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ON IMBALANCES IN ORIENTED BIPARTITE GRAPHS

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Abstract. An oriented bipartite graph is the result of assigning a direction to each edge of a simple bipartite graph. For any vertex x in an oriented bipartite graph $D(U, V)$, let d_x^+ and d_x^- respectively denote the outdegree and indegree of x. Define $a_{u_i} = d_{u_i}^+ - d_{u_i}^-$ and $b_{v_j} = d_{v_j}^+ - d_{v_j}^-$ respectively as the imbalances of vertices u_i in U and v_j in V. In this paper, we obtain constructive and existence criteria for a pair of sequences of integers to be the imbalances of some oriented bipartite graph. We also show the existence of a bipartite oriented graph with given imbalance set.

1 Introduction

A simple digraph is a digraph that is without loops and with no multiarcs. The imbalance of a vertex v_i in a digraph is b_{v_i} (or simply b_i) $= d_{v_i}^+ - d_{v_i}^-$, where $d_{v_i}^+$ and $d_{v_i}^-$ are respectively, the outdegree and indegree of v_i. The imbalance sequence of a simple digraph is formed by listing the vertex imbalances in non-increasing or non-decreasing order. A sequence of integers $F = [f_1 f_2, \cdots, f_n]$ with $f_1 \geq f_2 \geq \cdots \geq f_n$ is feasible if it has sum zero and satisfies

$$\sum_{i=1}^{k} f_i \leq k(n-k),$$

for $1 \leq k < n$.

One can verify that a sequence is realizable as an imbalance sequence if and only if it is feasible.

This is equivalent to saying that a sequence of integers $B = [b_1, b_2, \cdots, b_n]$ with $b_1 \geq b_2 \geq \cdots \geq b_n$ is an imbalance sequence of a simple digraph if and only if

$$\sum_{i=1}^{k} b_i \leq k(n-k),$$

for $1 \leq k < n$ with equality when $k = n$.

On arranging the imbalance sequence in non-decreasing order, we observe that a sequence of integers $B = [b_1, b_2, \cdots, b_n]$ with $b_1 \le b_2 \le \cdots \le b_n$ is an imbalance sequence of a simple digraph if and only if for $1 \le k < n$

$$\sum_{i=1}^{k} b_i \ge k(k-n),$$

with equality when $k = n$.

Various results for imbalances in simple digraphs and oriented graphs can be found in [1, 3].

2 Imbalances in oriented bipartite graphs

An oriented bipartite graph is the result of assigning a direction to each edge of a simple bipartite graph. Let $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$ be the parts of an oriented bipartite graph $D(U,V)$. For any vertex x in $D(U,V)$, let d_x^+ and d_x^- denote the outdegree and indegree of x. Define a_{u_i}(or simply a_i) $= d_{u_i}^+ - d_{u_i}^-$ and b_{v_j}(or simply b_j) $= d_{v_j}^+ - d_{v_j}^-$ respectively, as imbalances of the vertices u_i in U and v_j in V. The sequences $A = [a_1, a_2, \cdots, a_p]$ and $B = [b_1, b_2, \cdots, b_q]$ in non-decreasing order is a pair of imbalance sequences of $D(U,V)$.

In any oriented bipartite graph $D(U,V)$, we have one of the following possibilities for a vertex u in U and a vertex v in V.
(i) An arc directed from u to v, denoted by $u(1-0)v$.
(ii) An arc directed from v to u, denoted by $u(0-1)v$.
(iii) There is no arc from u to v and there is no arc from v to u and this is denoted by $u(0-0)v$.

A tetra in an oriented bipartite graph is an induced sub-oriented graph with two vertices from each part. Define tetras of the form $u_1(1-0)v_1(1-0)u_2(1-0)v_2(1-0)u_1$ and $u_1(1-0)v_1(1-0)u_2(1-0)v_2(0-0)u_1$ to be of α-type, and all other tetras to be of β-type. An oriented bipartite graph is said to be of α-type or β-type according as all of its tetras are of α-type or β-type respectively.

