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WEAK CONTINUITY OF JACOBIANS OF W_ν^1-HOMEOMORPHISMS ON CARNOT GROUPS

S.V. Pavlov, S.K. Vodop'yanov

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Key words: Carnot group, Sobolev mapping, Jacobian, continuity property.

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Abstract. The limit of a locally uniformly converging sequence of analytic functions is an analytic function. Yu.G. Reshetnyak obtained a natural generalization of that in the theory of mappings with bounded distortion: the limit of every locally uniformly converging sequence of mappings with bounded distortion is a mapping with bounded distortion, and established the weak continuity of the Jacobians.

In this article, similar problems are studied for a sequence of Sobolev-class homeomorphisms defined on a domain in a two-step Carnot group. We show that if such a sequence converges to some homeomorphism locally uniformly, the sequence of horizontal differentials of its terms is bounded in $L_{\nu,\mathrm{loc}}$, and the Jacobians of the terms of the sequence are nonnegative almost everywhere, then the sequence of Jacobians converges to the Jacobian of the limit homeomorphism weakly in $L_{1,\mathrm{loc}}$; here ν is the Hausdorff dimension of the group.

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1 Introduction

Consider a mapping $f = (f^1, \dots, f^n)$ of class $W^1_{1,\mathrm{loc}}(\Omega; \mathbb{R}^n)$, where Ω is a domain in $\mathbb{R}^n$. Given two multi-indices $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ of length $k \le n$ with $i_1 < i_2 < \ldots < i_k$ and $j_1 < j_2 < \ldots < j_k$, denote the (I, J)-minor of the generalized differential Df of f by

$$\frac{\partial f^I}{\partial x_J} = \det\left(\frac{\partial f^{i_\alpha}}{\partial x_{j_\beta}}\right)^k_{\alpha,\beta=1}.$$

The following nontrivial property holds for the differentials of Sobolev-class mappings: the $*$-weak continuity of their minors.

Theorem 1.1. *Given a positive integer $k \le n$ and some domain $\Omega \subset \mathbb{R}^n$, consider a sequence $\{f_m : \Omega \to \mathbb{R}^n\}$ of mappings of class $W^1_{p,\mathrm{loc}}(\Omega; \mathbb{R}^n)$ bounded in $W^1_{p,\mathrm{loc}}(\Omega; \mathbb{R}^n)$, where $p \ge k$. If the sequence $\{f_m\}$ converges in $L_{1,\mathrm{loc}}(\Omega; \mathbb{R}^n)$ to some mapping f_0, then for every pair of multi-indices (I, J) of length k the sequence $\left\{\frac{\partial f^I_m}{\partial x_J}\right\}$ converges in the sense of distributions to the (I, J)-minor of the generalized differential of f_0, that is*

$$\lim_{m\to\infty} \int\limits_\Omega \frac{\partial f^I_m}{\partial x_J}(x)\theta(x)\,dx = \int\limits_\Omega \frac{\partial f^I_0}{\partial x_J}(x)\theta(x)\,dx \tag{1.1}$$

for all functions $\theta \in C^\infty_0(\Omega)$.

This property was obtained in the case $n = 2$ in [2], while in the form presented above it was established in [15, Chapter II, Lemma 4.9] and [12]. Note that in [15] Theorem 1.1 appears as a corollary to a claim about the convergence of transported exterior differential k-forms. See [9, Theorem 8.2.1] as well.

Recall that for $1 < q < \infty$ the dual space to the Lebesgue space $L_q(D)$ is the Lebesgue space $L_{q'}(D)$, where the Hölder exponent q' dual to q is determined by the condition $\frac{1}{q'} + \frac{1}{q} = 1$, while the dual space to $L_1(D)$ is the space $L_\infty(D)$ of essentially bounded functions.

Since the space of C_0^∞ functions on a domain $D \Subset \Omega$ is dense in $L_r(D)$ for each $1 \le r < \infty$, and the hypotheses of Theorem 1.1 imply that the sequence of minors $\{\frac{\partial f_m^I}{\partial x_J}\}$ is bounded in $L_{p/k,\mathrm{loc}}(\Omega)$, we conclude that for $p > k$ it is not difficult to extend (1.1) to all functions $\theta \in L_{(p/k)'}(\Omega)$ with compact supports in Ω. The latter means that for $p > k$ the sequence of (I, J)-minors of the differentials of f_m converges *weakly* in the space $L_{p/k,\mathrm{loc}}(\Omega)$ to the (I, J)-minor of the differential of the limit mapping f_0.

At the same time, continuous functions do not constitute a dense subspace in $L_\infty(D)$. Therefore, for $p = k = n$ the transition in (1.1) from C_0^∞ functions to all functions $\theta \in L_\infty(\Omega)$ with compact supports in Ω is not obvious. However, that turns out feasible if we assume in addition that the Jacobians are nonnegative: $\det Df_m \ge 0$ almost everywhere. In this case the local uniform integrability of the sequence $\{\det Df_m\}$ established in [13] plays a key role.

Note that the conditions imposed on the sequence $\{f_m\}$ in Theorem 1.1 are equivalent to the weak convergence of $\{f_m\}$ to f_0 in the space $W^1_{p,\mathrm{loc}}(\Omega; \mathbb{R}^n)$.

The main result of this article is the following generalization of Theorem 1.1 to the case of Carnot groups, where ν stands for the homogeneous dimension of the group $\mathbb{G}$; see also [20], where a similar result on Carnot groups is established for sequences of mappings with bounded distortion.

Theorem 1.2. *Consider domains Ω, Ω'_0, $\Omega'_1, \dots$ in a two-step Carnot group $\mathbb{G}$ and a sequence $\{\varphi_k : \Omega \to \Omega'_k\}_{k=1}^\infty$ of homeomorphisms of class $W^1_{\nu,\mathrm{loc}}(\Omega; \mathbb{G})$. Suppose that $\{\varphi_k\}$ converges to some homeomorphism $\varphi_0 : \Omega \to \Omega'_0$ locally uniformly in Ω, the sequence $\{|D_h\varphi_k|\}_{k=1}^\infty$ is bounded in $L_{\nu,\mathrm{loc}}(\Omega)$, and $\det \widehat{D}\varphi_k \ge 0$ almost everywhere, for $k = 1, 2, \dots$.*

Then the sequence of Jacobians $\{\det \widehat{D}\varphi_k\}$ converges to $\det \widehat{D}\varphi_0$ weakly in $L_{1,\mathrm{loc}}(\Omega)$, that is,

$$\lim_{k\to\infty} \int_\Omega \theta(x) \det \widehat{D}\varphi_k(x)\, dx = \int_\Omega \theta(x) \det \widehat{D}\varphi_0(x)\, dx$$

for each function $\theta \in L_\infty(\Omega)$ vanishing almost everywhere outside some compact set $K \subset \Omega$.

