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A DISCRETE MODEL OF A TRANSMISSION LINE AND THE FABER POLYNOMIALS

V.G. Kurbatov

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Key words: discrete transmission line model, interconnection, matrix exponential, Faber polynomial, spectrum.

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Abstract. The spectrum of the matrix coefficient A corresponding to a discrete model of a transmission line often has the shape of a cross. The paper suggests to use the Faber series instead of the Taylor series when calculating the matrix exponential of A. This method can enlarge the accuracy and speed up calculations.

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1 Introduction

Transmission lines play the role of interconnections in electrical circuits. Discrete transmission line models (see an example in Figure 5) are often used in circuit theory, see, e. g., [6, 7, 11, 22, 25]. Discrete modelling of a transmission line may be more convenient than a more accurate partial differential equation description because, together with equations of other circuit elements, we obtain a system of ordinary differential (and possibly algebraic) equations only. Furthermore, approximate solving of partial differential equations usually also involves passing to a discrete model, which leads to a similar loss of accuracy.

A linear discrete stationary circuit is described (after eliminating algebraic equations) by an ordinary differential equation of the form $x'(t) = Ax(t) + f(t)$ with a matrix coefficient A. Its solving is reduced to finding the matrix exponential e^{At}, see Section 2. In turn, approximate calculation of e^{At} is usually based [10, 15, 20, 21] on approximation of the function $\exp_t(\lambda) = e^{\lambda t}$ by a polynomial (or a rational function) p_t on the spectrum $\sigma(A)$ of A and subsequent substitution of A into p_t.

An approximation of $\exp_t$ on a set wider than $\sigma(A)$ is not necessary. Moreover, it usually decreases the accuracy of approximation (by a polynomial of the same degree); an example of this phenomena is demonstrated in Figures 6-8. Using the Faber polynomials (see the definition in Section 3) allows us to restrict the set of approximation to (almost) $\sigma(A)$, i. e., the minimal possible. The idea of using the Faber polynomials to calculate matrix functions has been employed by many authors, see, e. g., [3, 4, 5, 14, 16, 26, 27, 28, 29, 31].

We propose to apply the Faber polynomials for approximate solving equations (Section 6) of a discrete model of a transmission line. In this case, the spectrum $\sigma(A)$ has a cross shape, see Figure 1. Our numerical experiments (Section 7) demonstrate that using the Faber expansion instead of the Taylor expansion can increase the accuracy by a factor of 100–1000. The main results of the paper are the exact formulas for the Faber functions Ψ and Φ for the cross (Section 4), and the algorithm (Section 5) that calculates the coefficients for expansion (3.6) of the exponential function in the Faber series.

For numerical calculations we use ‘Wolfram Mathematica’ [34].

2 Functions of matrices

In this section, we recall the definition of a matrix function and its main application.

Let A be a complex square matrix. The spectrum of A is the set $\sigma(A) \subseteq \mathbb{C}$ of all its eigenvalues. Let f be a complex-valued holomorphic function defined in a neighbourhood of $\sigma(A)$. The function f applied to A is [10, 15, 20, 21] the matrix

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)\, (\lambda \mathbf{1} - A)^{-1}\, d\lambda,$$

where the contour Γ surrounds $\sigma(A)$ and $\mathbf{1}$ is the identity matrix.

The exponential function $\exp_t(\lambda) = e^{\lambda t}$ is the most important for applications. It depends on the parameter $t \in \mathbb{R}$. Its importance is explained by the fact that the solution of the initial value problem

$$\begin{aligned} x'(t) &= Ax(t) + f(t), \\ x(t_0) &= x_0 \end{aligned}$$

can be represented in the form

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)} f(t)\, d\tau.$$

More generally, let the relation between the input vector function u and the output vector function y be described by the relations

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ x(t_0) &= x_0, \end{aligned}$$

where A, B, C, D — are matrices of compatible sizes. Then the dependence of y on u can [1, p. 65] be expressed as

$$y(t) = C\left(e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-r)} Bu(r)\, dr \right) + Du(t).$$

Usually, the matrix exponential e^{At} can be calculated only numerically. There is vast literature on approximate calculation of e^{At}, see, e. g., [10, 15, 20, 21]. The main goal is fast and accurate calculations; it is clear that these two goals are contradictory. Most methods for approximate calculation of $f(A)$ are based on approximating f by a polynomial or a rational function and substituting A into it. In this paper, we consider a special case when the differential equation $x'(t) = Ax(t) + f(t)$ describes a discrete model of a transmission line (Section 6). In this case, the set $\sigma(A)$ has the shape of a cross, see Figure 1. We use the Faber polynomials generated by the cross to reduce the order of the approximating polynomial.

3 Faber polynomials

A detailed exposition of the theory of Faber series and expansions can be found in [13, 19, 24, 30, 32]. Here we only recall the facts that are necessary for our aims.