Some results on oriented bipartite graphs can be found in [2, 4]. The following observation is an immediate consequence of the above definitions and facts.

Theorem 1. *Among all oriented bipartite graphs with given imbalance sequence, those with the fewest arcs are of β-type.*

A transmitter is a vertex with indegree zero. In a β-type oriented bipartite graph with imbalance sequences $A = [a_1, a_2, \cdots, a_p]$ and $B = [b_1, b_2, \cdots, b_q]$, either the vertex with imbalance a_p, or the vertex with imbalance b_q, or both may act as transmitters.

The next result provides a useful recursive test allowing to determine whether given sequences are the imbalance sequences of an oriented bipartite graph.

Theorem 2. *Let $A = [a_1, a_2, \cdots, a_p]$ and $B = [b_1, b_2, \cdots, b_q]$ be sequences of integers in non-decreasing order with $a_p > 0$, $a_p \le q$ and $b_q \le p$. Let A' be obtained from A by* *deleting one entry a_p, and B' be obtained from B by increasing a_p smallest entries of B by 1 each. Then A and B are the imbalance sequences of some oriented bipartite graph if and only if A' and B' are the imbalance sequences.*

Proof. Let A' and B' be the imbalance sequences of some oriented bipartite graph D' with parts U' and V'. Then an oriented bipartite graph D with imbalance sequences A and B can be obtained by adding a transmitter u_p in U' such that $u_p(1-0)v_i$ for those vertices v_i in V' whose imbalances are increased by 1 in going from A and B to A' and B'.

Conversely, suppose A and B be the imbalance sequences of an oriented bipartite graph D with parts U and V. Without loss of generality, we chose D to be of β-type. Then there is a vertex u_p in U with imbalance a_p(or a vertex v_q in V with imbalance b_q, or both u_p and v_q) which is a transmitter. Let the vertex u_p in U with imbalance a_p be a transmitter. Clearly, $d^+_{u_p} \geq 0$ and $d^-_{u_p} = 0$ so that $a_p = d^+_{u_p} - d^-_{u_p} \geq 0$. Also, $d^+_{v_q} \leq p$ and $d^-_{v_q} \geq 0$ so that $b_q = d^+_{v_q} - d^-_{v_q} \leq p$.

Let V_1 be the set of a_p vertices of smallest imbalances in V, and let $W = V - V_1$. Construct D such that $u_p(1-0)v_i$ for all $v_i \in V_i$. Clearly, $D - \{u_p\}$ realizes A' and B'. □

Theorem 2 provides an algorithm for determining whether two sequences of integers in non-decreasing order are the imbalance sequences, and for constructing a corresponding oriented bipartite graph. Suppose $A = [a_1, a_2, \cdots, a_p]$ and $B = [b_1, b_2, \cdots, b_q]$ are imbalance sequences of an oriented bipartite graph with parts $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$, where $a_p > 0$, $a_p \leq q$ and $b_q \leq p$. Deleting a_p, and increasing a_p smallest entries of B by 1 each to form $B' = [b'_1, b'_2, \cdots, b'_q]$. Then arcs are defined by $u_p(1-0)v_j$ for which $b'_{v_j} = b_{v_j} + 1$. Now, if the condition $a_p > 0$ does not hold, then we delete b_q(obviously $b_q > 0$), and increase b_q smallest entries of A by 1 each to form $A' = [a'_1, a'_2, \cdots, a'_p]$. In this case, arcs are defined by $v_q(1-0)u_i$ for which $a'_{u_i} = a_{u_i} + 1$. If this method is applied successively, then (i) it tests whether A and B are the imbalance sequences and, if A and B are the imbalance sequences (ii) an oriented bipartite graph $D(A, B)$ with imbalance sequences A and B is constructed.