In the case of H-type Carnot groups the local uniform convergence of a sequence $\{\varphi_k\}$ of homeomorphisms of class $W^1_{\nu,\mathrm{loc}}(\Omega; \mathbb{G})$, the horizontal differentials of whose terms are bounded in $L_{\nu,\mathrm{loc}}(\Omega)$, to some mapping φ_0 is equivalent to the convergence of $\{\varphi_k\}$ to φ_0 in $L_{1,\mathrm{loc}}(\Omega; \mathbb{G})$ because this sequence possesses a common local continuity modulus [22].

The weak continuity of minors of the differentials of Sobolev-class mappings is one of the main arguments when studying the existence of solutions to nonlinear elasticity problems. Namely, it is related to the possibility of applying Mazur's lemma to establish the semicontinuity of the functionals satisfying the polyconvexity condition, which is a generalized convexity condition, see [1], [4], [13], and [11] for instance.

Even though Theorem 1.2 assumes that the limit mapping φ_0 is bijective, this variation of Theorem 1.1 turns out suitable for deriving theorems about the existence of solutions to the model problems of elasticity on Carnot groups which will be considered by the authors in forthcoming articles.

2 Preliminaries

Carnot groups. Recall that a *stratified graded nilpotent group* or a *Carnot group*, see [5, Chapter 1] for instance, is a *connected simply-connected* Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ of left-invariant vector fields decomposes as a direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m$ of subspaces $\mathfrak{g}_i$ satisfying the conditions $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$ for $i = 1, \dots, m-1$ and $[\mathfrak{g}_1, \mathfrak{g}_m] = \{0\}$. A Carnot group $\mathbb{G}$ is called *two-step* whenever $m = 2$.

Fix some inner product in $\mathfrak{g}$. The subspace $\mathfrak{g}_1 \subset \mathfrak{g}$ is called the *horizontal space* of the algebra $\mathfrak{g}$, and its elements are *horizontal vector fields*. Put $N = \dim \mathfrak{g}$ and $n_i = \dim \mathfrak{g}_i$ for $i = 1, \dots, m$. For convenience also put $n = n_1$. Fix an orthonormal basis $X_{i1}, \dots, X_{in_i}$ of $\mathfrak{g}_i$. Since the exponential mapping

$$g = \exp\Big(\sum_{i=1}^{m}\sum_{j=1}^{n_i} x_{ij}X_{ij}\Big)(e),$$

where e is the neutral element of $\mathbb{G}$, is a global diffeomorphism of $\mathfrak{g}$ onto $\mathbb{G}$ [5, Proposition 1.2], we can identify the point $g \in \mathbb{G}$ with the point $x = (x_{ij}) \in \mathbb{R}^N$. Then $e = 0$ and $x^{-1} = -x$. The dilation δ_λ specified as $\delta_\lambda(x_{ij}) = (\lambda^i x_{ij})$ is an automorphism of the group for all $\lambda > 0$.

A *homogeneous norm* on $\mathbb{G}$ is a continuous function $\rho : \mathbb{G} \to [0, +\infty)$ of class $C^\infty(\mathbb{G} \setminus \{0\})$ such that

(a) $\rho(x) = 0$ if and only if $x = 0$;

(b) $\rho(x^{-1}) = \rho(x)$ and $\rho(\delta_\lambda x) = \lambda\rho(x)$.

This definition also implies [5, Proposition 1.6] the following properties:

(c) there exists a number $c > 0$ such that $\rho(xy) \le c\big(\rho(x) + \rho(y)\big)$ for all $x, y \in \mathbb{G}$;

(d) two arbitrary homogeneous norms are equivalent, that is, given two homogeneous norms ρ_1 and ρ_2, there are numbers $0 < \alpha \le \beta < \infty$ such that $\alpha\rho_1(x) \le \rho_2(x) \le \beta\rho_1(x)$ for all $x \in \mathbb{G}$.

Example 1. Given some point $x = (x_{ij}) \in \mathbb{G}$ and some index $i = 1, \dots, m$, define $X^{(i)} \in \mathfrak{g}_i$ as $\sum\limits_{j=1}^{n_i} x_{ij}X_{ij}$. The equality

$$\rho(x) = \Big(\sum_{i=1}^{m} |X^{(i)}|^{2m!/i}\Big)^{\frac{1}{2m!}}, \tag{2.1}$$

where $|X^{(i)}|$ is the Euclidean norm in $\mathfrak{g}_i$, defines a homogeneous norm $\rho : \mathbb{G} \to [0; +\infty)$.

A piecewise smooth curve $\gamma : [a; b] \to \mathbb{G}$ is called *horizontal* whenever $\dot\gamma(t) \in \mathfrak{g}_1(\gamma(t))$ for almost all t. The *Carnot–Carathéodory distance* $d_{cc}(x, y)$ between two points x, $y \in \mathbb{G}$ is the greatest lower bound of the lengths $\int_a^b |\dot\gamma(t)|\,dt$ of horizontal curves with endpoints x and y. According to the Rashevskiĭ–Chow theorem, see [7, §0.4, §1.1] for instance, we can connect two arbitrary points with a piecewise smooth horizontal curve of finite length. The metric d_{cc} and every homogeneous norm ρ are equivalent: there exist positive constants α and β such that

$$\alpha d_{cc}(x, y) \le \rho(y^{-1}x) \le \beta d_{cc}(x, y). \tag{2.2}$$

The Lebesgue measure dx on $\mathbb{R}^N$ is a bi-invariant Haar measure on $\mathbb{G}$ and $d(\delta_\lambda x) = \lambda^\nu dx$, where $\nu = \sum\limits_{i=1}^{m} i\,n_i$ is the *homogeneous dimension* of the group $\mathbb{G}$. The measure is normalized by choosing its value on the unit ball: $|B(0, 1)| = 1$. Here $B(x, r) = \{y \in \mathbb{G} \mid d_{cc}(x, y) < r\}$ is a ball with respect to the Carnot–Carathéodory metric. We denote balls and spheres in the homogeneous norm ρ by $B_\rho(x, r) = \{y \in \mathbb{G} \mid \rho(y^{-1}x) < r\}$ and $S_\rho(x, r) = \{y \in \mathbb{G} \mid \rho(y^{-1}x) = r\}$ respectively.

