_n - x_n\| + \|S^n x_{n+1} - S^n x_n\|\right) \\
&\leq \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| + L\left[(k_n + 1)\|x_n - x_{n+1}\| + \|S^n x_n - x_n\|\right],
\end{aligned}$$

which, in view of (6) and (7), gives that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{8}$$

Similarly we can prove that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{9}$$

Since C is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to q$ (say) in C. Continuity of S and T gives $Sx_{n_i} \to Sq$ and $Tx_{n_i} \to Tq$ as $n_i \to \infty$. Then, by (8) and (9),

$$\|Sq - q\| = 0 = \|Tq - q\|.$$

This yields $q \in F(S) \cap F(T)$. As proved above, $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F(S) \cap F(T)$, therefore $\{x_n\}$ must itself converge to $q \in F(S) \cap F(T)$. $\square$

Remark 1. *The above theorem improves the results of [4], [8] and [9] to the case of two asymptotically nonexpansive mappings.*

Recall that a self-mapping T on a nonempty subset C of a Banach space X is demicompact (respectively, semicompact) if every bounded sequence $\{x_n\}$ in C such that the sequence $\{\|Tx_n - x_n\|\}$ converges (respectively, $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$) contains a convergent subsequence. Thus the above theorem can be proved for demicompact and semicompact mappings.

Remark 2. *All the results proved above can also be proved with error terms. In this case our scheme takes the shape:*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = a_n x_n + b_n S^n y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T^n x_n + c'_n v_n, \quad n \geq 1, \end{cases} \tag{10}$$

with $\{u_n\}, \{v_n\}$ bounded sequences in C, $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ sequences in $[0,1]$ such that $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ for all $n \geq 1$. This generalizes the scheme due to Xu [10] to the case of two asymptotically nonexpansive mappings. The results proved with (10) generalize several results in literature regarding approximation of common fixed points of two asymptotically nonexpansive mappings.

References

[1] G. Das, J.P. Debata, *Fixed points of quasi-nonexpansive mappings,* Indian J. Pure. Appl. Math., 17(1986), 1263 – 1269.

[2] K. Goebel, W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., 35, no. 1 (1972), 171 – 174.

[3] D.I. Igbokwe, *Approximation of fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces,* J. Ineq. Pure and Appl. Math., 3, no. 1 (2002), Article3. [Online: http://jipam.vu.edu.au/v3n1/043_01.html].

[4] S.H. Khan, W. Takahashi, *Iterative approximation of fixed points of asymptotically nonexpansive mappings with compact domains*, PanAmer. Math. J., 11, no. 1 (2001), 19 – 24.

[5] W.A. Kirk, C.M. Yanez, S.S. Shin, *Asymptotically nonexpansive mappings,* Nonlinear Anal. TMA, 33, no. 1 (1998), 1 – 12.

[6] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc., 43 (1991), 153 – 159.

[7] W. Takahashi, *Iterative methods for approximation of fixed points and thier applications*, J. Oper. Res. Soc. Jpn., 43, no. 1 (2000), 87 – 108.

[8] W. Takahashi, G.E. Kim, *Approximating fixed points of nonexpansive mappings in Banach spaces*, Math. Japon., 48 (1998), 1 – 9.

[9] W. Takahashi, N. Tsukiyama, *Approximating fixed points of nonexpansive mappings with compact domains*, Comm. Appl. Nonlinear Anal., 7, no. 4 (2000), 39 – 47.

[10] Y. Xu, *Ishikawa and Mann iteration process with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl., 224 (1998), 91 – 101.

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