Let $K \subset \mathbb{C}$ be a compact simply connected set containing more than one point. We denote by G the complement $\mathbb{C} \setminus K$. Let

$$D = \{\, w \in \mathbb{C} : \, |w| > 1 \,\}.$$

It is known [24, p. 104] that there exists a unique mapping $\Phi : G \to D$ such that (i) Φ is bijective, (ii) Φ has a complex derivative at all points $z \in G$ with $\Phi'(z) \neq 0$, and (iii) there exists a number $\gamma > 0$ such that

$$\lim_{z\to\infty} \Phi(\infty) = \infty \qquad \text{and} \qquad \lim_{z\to\infty} \frac{\Phi(z)}{z} = \gamma. \tag{3.1}$$

The number γ is called the *capacity* of K. Evidently, in a neighborhood of infinity, the function Φ possesses the Laurent expansion

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \frac{\gamma_3}{z^3} + \dots, \tag{3.2}$$

where $\gamma > 0$ is the same as in (3.1). The general theory of Laurent series states that series (3.2) converges absolutely for all z such that $z \in G_0$, where G_0 is the outer part of the smallest circle with center at zero containing K:

$$G_0 = \{\, z \in \mathbb{C} : \, |z| > |\zeta| \text{ for all } \zeta \in K \,\}.$$

For $z \in G_0$, we have the representation

$$\Phi^n(z) = \Big(\gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \frac{\gamma_3}{z^3} + \dots\Big)^n.$$

Due to absolute convergence, the Laurent series in the parentheses can be multiplied and summed in any order. As a result we obtain the Laurent expansion of the function Φ^n. Removing the parentheses we see that the Laurent series of Φ^n has the form

$$\Phi^n(z) = \gamma^n z^n + a_{n-1}^{(n)} z^{n-1} + \dots + a_1^{(n)} z + a_0^{(n)} + \frac{b_1^{(n)}}{z} + \frac{b_2^{(n)}}{z^2} + \frac{b_3^{(n)}}{z^3} + \dots. \tag{3.3}$$

The polynomials

$$\Phi_n(z) = \gamma^n z^n + a_{n-1}^{(n)} z^{n-1} + \dots + a_1^{(n)} z^1 + a_0^{(n)} \tag{3.4}$$

containing the terms with nonnegative powers of z in Laurent expansions (3.3) of Φ^n are called [24, p. 105], [32, p. 33] the *Faber polynomials generated by* K. By definition, $\Phi_0(z) = 1$.

We denote by $\Psi : D \to G$ the inverse of $\Phi : G \to D$. It is easy to show that Ψ has the Laurent expansion of the form

$$\Psi(w) = \beta w + \beta_0 + \frac{\beta_1}{w} + \frac{\beta_2}{w^2} + \frac{\beta_3}{w^3} + \dots, \tag{3.5}$$

with $\beta = 1/\gamma$. Series (3.5) converges absolutely for all $w \in D$.

Often a holomorphic functions f can be represented as the *Faber series*

$$f(z) = \sum_{n=0}^{\infty} c_n \Phi_n(z),$$

and such an expansion is unique. For our aims, it is important that the Faber series converges faster [13, 19, 24, 30, 32] than the Taylor series. An accurate formulation of the existence of the Faber series expansion is presented in the following theorem.

Theorem 3.1 ([32, Chapter III, § 2]). *Let f be a holomorphic function defined on an open neighbourhood U of K. Let the function Ψ possess a continuous extension to the closure*

$$\overline{D}=\{\,w\in\mathbb{C}:\ |w|\geq 1\,\}.$$

Then the function f can be expanded into the Faber series

$$f(z)=\sum_{n=0}^{\infty}c_n\Phi_n(z), \tag{3.6}$$

which uniformly converges on compact subsets of U. The coefficients c_n can be found by the formula

$$c_n=\frac{1}{2\pi i}\int_{|w|=1}\frac{f\big(\Psi(w)\big)}{w^{n+1}}\,dw,\qquad n=0,1,\dots. \tag{3.7}$$

It is known [19, § 18.2.V] that the approximation of f by partial sums of series (3.6) is close to the best uniform approximation on K by polynomials. This fact explains the efficiency of the transition from the Taylor approximation to the Faber one.

For our goal, it is important that expansion (3.6) extends to functions of matrices.

Corollary 3.1 ([16, Theorem 3.1]). *Let assumptions of Theorem 3.1 be satisfied. Let A be a square complex matrix with $\sigma(A)\subseteq K$. Then*

$$f(A)=\sum_{n=0}^{\infty}c_n\Phi_n(A).$$

4 The functions Ψ and Φ for the cross

For some sets K, the Faber polynomials can be calculated explicitly. Examples can be found in [2, 8, 9, 17, 24, 32]. In this section, we restrict ourselves to the case, which is related to our problem.

Let $a,b>0$ and $c\in\mathbb{R}$ be some numbers. We consider the set $K\subseteq\mathbb{C}$ shown in left Figure 1 and having the shape of a cross. It consists of two segments intersecting at the point c on the real axis. The endpoints of one segment are the points $c-a$ and $c+a$, the endpoints of the second segment are the points $c-ib$ and $c+ib$. In our situation, K contains the spectrum of our matrix A, see Section 7.