Example 2. The Heisenberg group $\mathbb{H}^k = (\mathbb{R}^{2k+1}, *)$ with the group operation

$$(x,y,z)*(x',y',z') = \big(x+x', y+y', z+z'+\tfrac{x\cdot y'-x'\cdot y}{2}\big), \quad x,x',y,y' \in \mathbb{R}^k,\ \ z,z' \in \mathbb{R},$$

is the classical example of a nonabelian Carnot group. Its Lie algebra $\mathfrak{h}^k$ is formed by the vector fields

$$X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2}\frac{\partial}{\partial z}, \quad Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial z}, \quad i = 1,\dots,k, \quad Z = \frac{\partial}{\partial z}.$$

Here $\mathfrak{h}_1^k = \operatorname{span}\{X_i, Y_i \mid i = 1,\dots,k\}$ and $\mathfrak{h}_2^k = \operatorname{span}\{Z\}$, while the only nontrivial Lie brackets are $[X_i, Y_i] = Z$ for $i = 1,\dots,k$. The homogeneous dimension of $\mathbb{H}^k$ equals $\nu = 2k+2$.

Sobolev-class mappings. Consider a domain $\Omega \subset \mathbb{G}$, which is a nonempty connected open subset of $\mathbb{G}$. The space $L_p(\Omega)$, where $p \in [1;\infty)$, consists of all measurable functions $u : \Omega \to \mathbb{R}$ integrable to power p. The norm on $L_p(\Omega)$ is defined as

$$\|u \mid L_p(\Omega)\| = \Bigg(\int\limits_\Omega |u(x)|^p\,dx\Bigg)^{\frac{1}{p}}.$$

The space $L_\infty(\Omega)$ consists of all measurable essentially bounded functions $u : \Omega \to \mathbb{R}$. The norm on $L_\infty(\Omega)$ is defined as

$$\|u \mid L_\infty(\Omega)\| = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|,$$

where $\operatorname*{ess\,sup}_{x\in\Omega} |u(x)|$ is the essential supremum of u. A function u belongs to $L_{p,\mathrm{loc}}(\Omega)$, where $p \in [1;\infty]$, whenever $u \in L_p(K)$ for every compact set $K \subset \Omega$.

Take some basis $X_1 = X_{11}, \dots, X_n = X_{1n}$ of the horizontal space $\mathfrak{g}_1$. Denote by Π_j the hyperplane $\{x \in \mathbb{G} \mid x_j = 0\}$, for $j = 1,\dots,n$, where $x_j = x_{1j}$ is the horizontal coordinate of the point $x = (x_{ij})$. The measure $d\mu_j = \imath(X_j)dx$ on Π_j is determined by the contraction of X_j with the volume form. Associated to each $y \in \Pi_j$ there is the integral line $\gamma_j(t) = \exp(tX_j)(y)$. A mapping $\varphi : \Omega \to M$ from some domain $\Omega \subset \mathbb{G}$ into some metric space M is *absolutely continuous on almost all lines*, briefly $\varphi \in \mathrm{ACL}(\Omega; M)$, if we can modify it on a measure zero set so that for each $j = 1,\dots,n$ it becomes absolutely continuous on the integral line $\{\exp(tX_j)(y) \mid t \in \mathbb{R}\} \cap \Omega$ of the vector field X_j for μ_j-almost all $y \in \Pi_j$. Put $\mathrm{ACL}(\Omega) = \mathrm{ACL}(\Omega;\mathbb{R})$.

The space $L_p^1(\Omega)$, where $p \in [1;\infty]$, consists of all functions $u \in L_{1,\mathrm{loc}}(\Omega) \cap \mathrm{ACL}(\Omega)$ with the classical derivatives[1] $X_j u$ lying in $L_p(\Omega)$ for all $j = 1,\dots,n$. The seminorm of the function $u \in L_p^1(\Omega)$ equals $\|u \mid L_p^1(\Omega)\| = \big\| |\nabla_h u| \mid L_p(\Omega)\big\|$, where $\nabla_h u = (X_1 u, \dots, X_n u) = \sum\limits_{j=1}^n (X_j u) X_j$ is the *horizontal gradient* of u. Henceforth, instead of $\big\| |\nabla_h u| \mid L_p(\Omega)\big\|$ we write $\|\nabla_h u \mid L_p(\Omega)\|$.

An equivalent definition of the space $L_p^1(\Omega)$ relies on the concept of generalized derivative in the sense of Sobolev: a locally summable function $u_j : \Omega \to \mathbb{R}$ is called the *generalized derivative of the function $u \in L_{1,\mathrm{loc}}(\Omega)$ along the vector field X_j*, for $j = 1,\dots,n$, whenever

$$\int\limits_\Omega u_j(x) v(x)\,dx = -\int\limits_\Omega u(x) X_j v(x)\,dx$$

for every test function $v \in C_0^\infty(\Omega)$. A locally summable function $u : \Omega \to \mathbb{R}$ belongs to $L_p^1(\Omega)$ if and only if its generalized derivatives $u_j \in L_p(\Omega)$ exist for $j = 1\dots,n$. Moreover, $u_j = X_j u$ almost

[1] More exactly, the derivatives of a representative of the function u which is absolutely continuous on almost all integral lines of $X_1,\dots,X_n$. The classical derivatives of this representative exist almost everywhere.

everywhere, where $X_j u$ are the classical derivatives of the function[2] $u \in \mathrm{ACL}(\Omega)$, which exist almost everywhere.