We illustrate this reduction and the resulting construction as follows, beginning with sequences A_1 and B_1

$A_1 = [-3, 1, 2, 2]$ $B_1 = [-3, -1, 0, 1, 1]$
$A_2 = [-3, 1, 2]$ $B_2 = [-2, 0, 0, 1, 1]$ $u_4(1-0)v_1,\ u_4(1-0)v_2$
$A_3 = [-3, 1]$ $B_3 = [-1, 1, 0, 1, 1]$ $u_3(1-0)v_1,\ u_3(1-0)v_2$
or $A_3 = [-3, 1]$ $B_3 = [-1, 0, 1, 1, 1]$
$A_4 = [-3]$ $B_4 = [0, 0, 1, 1, 1]$ $u_2(1-0)v_1$
$A_5 = [-2]$ $B_5 = [0, 0, 1, 1]$ $v_5(1-0)u_1$
$A_6 = [-1]$ $B_6 = [0, 0, 1]$ $v_4(1-0)u_1$
$A_7 = [0]$ $B_7 = [0, 0]$ $v_2(1-0)u_1$

Obviously, an oriented bipartite graph D with parts $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2, v_3, v_4, v_5\}$ in which $u_4(1-0)v_1,\ u_4(1-0)v_2,\ u_3(1-0)v_1,\ u_3(1-0)v_2,\ u_2(1-0)v_1,\ v_5(1-0)u_1,\ v_4(1-0)u_1,\ v_2(1-0)u_1$ are arcs has imbalance sequences $[-3, 1, 2, 2]$ and $[-3, -1, 0, 1, 1]$.

The following result is a combinatorial criterion for determining whether the sequences are realizable as imbalances.

Theorem 3. *Two sequences $A = [a_1, a_2, \cdots, a_p]$ and $B = [b_1, b_2, \cdots, b_q]$ of integers in non-decreasing order are the imbalance sequences of some oriented bipartite graph if and only if*

$$\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j \geq 2kl - kq - lp, \tag{1}$$

for $1 \leq k \leq p$, $1 \leq l \leq q$ with equality when $k = p$ and $l = q$.

Proof. The necessity follows from the fact that an oriented sub-bipartite graph induced by k vertices from the first part and l vertices from the second part has a sum of imbalances $2kl - kq - lp$.

For sufficiency, assume that $A = [a_1, a_2, \cdots, a_p]$ and $B = [b_1, b_2, \cdots, b_q]$ are the sequences of integers in non-decreasing order satisfying conditions (1) but are not the imbalance sequences of any oriented bipartite graph. Let these sequences be chosen in such a way that p and q are the smallest possible and a_1 is the least with that choice of p and q. We consider the following two cases.

Case(i). Suppose equality in (1) holds for some $k \leq p$ and $l < q$, so that

$$\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j = 2kl - kq - lp.$$

By the minimality of p and q, $A = [a_1, a_2, \cdots, a_p]$ and $B = [b_1, b_2, \cdots, b_q]$ are the imbalance sequences of some oriented bipartite graph $D_1(U_1, V_1)$. Let $A_2 = [a_{k+1}, a_{k+2}, \cdots, a_p]$ and $B_2 = [b_{l+1}, b_{l+2}, \cdots, b_q]$.

Now,

$$\begin{aligned}
\sum_{i=1}^{f} a_{k+i} + \sum_{j=1}^{g} b_{l+j} &= \sum_{i=1}^{k+f} a_i + \sum_{j=1}^{l+g} b_j - (\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j) \\
&\geq 2(k+f)(l+g) - (k+f)q - (l+g)p - 2kl + kq + lp \\
&= 2kl + 2kg + 2fl + 2fg - kq - fq - lp - gp - 2kl + kq + lp \\
&= 2fg - fq - gp + 2kg + 2fl \\
&\geq 2fg - fq - gp,
\end{aligned}$$

for $1 \leq f \leq p - k$ and $1 \leq g \leq q - l$, with equality when $f = p - k$ and $g = q - l$. So, by the minimality for p and q, the sequences A_2 and B_2 form the imbalance sequences of some oriented bipartite graph $D_2(U_2, V_2)$. Now construct a new oriented bipartite graph $D(U, V)$ as follows.