Theorem 4.1. *For the cross shown in the left Figure 1 with parameters $a>0$, $b>0$ and $c\in\mathbb{R}$, the function Ψ has the form*

$$\Psi(w)=c+w\sqrt{\frac{a^2+b^2}{2}}\sqrt{\frac{a^2-b^2}{a^2+b^2}\,\frac{1}{w^2}+\frac{1}{2}\Big(1+\frac{1}{w^4}\Big)}, \tag{4.1}$$

where the square root means the principal value, i. e. $\sqrt{\cdot}$ takes values in the right complex plane

$$\mathbb{C}_r=\{\,z\in\mathbb{C}:\ \operatorname{Re}z>0\,\}\cup\{0\}.$$

The function Ψ is bijective and holomorphic on D, and is continuous on the closure

$$\overline{D}=\{\,w\in\mathbb{C}:\ |w|\geq 1\,\}.$$

The conditions

$$\lim_{w\to\infty}\Psi(w)=\infty\qquad\text{and}\qquad\lim_{w\to\infty}\frac{\Psi(w)}{w}=\frac{\sqrt{a^2+b^2}}{2}. \tag{4.2}$$

are satisfied.

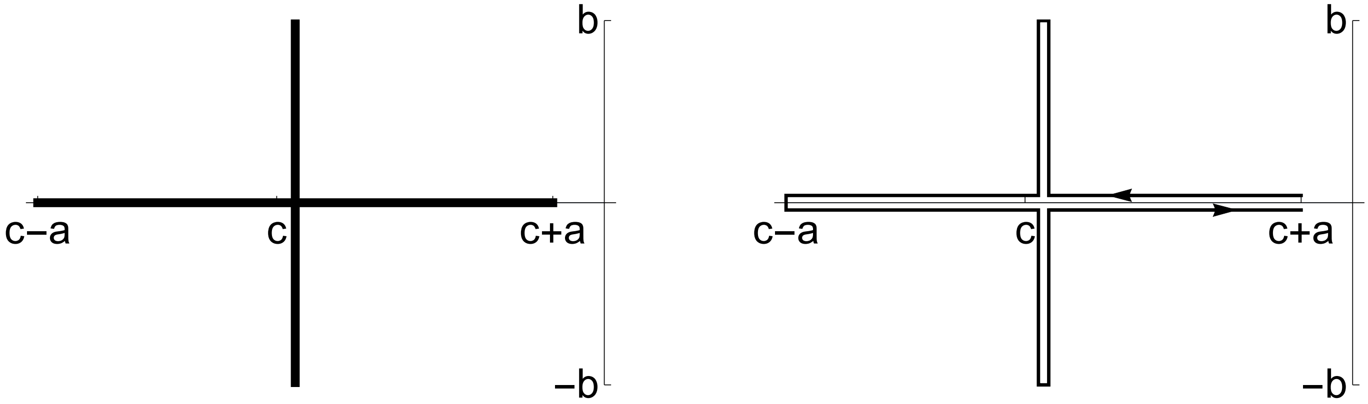


Figure 1: Left: set K having the cross shape; right: the image of the set $\partial D = \{\, w \in \mathbb{C} : \, |w| = 1 \,\}$ under the action of the function Ψ

Proof. The considered principal value of the square root function $\sqrt{\cdot}$ is defined and continuous on the complement of the open half-line

$$L = \{\, z \in \mathbb{C} : \, z \in \mathbb{R} \text{ and } z < 0 \,\}$$

and holomorphic on the complement of the closed half-line

$$\overline{L} = \{\, z \in \mathbb{C} : \, z \in \mathbb{R} \text{ and } z \leq 0 \,\}.$$

Therefore, Ψ is (defined and) holomorphic at $w \in \mathbb{C}$ as long as $\zeta(w) \notin \overline{L}$, where

$$\zeta(w) = \frac{a^2 - b^2}{a^2 + b^2} \frac{1}{w^2} + \frac{1}{2}\Big(1 + \frac{1}{w^4}\Big),$$

and Ψ is (defined and) continuous at $w_0 \in \mathbb{C}$ if $w \notin L$ for all w in a neighbourhood of w_0.

Let us find out when $\zeta(w) \in L$ and $\zeta(w) \in \overline{L}$. For brevity we set $g = \frac{a^2-b^2}{a^2+b^2}$; obviously, g can take any value from $(-1, 1)$. We represent w in the form $w = r(\cos t + i \sin t)$, where $r > 0$ and $t \in (-\pi/2, \pi/2]$, and substitute it into the definition of ζ:

$$\begin{aligned}
\zeta(w) &= g\frac{1}{w^2} + \frac{1}{2}\Big(1 + \frac{1}{w^4}\Big) \\
&= \frac{2gw^2 + w^4 + 1}{2w^4} \\
&= \frac{2gr^2(\cos t + i \sin t)^2 + r^4(\cos t + i \sin t)^4 + 1}{2r^4(\cos t + i \sin t)^4} \\
&= \frac{\big(2gr^2(\cos t + i \sin t)^2 + r^4(\cos t + i \sin t)^4 + 1\big)(\cos t - i \sin t)^4}{2r^4} \\
&= \frac{2gr^2(\cos t - i \sin t)^2 + r^4 + (\cos t - i \sin t)^4}{2r^4} \\
&= \frac{-2gr^2 \sin^2 t + 2gr^2 \cos^2 t + r^4 + \sin^4 t + \cos^4 t - 6 \sin^2 t \cos^2 t}{2r^4} \\
&\quad + i\frac{-4gr^2 \sin t \cos t - 4 \sin t \cos^3 t + 4 \sin^3 t \cos t}{2r^4} \\
&= \frac{2gr^2 \cos 2t + r^4 + 2\cos^2 2t - 1}{2r^4} - i\frac{\big(gr^2 + \cos 2t\big) \sin 2t}{r^4}.
\end{aligned} \tag{4.3}$$