The Sobolev space $W_p^1(\Omega)$ consists of all functions $u \in L_p(\Omega) \cap L_p^1(\Omega)$ and is equipped with the norm

$$\|u \mid W_p^1(\Omega)\| = \|u \mid L_p(\Omega)\| + \|u \mid L_p^1(\Omega)\|.$$

Given two Carnot groups $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ and a domain $\Omega \subset \mathbb{G}$, consider $\varphi \in \mathrm{ACL}(\Omega; \widetilde{\mathbb{G}})$. Then $X_j\varphi(x) \in \tilde{\mathfrak{g}}_1(\varphi(x))$ for almost all $x \in \Omega$ [14, Proposition 4.1]. The matrix $D_h\varphi(x) = (X_i\varphi_j)$, where $i = 1, \dots, n$ and $j = 1, \dots, \widetilde{n}$, determines the linear operator $D_h\varphi(x) : \mathfrak{g}_1 \to \tilde{\mathfrak{g}}_1$ called the *horizontal differential* of φ. It is known [18, Theorem 1.2] that for almost all $x \in \Omega$ the linear operator $D_h\varphi(x)$ is defined and extends to a Lie algebra homomorphism $\widehat{D}\varphi(x) : \mathfrak{g} \to \widetilde{\mathfrak{g}}$, which we can also consider as a linear operator $\widehat{D}\varphi(x) : T_x\mathbb{G} \to T_{\varphi(x)}\widetilde{\mathbb{G}}$. The norms of both operators satisfy

$$\bigl|D_h\varphi(x)\bigr| \le \bigl|\widehat{D}\varphi(x)\bigr| \le C\bigl|D_h\varphi(x)\bigr|, \tag{2.3}$$

where C depends only on the group structures. Here the norm of $\widehat{D}\varphi(x)$ is defined as

$$\sup\bigl\{\widetilde{\rho}\bigl(\widehat{D}\varphi(x)\langle X\rangle\bigr) \mid X \in \mathfrak{g},\ \rho(X) \le 1\bigr\}, \tag{2.4}$$

where we put $\rho(X) = \rho(\exp(X))$ and $\widetilde{\rho}(\widetilde{X}) = \widetilde{\rho}(\widetilde{\exp}(\widetilde{X}))$ for $X \in \mathfrak{g}$ and $\widetilde{X} \in \widetilde{\mathfrak{g}}$ for brevity. Corresponding to $\widehat{D}\varphi(x)$, there is the group homomorphism

$$D_{\mathcal{P}}\varphi(x) = \widetilde{\exp} \circ \widehat{D}\varphi(x) \circ \exp^{-1}$$

known as the *Pansu differential*, which is the approximative differential of φ with respect to the group structure [18].

Definition 1. The class $W_p^1(\Omega; \widetilde{\mathbb{G}})$ of Sobolev mappings consists of all measurable mappings $\varphi \in \mathrm{ACL}(\Omega; \widetilde{\mathbb{G}})$ for which

$$\bigl\|\varphi \mid W_p^1(\Omega)\bigr\| = \|\rho \circ \varphi \mid L_p(\Omega)\| + \|\,|D_h\varphi| \mid L_p(\Omega)\|$$

is finite. A mapping φ belongs to $W^1_{p,\mathrm{loc}}(\Omega; \widetilde{\mathbb{G}})$ whenever $\varphi \in W_p^1(U; \widetilde{\mathbb{G}})$ for every compactly embedded domain $U \Subset \Omega$. Henceforth we write $\|D_h\varphi \mid L_p(\Omega)\|$ instead of $\|\,|D_h\varphi| \mid L_p(\Omega)\|$.

Some equivalent descriptions of Sobolev-class mappings of Carnot groups appeared in [18, Proposition 4.2]. If $\varphi \in W_p^1(\Omega; \widetilde{\mathbb{G}})$ then all coordinate functions φ_i for $i = 1, \dots, \widetilde{N}$ belong to $W_p^1(\Omega)$.

3 Uniform integrability and weak continuity of the determinant of the Pansu differential

First, we establish the uniform integrability of the Jacobians of a sequence of orientation-preserving mappings whose horizontal differentials are bounded in $L_{\nu,\mathrm{loc}}$. In connection with that we generalize to the case of Carnot groups the results of paper [16], in which the $L \log L$-norm of an arbitrary summable function f is estimated via the L_1-norm of its maximal function Mf, as well as the results of paper [13], in which the L_1-norm of the maximal function of the Jacobian of an arbitrary orientation-preserving mapping of class W_n^1 is estimated.

[2] Namely, the derivatives of a representative of the function u which is absolutely continuous on almost all lines.

In order to reproduce the arguments of paper [16], we extend the widely known Calderon–Zygmund lemma [3, Lemma 1] to Carnot groups by replacing a system of binary cubes with a suitable system of Borel sets adequate for the geometry of Carnot groups.

Since in a Carnot group equipped with the Carnot–Carathéodory metric each open ball can be covered with a finite number, independent of the ball, of open balls of half the radius, [8, Theorem 2.2] directly implies the following lemma[3].

Lemma 3.1. *Given an arbitrary Carnot group $\mathbb{G}$, there exist collections $\{x_{k,i} \in \mathbb{G}\}_{i\in\mathbb{N}}$, for $k \in \mathbb{Z}$, of points and $\{Q_{k,i} \subset \mathbb{G}\}_{i\in\mathbb{N}}$, for $k \in \mathbb{Z}$, of Borel sets with the following properties:*

(1) *for all $k \in \mathbb{Z}$ the collection $\{Q_{k,i}\}_{i\in\mathbb{N}}$ is disjoint and $\mathbb{G} = \bigcup\limits_{i\in\mathbb{N}} Q_{k,i}$;*

(2) *if $m \geq k$ then either $Q_{m,j} \subset Q_{k,i}$ or $Q_{m,j} \cap Q_{k,i} = \emptyset$;*

(3) *for all $k \in \mathbb{Z}$ and $i \in \mathbb{N}$ we have the inclusions*

$$B\Big(x_{k,i}, c\frac{1}{24^k}\Big) \subset Q_{k,i} \subset B\Big(x_{k,i}, C\frac{1}{24^k}\Big),$$

where $c = \frac{1}{3}$ and $C = 4$.

Proposition 3.1. *Given an arbitrary Carnot group $\mathbb{G}$ and a nonnegative function $f \in L_1(\mathbb{G})$, for every $\alpha > 0$ the collection $\{Q_{k,i} \mid i \in \mathbb{N}, k \in \mathbb{Z}\}$ of Lemma 3.1 includes a disjoint subcollection[4] $\mathcal{Q} = \{Q_j\}$ of Borel sets such that*

$$\alpha \leq \frac{1}{|Q_j|} \int\limits_{Q_j} f(x)\,dx \leq 288^\nu \alpha \quad \text{for all } j, \tag{3.1}$$

and $f(x) \leq \alpha$ for almost all $x \notin \bigcup_j Q_j$.