Let $U = U_1 \cup U_2$, $V = V_1 \cup V_2$ with $U_1 \cap U_2 = \varphi$, $V_1 \cap V_2 = \varphi$ and the arc set containing those arcs which are between U_1 and V_1 and between U_2 and V_2. Then we obtain an oriented bipartite graph $D(U, V)$ with the imbalance sequences A and B, which is a contradiction.

Case (ii). Suppose that the strict inequality holds in (1) for some $k \neq p$ and $l \neq q$.

Let $A_1 = [a_1 - 1, a_2, \cdots, a_{p-1}, a_p]$ and $B_1 = [b_1, b_2, \cdots, b_q]$, so that A_1 and B_1 satisfy the conditions (1). Thus by the minimality of a_1, the sequences A_1 and B_1 are the imbalances sequences of some oriented bipartite graph $D_1(U_1, V_1)$. Let $a_{u_1} = a_1 - 1$ and $a_{u_p} = a_p + 1$. Since $a_{u_p} > a_{u_1} + 1$, therefore there exists a vertex $v_1 \in V_1$ such that $u_p(0-0)v_1(1-0)u_1$, or $u_p(1-0)v_1(0-0)u_1$, or $u_p(1-0)v_1(1-0)u_1$, or $u_p(0-0)v_1(0-0)u_1$, in $D_1(U_1, V_1)$ and if these are changed to $u_p(0-1)v_1(0-0)u_1$, or $u_p(0-0)v_1(0-1)u_1$, or $u_p(0-0)v_1(0-0)u_1$, or $u_p(0-1)v_1(0-1)u_1$ respectively, the result is an oriented bipartite graph with imbalances sequences A and B, which is a contradiction. This proves the result. □

The set of distinct imbalances of the vertices in an oriented bipartite graph is called its imbalance set.

Finally, we prove the existence of an oriented bipartite graph with a given imbalance set.

Theorem 4. *Let $A = [a_1, a_2, \cdots, a_n]$ and $B = [-b_1, -b_2, \cdots, -b_n]$, where $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are positive integers with $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$. Then there exists an oriented bipartite graph with imbalance set $A \cup B$.*

Proof. Construct an oriented bipartite graph $D(U, V)$ as follows. Let $U = U_1 \cup U_2 \cup \cdots \cup U_n$, $V = V_1 \cup V_2 \cup \cdots \cup V_n$ with $U_i \cap U_j = \varphi(i \neq j)$, $V_i \cap V_j = \varphi(i \neq j)$, $|U_i| = b_i$ for all i, $1 \leq i \leq n$ and $|V_j| = a_j$ for all j, $1 \leq j \leq n$. Let there be an arc from every vertex of U_i to each vertex of V_i for all i, $1 \leq i \leq n$, so that we obtain the oriented bipartite graph $D(U, V)$ with the imbalances of vertices as follows.

For $1 \leq i, j \leq n$,
$a_{u_i} = |V_i| - 0 = a_i$, for all $u_i \in U_i$ and $b_{v_j} = 0 - |U_j| = -b_j$, for all $v_i \in V_i$.

Therefore the imbalance set of $D(U, V)$ is $A \cup B$.

Obviously the oriented bipartite graph constructed above is not connected. In order to see the existence of oriented bipartite graph, whose underlying graph is connected, we proceed as under.

Taking $U_i = \{u_1, u_2, \cdots, u_{b_i}\}$ and $V_j = \{v_1, v_2, \cdots, v_{a_j}\}$, and let there be an arc from each vertex of U_i to every vertex of V_j except the arcs between u_{b_i} and v_{a_j}, that is $u_{b_i}(0-0)v_{a_j}$, $1 \leq i \leq n$ and $1 \leq j \leq n$. We take $u_{b_1}(0-0)v_{a_2}$, $u_{b_2}(0-0)v_{a_3}$, and so on $u_{b_{(n-1)}}(0-0)v_{a_n}$, $u_{b_n}(0-0)v_{a_1}$. The underlying graph of this oriented bipartite graph is connected. □

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