We observe that $\zeta(w) \in \overline{L}$ (respectively, $\zeta(w) \in L$) if and only if (i) $\operatorname{Im}\zeta(w) = 0$ and (ii) $\operatorname{Re}\zeta(w) \le 0$ (respectively, $\operatorname{Re}\zeta(w) < 0$). According to representation (4.3), conditions (i) and (ii) mean that

$$\begin{gathered}\left(gr^2 + \cos 2t\right)\sin 2t = 0,\\ 2gr^2\cos 2t + r^4 + 2\cos^2 2t - 1 \le 0 \quad (<0).\end{gathered}$$

The first condition is satisfied if and only if $t = 0, \pm\pi/2, \pi$ or $\cos 2t = -gr^2$ (provided $|gr^2| \le 1$).

After substituting $t = 0, \pm\pi/2, \pi$, the second condition turns into

$$\pm 2gr^2 + r^4 + 2 - 1 \le 0 \quad (<0)$$

or

$$(r^2-1)^2 + 2r^2(1 \pm g) \le 0 \quad (<0),$$

which is never true, because $|g| < 1$ and $r > 0$. Thus, in this case $\zeta(w) \notin \overline{L}$ and, moreover, $\zeta(w) \notin L$.

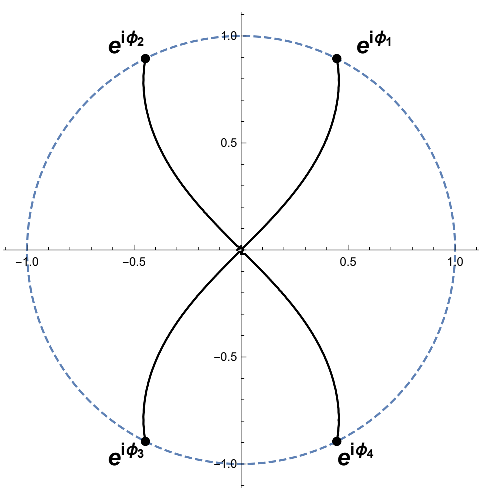
After substituting $\cos 2t = -gr^2$, the second condition turns into

$$r^4 - 1 \le 0 \quad (<0).$$

If $w \in D$, i. e. $r > 1$, then $r^4 - 1 > 0$ and $r^4 - 1 \le 0$ does not hold; thus $\zeta(w) \notin \overline{L}$ for all $w \in D$. Therefore, the function ζ is holomorphic on D.

However, if $w \in \overline{D}$, i. e. $r \ge 1$, then $r^4 - 1 \ge 0$ and only $r^4 - 1 < 0$ does not hold; thus $\zeta(w) \notin L$ for all $w \in \overline{D}$. Therefore, the function ζ is only continuous on $\overline{D}$.

For a curious reader, the entire set of points w at which $\zeta(w) \in \overline{L}$ (not only its intersection with $\overline{D}$) is shown in the left Figure 2.



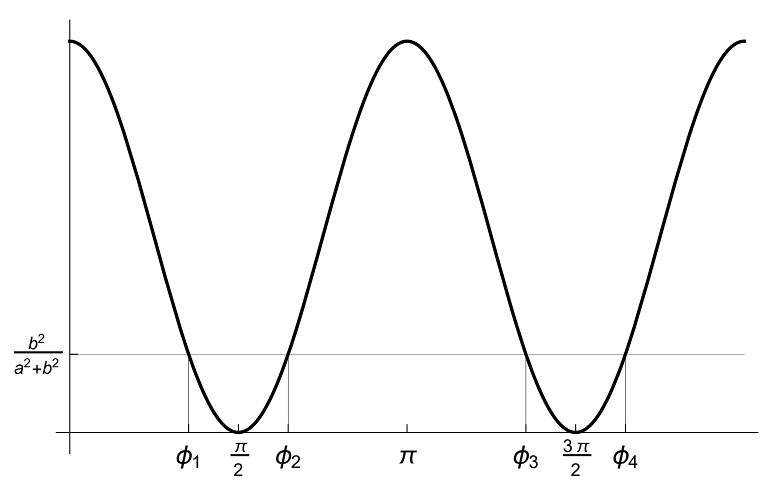


Figure 2: Left: the bold curves constitute the set of points w such that $\zeta(w) \in \overline{L}$; right: the solutions $\varphi_{1,2,3,4}$ of the equation $\cos^2 t - \frac{b^2}{a^2+b^2} = 0$ on $[0, 2\pi]$