Proof. Put $\mathcal{Q}_k = \{Q_{k,i}\}_{i\in\mathbb{N}}$ for $k \in \mathbb{Z}$. Since f is an integrable function and each $Q_{k,i}$ contains a ball of radius $\frac{1}{3} \cdot \frac{1}{24^k}$, there is $k_0 \in \mathbb{Z}$ such that

$$\frac{1}{|Q_{k_0,i}|} \int\limits_{Q_{k_0,i}} f(x)\,dx < \alpha$$

for all $i \in \mathbb{N}$. Fix such $k_0 \in \mathbb{Z}$ and an arbitrary $i \in \mathbb{N}$. Add to $\mathcal{Q}$ the sets $Q \in \mathcal{Q}_{k_0+1}$ included into $Q_{k_0,i}$ with

$$\frac{1}{|Q|} \int\limits_{Q} f(x)\,dx \geq \alpha.$$

For these Q claim (3) of Lemma 3.1 yields

$$\frac{1}{|Q|} \int\limits_{Q} f(x)\,dx \leq \Big(\frac{24C}{c}\Big)^\nu \frac{1}{|Q_{k_0,i}|} \int\limits_{Q_{k_0,i}} f(x)\,dx \leq 288^\nu \alpha.$$

Repeat this procedure taking instead of $Q_{k_0,i}$ each set $Q \in \mathcal{Q}_{k_0+1}$ with $Q \subset Q_{k_0,i}$ still not in $\mathcal{Q}$ while they exist. Continue this process by induction and take the union of the resulting collections over $i \in \mathbb{N}$.

[3] In Theorem 2.2 of [8] it suffices to put $A_0 = 1$, $c_0 = 1$, $C_0 = 2$, and $\delta = \frac{1}{24}$ and choose the families $\{x_{k,i}\}_i$ as maximal δ^k-sparse sets in $(\mathbb{G}, d_{cc})$. Each of these collections is obviously countable.

[4] The collection in question can be countable, finite, or empty.

The construction of the family $\mathcal{Q}$ immediately implies that (3.1) holds. It is clear also that if some set $Q \in \{Q_{k,i} \mid k \in \mathbb{Z},\ i \in \mathbb{N}\}$ is disjoint from all sets in $\mathcal{Q}$ then $\frac{1}{|Q|}\int\limits_Q f(x)\,dx < \alpha$.

Assuming now that $y \notin \bigcup \mathcal{Q}$ is a Lebesgue point of the functions $f\chi_{(\bigcup \mathcal{Q})^c}$ and $\chi_{(\bigcup \mathcal{Q})^c}$, verify that $f(y) \le \alpha$. Since every point $z \notin \bigcup \mathcal{Q}$ lies in some $Q_{k,i}$ for all sufficiently large k, it follows that for arbitrary $r > 0$ we can express the complement $Q(r) = B(y,r) \setminus \bigcup \mathcal{Q}$ as the union of a countable collection of disjoint sets $Q_{k,i}$ with

$$\frac{1}{|Q_{k,i}|}\int\limits_{Q_{k,i}} f(x)\,dx < \alpha.$$

Their union $Q(r) = \bigcup Q_{k,i}$ also satisfies

$$\frac{1}{|Q(r)|}\int\limits_{Q(r)} f(x)\,dx < \alpha.$$

Since $\lim\limits_{r\to 0}\frac{|Q(r)|}{|B(y,r)|} = 1$ and

$$\frac{1}{|B(y,r)|}\int\limits_{Q(r)} f(x)\,dx = \frac{1}{|B(y,r)|}\int\limits_{B(y,r)} (f\chi_{(\bigcup \mathcal{Q})^c})(x)\,dx \to f(y)$$

as $r \to 0$, we infer that $f(y) \le \alpha$. □

In the following statement we consider the maximal function in the sense of balls $B(x,r) = \{y \in \mathbb{G} \mid d_{cc}(x,y) < r\}$ with respect to the Carnot–Carathéodory metric:

$$Mf(x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |f(y)|\,dy.$$

Theorem 3.1. *Given an arbitrary Carnot group $\mathbb{G}$, let a function $f \in L_1(\mathbb{G})$ vanish almost everywhere outside the ball $B = B(0,R)$. If $Mf \in L_1(2B)$, then $|f|\log^+|f| \in L_1(B)$. Moreover, we have*

$$\big\||f|\log^+|f| \mid L_1(B)\big\| \le C_L\cdot\Big(\|Mf \mid L_1(2B)\| + \|f \mid L_1(B)\| + R^{-\nu}\|f \mid L_1(B)\|^2\Big),$$

where C_L depends only on the homogeneous dimension ν of $\mathbb{G}$.

Proof. We may assume that $f \ge 0$. Fix $\alpha > 0$. Choose $\{Q_j\}$ according to Proposition 3.1. Take $z \in Q_j$. By claim (3) of Lemma 3.1, we can choose $r > 0$ such that the ball $B(z,r)$ includes Q_j and $|B(z,r)| < c_0|Q_j|$, where the constant c_0 depends only on ν. The definition of the maximal function and the choice of $\{Q_j\}$ show that

$$Mf(z) \ge \frac{1}{|B(z,r)|}\int\limits_{B(z,r)} f(x)\,dx > \frac{c_0^{-1}}{|Q_j|}\int\limits_{Q_j} f(x)\,dx \ge c_0^{-1}\alpha.$$

This means that $\bigcup_j Q_j \subset \{z \in \mathbb{G} \mid Mf(z) > c_0^{-1}\alpha\}$. Since the collection $\{Q_j\}$ is disjoint, we infer that[5]

$$|\{z \in \mathbb{G} \mid Mf(z) > c_0^{-1}\alpha\}| \ge \sum_j |Q_j| \ge \frac{288^{-\nu}}{\alpha}\sum_j\int\limits_{Q_j} f(x)\,dx \ge \frac{288^{-\nu}}{\alpha}\int\limits_{f>\alpha} f(x)\,dx.$$

[5]If for this $\alpha > 0$ the collection $\{Q_j\}$ is empty then the required inequality is obvious because in this case $f \le \alpha$ almost everywhere.

Replacing α with $c_0\alpha$, we obtain

$$|\{z \in \mathbb{G} \mid Mf(z) > \alpha\}| \geq \frac{288^{-\nu} c_0^{-1}}{\alpha} \int\limits_{f > c_0\alpha} f(x)\,dx.$$

Integrate this over $\alpha \in (c_0^{-1}; \infty)$. Thanks to the Cavalieri–Lebesgue formula, the left-hand side equals $\int_{Mf > c_0^{-1}} Mf(x)\,dx$. Rearranging the integral in the right-hand side,

$$\int\limits_{c_0^{-1}}^{\infty} \int\limits_{f > c_0\alpha} \frac{1}{\alpha} f(x)\,dx d\alpha = \int\limits_{1}^{\infty} \int\limits_{f > \beta} \frac{1}{\beta} f(x)\,dx d\beta = \int\limits_{f > 1} \int\limits_{1}^{f(x)} \frac{1}{\beta} f(x)\,d\beta dx$$
$$= \int\limits_{f > 1} f(x) \log f(x)\,dx = \int\limits_{\mathbb{G}} f(x) \log^+ f(x)\,dx,$$

we arrive at the inequality

$$\int\limits_{\mathbb{G}} f(x) \log^+ f(x)\,dx \leq c_1 \int\limits_{Mf > c_2} Mf(x)\,dx,$$

where $c_1 = 288^\nu c_0$ and $c_2 = c_0^{-1}$.

In order to estimate $\int\limits_{Mf > c_2} Mf(x)\,dx$, take $z \notin \overline{B}$; therefore, $d_{cc}(z, 0) > R$. For $t < (d_{cc}(z, 0) - R)$ the intersection $B(z, t) \cap B$ is empty. Since the function f vanishes almost everywhere outside B, it follows that

$$Mf(z) \leq \frac{\|f \mid L_1(B)\|}{\big(d_{cc}(z, 0) - R\big)^\nu}. \tag{3.2}$$

Hence, for $d_{cc}(z, 0) \geq R_f = c_3 \|f \mid L_1(B)\|^{1/\nu} + R$, where $c_3 = c_2^{-1/\nu}$, the value of the maximal function Mf at z is at most c_2, and so the set $\{x \in \mathbb{G} \mid Mf(x) > c_2\}$ lies in the ball $B(0, R_f)$. Using (3.2), we infer that

$$\int\limits_{Mf > c_2} Mf(x)\,dx \leq \int\limits_{2B} Mf(x)\,dx + \int\limits_{B(0,R_f)\setminus 2B} Mf(x)\,dx$$
$$\leq \int\limits_{2B} Mf(x)\,dx + \frac{\|f \mid L_1(B)\|}{R^\nu} R_f^\nu$$
$$= \int\limits_{2B} Mf(x)\,dx + \frac{\|f \mid L_1(B)\|}{R^\nu} \big(c_3 \|f \mid L_1(B)\|^{1/\nu} + R\big)^\nu.$$

□

Consider the maximal function defined with respect to some homogeneous norm ρ:

$$M^\rho f(x) = \sup_{r > 0} \frac{1}{B_\rho(x, r)} \int\limits_{B_\rho(x,r)} |f(y)|\,dy.$$

Since the Carnot–Carathéodory metric d_{cc} is equivalent to every homogeneous norm, we have

$$aMf(x) \leq M^\rho f(x) \leq bMf(x) \quad \text{for all } x,$$

where the constants a and b depend only on ρ. This implies that Theorem 3.1 remains valid when we replace Mf by $M^\rho f$.

Definition 2. A sequence of integrable functions $\{f_k\}$ defined on a measurable space X endowed with some measure μ is called *uniformly integrable* whenever the sequence of integrals $\int\limits_X |f_k|\,d\mu$ is bounded and, given a positive number ε, there is positive δ such that

$$\int\limits_E |f_k|\,d\mu < \varepsilon$$

for all k and all measurable sets $E \subset X$ with $\mu(E) < \delta$.

By the commensurability of the maximal functions Mf and $M^\rho f$, Theorem 3.1 and the de la Vallée Poussin theorem directly imply the following corollary.

Corollary 3.1. *Given a domain Ω in an arbitrary Carnot group $\mathbb{G}$ and an arbitrary homogeneous norm ρ on $\mathbb{G}$, if $\{f_k \in L_{1,\mathrm{loc}}(\Omega)\}$ is a sequence of functions such that for each compact set $K \Subset \Omega$ the sequence $\{M^\rho(f_k\chi_K)\}$ is bounded in $L_{1,\mathrm{loc}}(\Omega)$, then the sequence $\{f_k\}$ is uniformly integrable on every compact subset of Ω.*

In the following two statements we denote by ρ homogeneous norm (2.1), while $\mathcal{H}^{\nu-1}$ stands for the spherical Hausdorff measure defined with respect to ρ. The adjoint operator $\mathrm{adj}\widehat{D}\varphi(y) : \mathfrak{g} \to \mathfrak{g}$ is determined by the condition

$$\widehat{D}\varphi(y) \cdot \mathrm{adj}\widehat{D}\varphi(y) = \det \widehat{D}\varphi(y) \cdot \mathrm{Id}$$

provided that the determinant of the $N \times N$ matrix $\widehat{D}\varphi(y)$ is nonzero and extended by continuity in the topology of $\mathbb{R}^{N\times N}$ otherwise. Its norm $|\mathrm{adj}\widehat{D}\varphi(y)|$ is defined by analogy with (2.4).

Lemma 3.2 ([21, Theorem 3.1])**.** *Given a two-step Carnot group $\mathbb{G}$ and a bounded domain $\Omega \subset \mathbb{G}$, consider $\varphi \in W^1_\nu(\Omega;\mathbb{G})$.*

Then for almost all $x \in \Omega$ and almost all $r \in (0;\mathrm{dist}_\rho(x,\partial\Omega))$ we have

$$\Big|\int\limits_{B_\rho(x,r)} \det \widehat{D}\varphi(y)\,dy\Big|^{\frac{\nu-1}{\nu}} \le C_I \int\limits_{S_\rho(x,r)} |\mathrm{adj}\widehat{D}\varphi(y)|\,d\mathcal{H}^{\nu-1}(y),$$

where the constant C_I is independent of φ.