Now let us move w along the boundary ∂D. From (4.3) we have

$$\begin{aligned}\zeta(e^{it}) &= \frac{1}{2}\left(2g\cos 2t + 2\cos^2 2t\right) - i\sin 2t(g + \cos 2t)\\ &= \cos 2t(g + \cos 2t) - i\sin 2t(g + \cos 2t)\\ &= (g + \cos 2t)e^{-2it}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Psi(e^{it}) &= c + e^{it}\frac{\sqrt{a^2+b^2}}{\sqrt{2}}\sqrt{(g+\cos 2t)e^{-2it}}\\ &= c + e^{it}\frac{\sqrt{a^2+b^2}}{\sqrt{2}}\sqrt{\Big(\frac{a^2-b^2}{a^2+b^2}+\cos 2t\Big)e^{-2it}}\\ &= c + e^{it}\frac{\sqrt{a^2+b^2}}{\sqrt{2}}\sqrt{\Big(\frac{a^2-b^2}{a^2+b^2}+2\cos^2 t-1\Big)e^{-2it}}\\ &= c + e^{it}\sqrt{a^2+b^2}\sqrt{\Big(\cos^2 t-\frac{b^2}{a^2+b^2}\Big)e^{-2it}}.\end{aligned}$$

We denote by $\varphi_{1,2,3,4}$ the solutions of the equation $\cos^2 t - \frac{b^2}{a^2+b^2} = 0$ (here the unknown is t) on $[0, 2\pi]$, see the right Figure 2. Then $\Psi(e^{it})$ can be represented as

$$\Psi(e^{it}) = c + e^{it}\sqrt{a^2+b^2}\sqrt{(\cos^2 t - \cos^2\varphi_1)e^{-2it}},$$

where

$$\varphi_1 = \arccos\frac{b}{\sqrt{a^2+b^2}}.$$

In Figure 3, the real and imaginary parts of the function $t \mapsto \Psi(e^{it}) - c$ are presented; the main features are the values (and signs!) at points of extremums.

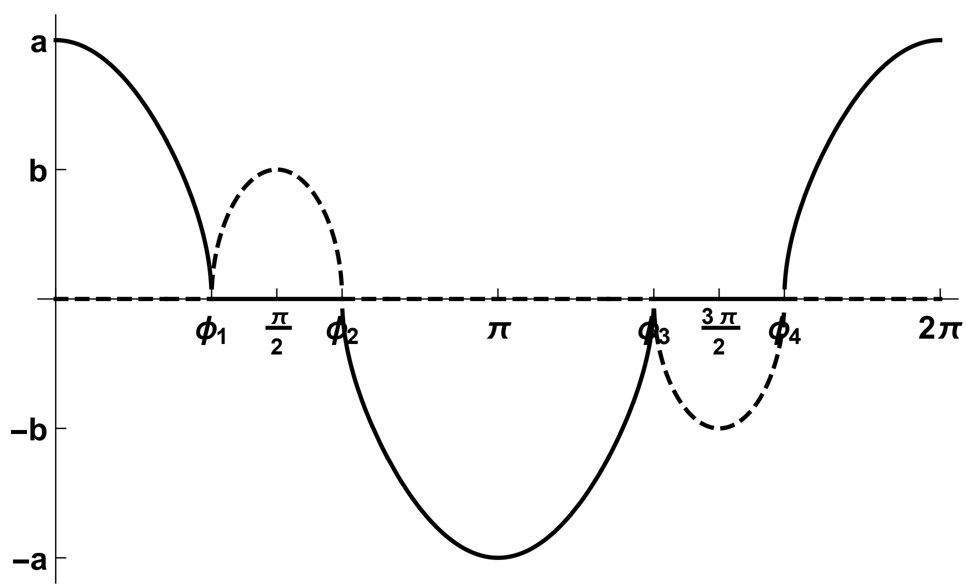


Figure 3: The real (solid line) and imaginary (dashed line) parts of the function $t \mapsto e^{it}\sqrt{a^2+b^2}\sqrt{(\cos^2 t - \cos^2\varphi_1)e^{-2it}}$

Let us describe the curve $z(t) = \Psi(e^{it})$, $t \in [0, 2\pi]$, see the right Figure 1. When $t \in [0, \varphi_1]$, the number $z(t)$ is real and moves from $c + a$ to c. When $t \in [\varphi_1, \pi/2]$, the number $z(t)$ is imaginary and varies form c to $c + ib$. When $t \in [\pi/2, \varphi_2]$, the number $z(t)$ remains imaginary and varies back form $c + ib$ to c. When $t \in [\varphi_2, \pi]$, the number $z(t)$ becomes real and varies form c to $c - a$. When $t \in [\pi, \varphi_3]$, the number $z(t)$ remains real and varies form $c - a$ to c. When $t \in [\varphi_3, 3\pi/2]$, the number $z(t)$ varies form c to $c - ib$. When $t \in [3\pi/2, \varphi_4]$, the number $z(t)$ varies form $c - ib$ to c. When $t \in [\varphi_4, 2\pi]$, the number $z(t)$ varies form c to $c + a$ and thus returns back.

The fulfillment of (4.2) immediately follows from (4.1).