Proposition 3.2. *Given a two-step Carnot group $\mathbb{G}$ and a bounded domain $\Omega \subset \mathbb{G}$, consider $\varphi \in W^1_\nu(\Omega;\mathbb{G})$ with $\det \widehat{D}\varphi \ge 0$ almost everywhere. Then for every measurable set $K \Subset \Omega$ there is a constant $C(K)$ independent of φ such that*

$$\|M^\rho(\chi_K \det \widehat{D}\varphi) \mid L_1(\Omega)\| \le C(K)\big(\|D_h\varphi \mid L_\nu(\Omega)\|^{\frac{\nu}{\nu-1}} + |\Omega| \cdot \|D_h\varphi \mid L_\nu(\Omega)\|\big).$$

Proof. Fix a measurable set $K \Subset \Omega$. Put $g = \chi_K \det \widehat{D}\varphi$ and $d = \mathrm{dist}_\rho(K,\partial\Omega)$, as well as $\alpha = \frac{1}{2c}$, and $\beta = \frac{1}{6c^2}$, where c is the constant involved in the generalized triangle inequality. To estimate $\frac{1}{|B_\rho(x,R)|}\int\limits_{B_\rho(x,R)} g(y)\,dy$, make a brute-force search of the cases.

If $x \in \Omega$ is an arbitrary point and $R > \beta d$, then

$$\frac{1}{|B_\rho(x,R)|}\int\limits_{B_\rho(x,R)} g(y)\,dy \le c_1(\beta d)^{-\nu}\int\limits_\Omega \det \widehat{D}\varphi(y)\,dy.$$

However, if $\mathrm{dist}_\rho(x,K) > \alpha d$ and $R \le \beta d$, then by the choice of α and β the intersection $K \cap B_\rho(x,R)$ is empty, and so $\frac{1}{|B_\rho(x,R)|} \int\limits_{B_\rho(x,R)} g(y)\,dy = 0$.

Assume now that $\mathrm{dist}_\rho(x,K) \le \alpha d$ and $R \le \beta d$. In this case the ball $B_\rho(x,2R)$ lies in Ω, and so for almost all x with $\mathrm{dist}_\rho(x,K) \le \alpha d$ and almost all $r \in (R;2R)$ we have

$$\Big(\int\limits_{B_\rho(x,R)} g(y)\,dy\Big)^{\frac{\nu-1}{\nu}} \le \Big(\int\limits_{B_\rho(x,r)} \det \widehat{D}\varphi(y)\,dy\Big)^{\frac{\nu-1}{\nu}} \le C_I \int\limits_{S_\rho(x,r)} |\mathrm{adj}\widehat{D}\varphi(y)|\,d\mathcal{H}^{\nu-1}(y).$$

Integrate the last inequality over $r \in (R;2R)$, see the coarea formula [10, Theorem 6.1], and divide by $|B_\rho(x,R)|$. Taking into account the local boundedness of the horizontal gradient of the function ρ and making some easy rearrangements, we obtain[6]

$$\Big(\frac{1}{|B_\rho(x,r)|}\int\limits_{B_\rho(x,R)} g(y)\,dy\Big)^{\frac{\nu-1}{\nu}} \le \frac{C}{|B_\rho(x,2R)|}\int\limits_{B_\rho(x,2R)} |\mathrm{adj}\widehat{D}\varphi(y)|\,dy \le CM^\rho f(x),$$

where $f = |\mathrm{adj}\widehat{D}\varphi| \in L_{\frac{\nu}{\nu-1}}(\Omega)$, while C is a constant independent of φ, x and r. Adding the resulting estimates, we see that

$$M^\rho g(x) \le C(K)\big((M^\rho f(x))^{\frac{\nu}{\nu-1}} + \|\det \widehat{D}\varphi \mid L_1(\Omega)\|\big)$$

for almost all $x \in \Omega$. Integrating this over $x \in \Omega$, we obtain

$$\|M^\rho g \mid L_1(\Omega)\| \le C(K)\big(\|M^\rho f \mid L_{\frac{\nu}{\nu-1}}(\Omega)\|^{\frac{\nu}{\nu-1}} + |\Omega|\cdot\|\det \widehat{D}\varphi \mid L_1(\Omega)\|\big).$$

It remains to observe that the Hardy–Littlewood theorem [17, Chapter I.3, Theorem 1], Hölder's inequality, and (2.3) yield

$$\|M^\rho f \mid L_{\frac{\nu}{\nu-1}}(\Omega)\| \le C\|\mathrm{adj}\widehat{D}\varphi \mid L_{\frac{\nu}{\nu-1}}(\Omega)\| \le C\|D_h\varphi \mid L_\nu(\Omega)\|,$$

$$\|\det \widehat{D}\varphi \mid L_1(\Omega)\| \le C\|D_h\varphi \mid L_\nu(\Omega)\|.$$

□

Corollary 3.1 and Proposition 3.2 directly imply the following statement.

Theorem 3.2. *Given a domain Ω in a two-step Carnot group $\mathbb{G}$, if $\{\varphi_k : \Omega \to \mathbb{G}\}$ is a sequence of mappings of class $W^1_{\nu,\mathrm{loc}}(\Omega;\mathbb{G})$ such that $\det \widehat{D}\varphi_k \ge 0$ almost everywhere and the sequence $\{|D_h\varphi_k|\}$ is bounded in $L_{\nu,\mathrm{loc}}(\Omega)$, then the sequence $\{\det \widehat{D}\varphi_k\}$ of Jacobians is uniformly integrable on every compact set $K \Subset \Omega$.*

Let us use the following particular case of Theorem 1 of [6].

Lemma 3.3. *Consider domains Ω, Ω'_0, $\Omega'_1, \dots$ in $\mathbb{R}^N$ and a sequence of homeomorphisms $\{\varphi_k : \Omega \to \Omega'_k\}_{k=1}^\infty$ converging locally uniformly in Ω to some homeomorphism $\varphi_0 : \Omega \to \Omega'_0$.*

Then every compact set $K \Subset \Omega'_0$ lies in Ω'_k for all sufficiently large k, while the sequence $\{\varphi_k^{-1}\}$ of the inverse homeomorphisms converges to φ_0^{-1} locally uniformly on Ω'_0.

[6]In these estimates we use the property that the full-dimensional Hausdorff ν-measure and the Hausdorff measure on the level lines of the function $\varphi(y) = \rho(y^{-1}x)$ considered in [10, Theorem 6.1] are equivalent respectively to the Lebesgue measure and the Hausdorff measure defined with respect to homogeneous norm (2.1).