It remains to prove that $\Psi : D \to G$ is bijective. Let us take an arbitrary $z \in \mathbb{C} \setminus K$ and consider the equation $\Psi(w) = z$. We have to prove that the equation $\Psi(w) = z$ has exactly one solution $w \in D$ for any $z \in G$.

We take an arbitrary point $z \in G$. We denote by S_R the circle $\{ w \in \mathbb{C} : |w| = R \}$ of large radius R centered at 0 and oriented counterclockwise, and we denote by $-S_1$ the circle $\{ w \in \mathbb{C} : |w| = 1 \}$ of the radius 1 centered at 0 and oriented clockwise, see Figure 4. We denote by $S_R - S_1$ the contour consisting of S_R and $-S_1$. It is clear that $S_R - S_1$ is the oriented boundary of the annulus $A_R = \{ w \in \mathbb{C} : 1 \le |w| \le R \}$. From (4.2) it follows that the point z lies inside the image $\Psi(S_R)$ of the circle S_R under the action of Ψ if R is large enough.

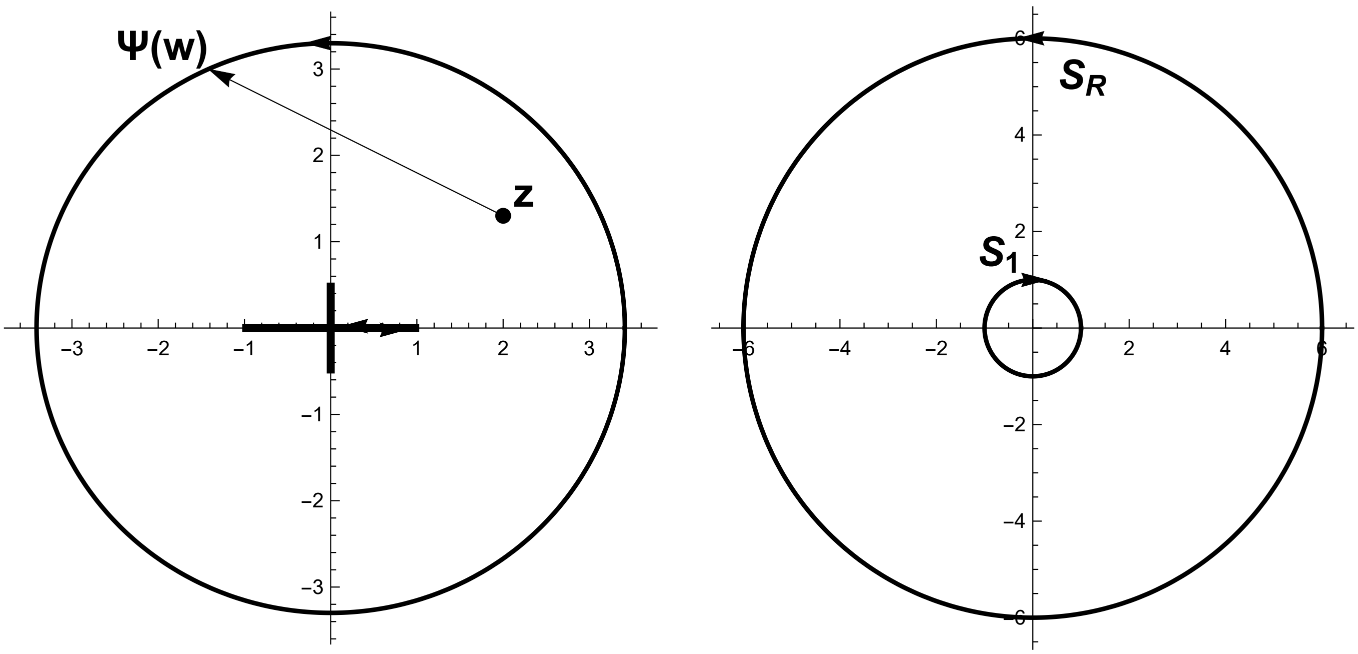


Figure 4: The circles S_R and S_1 (right) and their images (left) under the action of Ψ

We make use of the argument principle [23, p. 48, Theorem 2.3], [18, p. 278, Theorem 4.10a]: the number of solutions w of the equation $\Psi(w) = z$ in the annulus A_R is equal to the increment of the argument of the complex number $\Psi(w) - z$ along the oriented boundary $S_R - S_1$ divided by 2π. Since z lies outside $K = \Psi(S_1)$, the increment of the argument of $\Psi(w) - z$ along S_1 equals zero. On the other hand, from formula (4.1) and the Rouche theorem [23, p. 48, Theorem 2.4], [18, p. 280, Theorem 4.10b] (more correctly, from the proof of the Rouche theorem) it is seen that the increment of the argument of $\Psi(w) - z$ along S_R equals the increment of the argument of $\Psi_1(w) - z$ along S_R, where

$$\Psi_1(w) = c + w\frac{\sqrt{a^2 + b^2}}{2},$$

provided R is large enough. But the increment of the argument of $\Psi_1(w) - z$ along S_R is obviously equal to 2π. Thus, for all R large enough, there is exactly one solution w of the equation $\Psi(w) = z$ in the annulus A_R. Hence, there is exactly one solution in D. □

Corollary 4.1. *Let the assumptions of Theorem* 4.1 *be satisfied. Then for the function* $\Phi : G \to D$, *inverse to* Ψ, *conditions* (3.1) *are satisfied with* $\gamma = \frac{2}{\sqrt{a^2+b^2}}$.