Recall that the space $C_0(\Omega)$ consists of all continuous functions $\theta : \Omega \to \mathbb{R}$ with compact support in Ω.

Lemma 3.4. *Consider domains Ω, Ω'_0, $\Omega'_1, \dots$ in some Carnot group $\mathbb{G}$ and a sequence $\{\varphi_k : \Omega \to \Omega'_k\}_{k=1}^{\infty}$ of homeomorphisms of class $W^1_{\nu,\mathrm{loc}}(\Omega;\mathbb{G})$ such that $\{\varphi_k\}$ converges to some homeomorphism $\varphi_0 : \Omega \to \Omega'_0$ locally uniformly in Ω, the sequence $\{|D_h\varphi_k|\}_{k=1}^{\infty}$ is bounded in $L_{\nu,\mathrm{loc}}(\Omega)$, and $\det \widehat{D}\varphi_k \geq 0$ almost everywhere, for $k = 1, 2, \dots$.*

Then the sequence $\{\det \widehat{D}\varphi_k\}$ of Jacobians converges $*$-weakly in $L_{1,\mathrm{loc}}(\Omega)$ to $\det \widehat{D}\varphi_0$, that is

$$\lim_{k\to\infty} \int\limits_{\Omega} \theta(x) \det \widehat{D}\varphi_k(x)\, dx = \int\limits_{\Omega} \theta(x) \det \widehat{D}\varphi_0(x)\, dx$$

for all functions $\theta \in C_0(\Omega)$.

Proof. Since the sequence $\{|D_h\varphi_k|\}$ is bounded in $L_{\nu,\mathrm{loc}}(\Omega)$, the limit homeomorphism φ_0 is also of class $W^1_{\nu,\mathrm{loc}}(\Omega;\mathbb{G})$ [22, Proposition 3.3].

For all quasi-monotone mappings $\varphi \in W^1_{\nu,\mathrm{loc}}(\Omega;\mathbb{G})$, in particular for all homeomorphisms, we have the following change-of-variables formula [19, Theorem 4]:

$$\int\limits_{D} (u \circ \varphi)(x) \det \widehat{D}\varphi(x)\, dx = \int\limits_{\mathbb{G}} u(y)\mu(y, \varphi, D)\, dy, \tag{3.3}$$

where $D \Subset \Omega$ is a compactly embedded subdomain such that $|\varphi(\partial D)| = 0$, while $\mu(y, \varphi, D)$ is the topological degree of the mapping φ at $y \notin \varphi(\partial D)$ defined with respect to the domain D, while u is an arbitrary measurable function such that the function $y \mapsto u(y)\mu(y, \varphi, D)$ is integrable on $\mathbb{G}$.

According to [19, Theorem 3], all quasi-monotone mappings of class $W^1_{\nu,\mathrm{loc}}(\Omega;\mathbb{G})$ have Luzin's $\mathcal{N}$-property. Hence, for every finite collection of balls $B_j \Subset \Omega$ and arbitrary $k = 0, 1, \dots$ the measure of the set $\varphi_k(\partial \bigcup_j B_j)$ vanishes. Consequently, we can put $D = \bigcup_j B_j$ in (3.3).

The degree $\mu(\cdot, \varphi, D)$ of each homeomorphism $\varphi : \overline{D} \to \mathbb{G}$ is a constant on the image $\varphi(D)$ and equals either 1 or -1. Since $\det \widehat{D}\varphi_k \geq 0$ almost everywhere on Ω, for $k = 1, 2, \dots$, we find that (3.3) applied to the mapping $\varphi = \varphi_k$ and the functions $u = \chi_{\varphi_k(D)}$ and $D = \bigcup_j B_j$ for $k = 1, 2, \dots$ implies that $\mu(y, \varphi_k, D) = 1$ for $y \in \varphi_k(D)$.

Furthermore, the continuity of the degree of a mapping under uniform convergence also implies that $\mu(y, \varphi_0, D) = 1$ for $y \in \varphi_0(D)$. Now put $\varphi = \varphi_0$ and $u = \chi_U$ in (3.3), where $U \subset \varphi_0(D)$ is an arbitrary open set. This yields

$$\int\limits_{\varphi_0^{-1}(U)} \det \widehat{D}\varphi_0(x)\, dx = \int\limits_{U} \mu(y, \varphi_0, D)\, dy = |U| > 0. \tag{3.4}$$

Since φ_0 is a homeomorphism, while the open set $U \subset \varphi_0(D)$ and the balls $B_j \Subset \Omega$ which constitute the subdomain D are arbitrary, (3.4) implies that $\det \widehat{D}\varphi_0$ is nonnegative almost everywhere on Ω.

Put $\psi_k = \varphi_k^{-1}$. For $\theta \in C_0(\Omega)$ and $k = 0, 1, 2, \dots$ the change-of-variables formula (3.3) yields[7]

$$\int\limits_{\Omega} \theta(x) \det \widehat{D}\varphi_k(x)\, dx = \int\limits_{\Omega'_k} \theta(\psi_k(y))\, dy.$$

[7]As D we should consider a finite union of compactly embedded balls in Ω covering the support of the function θ.

Since $\{\varphi_k\}_{k=1}^{\infty}$ converges uniformly to φ_0 on the support of θ, according to Lemma 3.3 the supports of the functions $\theta \circ \psi_k$ for all sufficiently large k lie in some compact set $K \Subset \Omega'_0$. The uniform convergence of $\{\psi_k\}$ to ψ_0 on K implies that

$$\lim_{k\to\infty} \int\limits_{\Omega} \theta(x) \det \widehat{D}\varphi_k(x)\, dx = \lim_{k\to\infty} \int\limits_{\Omega'_k} \theta(\psi_k(y))\, dy = \lim_{k\to\infty} \int\limits_{K} \theta(\psi_k(y))\, dy$$
$$= \int\limits_{K} \theta(\psi_0(y))\, dy = \int\limits_{\Omega'_0} \theta(\psi_0(y))\, dy = \int\limits_{\Omega} \theta(x) \det \widehat{D}\varphi_0(x)\, dx.$$

Finally, Theorem 3.2 and Lemma 3.4 imply Theorem 1.2 in the standard fashion. □

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