Proof. The Laurent series of Ψ in a neighborhood of infinity has the form (3.5) and converges at all points of D. Hence, the series

$$h(w)=\beta_0+\frac{\beta_1}{w}+\frac{\beta_2}{w^2}+\frac{\beta_3}{w^3}+\dots$$

also converges in D and is bounded in $\overline{D_2}=\{\,w\in\mathbb{C}:\,|w|\geq 2\,\}$. At the same time, by Theorem 4.1, Ψ and, consequently, h are bounded in the annulus $A_2=\{\,w\in\mathbb{C}:\,1\leq|w|\leq 2\,\}$. Therefore h is bounded in D. Then from (4.2) it follows that $\Psi(w)\to\infty$, $w\in D$, implies that $w\to\infty$.

Let us calculate $\lim_{z\to\infty}\Phi(z)$. We set $w=\Phi(z)$ or $z=\Psi(w)$. By the proved, when $z=\Psi(w)\to\infty$, we also have $\Phi(z)=w\to\infty$. This shows that $\lim_{z\to\infty}\Phi(z)=\infty$.

Now, with the same change $w=\Phi(z)$ or $z=\Psi(w)$, we have $\lim_{z\to\infty}\frac{\Phi(z)}{z}=\lim_{w\to\infty}\frac{w}{\Psi(w)}=\frac{2}{\sqrt{a^2+b^2}}$. ☐

Corollary 4.2. *Let the assumptions of Theorem 4.1 be satisfied. Then the function $\Phi:G\to D$, inverse to Ψ, possesses the representation*

$$\Phi(z)=(z-c)\sqrt{\frac{\frac{b^2-a^2}{(z-c)^2}+2\sqrt{\left(1-\frac{a^2}{(z-c)^2}\right)\left(\frac{b^2}{(z-c)^2}+1\right)}+2}{a^2+b^2}}.$$

Proof. For brevity, we temporary set $g=\frac{a^2-b^2}{a^2+b^2}$ and $h=\frac{a^2+b^2}{2}$. To find Φ we solve the equation $z=\Psi(w)$ (from Theorem 4.1 we know that the solution exists and unique):

$$z=c+w\sqrt{h}\sqrt{g\frac{1}{w^2}+\frac{1}{2}\Big(1+\frac{1}{w^4}\Big)},$$

$$\frac{z-c}{w\sqrt{h}}=\sqrt{g\frac{1}{w^2}+\frac{1}{2}\Big(1+\frac{1}{w^4}\Big)},$$

$$\frac{(z-c)^2}{w^2h}=g\frac{1}{w^2}+\frac{1}{2}\Big(1+\frac{1}{w^4}\Big),$$

$$0=\frac{1}{2}+\Big(g-\frac{(z-c)^2}{h}\Big)\frac{1}{w^2}+\frac{1}{2}\frac{1}{w^4},$$

$$0=\frac{1}{2}w^4+\Big(g-\frac{(z-c)^2}{h}\Big)w^2+\frac{1}{2},$$

$$w^2=-g+\frac{(z-c)^2}{h}\pm\sqrt{\Big(-g+\frac{(z-c)^2}{h}\Big)^2-1},$$

$$w^2=\frac{b^2-a^2+2(z-c)^2}{a^2+b^2}\pm\sqrt{\Big(\frac{b^2-a^2+2(z-c)^2}{a^2+b^2}\Big)^2-1},$$

$$w^2=\frac{b^2-a^2+2(z-c)^2}{a^2+b^2}\pm\frac{1}{a^2+b^2}\sqrt{\big(b^2-a^2+2(z-c)^2\big)^2-\big(a^2+b^2\big)^2},$$

$$w^2=\frac{b^2-a^2+2(z-c)^2}{a^2+b^2}\pm\frac{2}{a^2+b^2}\sqrt{((z-c)^2-a^2)((z-c)^2+b^2)},$$

$$w^2=\frac{b^2-a^2+2(z-c)^2\pm 2\sqrt{((z-c)^2-a^2)((z-c)^2+b^2)}}{a^2+b^2},$$

$$w^2 = (z-c)^2 \frac{\frac{b^2-a^2}{(z-c)^2} + 2 \pm 2\sqrt{\left(1 - \frac{a^2}{(z-c)^2}\right)\left(1 + \frac{b^2}{(z-c)^2}\right)}}{a^2+b^2},$$

$$w = \pm(z-c)\sqrt{\frac{\frac{b^2-a^2}{(z-c)^2} + 2 \pm 2\sqrt{\left(1 - \frac{a^2}{(z-c)^2}\right)\left(1 + \frac{b^2}{(z-c)^2}\right)}}{a^2+b^2}}.$$

We choose the signs + in the both $\pm$ because $\lim_{z\to\infty} \frac{\Phi(z)}{z} = \frac{2}{\sqrt{a^2+b^2}}$. □

Now we can easily calculate the Faber polynomials Φ_n for the cross. According to definition (3.4) we calculate the initial terms of the Laurent series of the function $z \mapsto \Phi^n(z)$ and take its polynomial part. Since we have an exact representation for Φ (Corollary 4.2), the calculations can be performed symbolically and thus Φ_n can be found explicitly. For example,

$$\begin{aligned}\Phi_{11}(z) = 2\left(\frac{1}{a^2+b^2}\right)^{11/2} &(z-c)\Big(-11\,a^{10} + 55\,a^8\left(5\,b^2 + 4(z-c)^2\right) \\ &- 44\,a^6\left(25\,b^4 + 50\,b^2(z-c)^2 + 28(z-c)^4\right) \\ &+ 44\,a^4\left(25\,b^6 + 100\,b^4(z-c)^2 + 140\,b^2(z-c)^4 + 64(z-c)^6\right) \\ &- 11\,a^2\left(5\,b^4 + 20\,b^2(z-c)^2 + 16(z-c)^4\right)^2 \\ &+ 11\,b^{10} + 220\,b^8(z-c)^2 + 1232\,b^6(z-c)^4 + 2816\,b^4(z-c)^6 \\ &+ 2816\,b^2(z-c)^8 + 1024\,(z-c)^{10}\Big).\end{aligned}$$

5 Calculating the Faber coefficients of the exponential function

We begin with the presentation of a simple algorithm for calculating the Faber coefficients c_m in the expansion

$$e^z = \sum_{m=0}^{\infty} c_m \Phi_m(z).$$

For doing it we use formula (3.7):

$$c_m = \frac{1}{2\pi i}\int_{|w|=1} \frac{f\big(\Psi(w)\big)}{w^{m+1}}\,dw = \frac{1}{2\pi}\int_0^{\pi} \exp\big(\Psi(e^{it})\big)e^{-imt}\,dt. \tag{5.1}$$

Since the function Φ has breaks at the points $\varphi_{1,2,3,4}$ (see Figure 3), it is reasonable to represent the integral as the sum of four ones:

$$\int_0^{\pi} = \int_{-\varphi_1}^{\varphi_1} + \int_{\varphi_1}^{\varphi_2} + \int_{\varphi_2}^{\varphi_3} + \int_{\varphi_3}^{\varphi_4}$$

and use for each integral the Gauss quadrature rule with the Chebyshev weight. Since we are going to substitute a matrix A instead of z, a high accuracy in c_m is desirable. The high accuracy of integral values can be archived by calculating the integrals with an increased number of significant digits (this will not lead to the significant loss of time compared to matrix operations to come later).

Remark 1. A useful idea is proposed in paper [12]. According to formula (5.1), the numbers c_m can be interpreted as the Fourier coefficients of the function $w \mapsto \exp\big(\Psi(e^{it})\big)$. This observation makes it possible to use the fast Fourier transform to calculate integrals of kind (5.1), which speeds up calculations.

The above algorithm for calculating c_m has a drawback: it calculates e^{At} only at one point $t = 1$. Nevertheless, it is often important to have the resulting matrix e^{At} in the form of an expression depending on t. Now we present another algorithm that is free from this shortcoming.

By formula (3.5), the function Ψ has the expansion

$$\Psi(w) = \beta w + \beta_0 + \frac{\beta_1}{w} + \frac{\beta_2}{w^2} + \frac{\beta_3}{w^3} + \dots,$$

which converges in the open exterior D of the unit circle. We consider the Laurent expansions for the powers Ψ^n of Ψ:

$$\Psi^n(w) = \Big(\beta w + \beta_0 + \frac{\beta_1}{w} + \frac{\beta_2}{w^2} + \frac{\beta_3}{w^3} + \dots\Big)^n,$$

and in analogy with the Faber polynomials Φ_n define Ψ_n as the polynomial part of Ψ^n:

$$\Psi_n(w) = b_n^{(n)} w^n + b_{n-1}^{(n)} w^{n-1} + \dots + b_1^{(n)} w^1 + b_0^{(n)}.$$

The polynomials Ψ_n and their coefficients $b_k^{(n)}$ can be calculated symbolically (and therefore explicitly) in the same way as was done for Φ_n.

We set

$$M = \max_{|w|=1} |\Psi(w)|.$$

Obviously,

$$|\Psi^n(w)| \le M^n, \qquad |w| = 1. \tag{5.2}$$

From the formula for the Laurent coefficients [23, p. 6, Theorem 1.2] we have

$$b_k^{(n)} = \frac{1}{2\pi i} \int_{|w|=1} \frac{\Psi_n(w)}{w^{k+1}}\, dw = \frac{1}{2\pi i} \int_{|w|=1} \frac{\Psi^n(w)}{w^{k+1}}\, dw, \qquad 0 \le k \le n.$$

which implies

$$|b_k^{(n)}| \le M^n. \tag{5.3}$$

We consider the function $\exp_t(z) = e^{tz}$. For it, expansion (3.6) looks like

$$\exp_t(z) = \sum_{m=0}^{\infty} c_m(t) \Phi_m(z).$$

For the coefficients $c_m(t)$, from formula (3.7) we have (due to estimates (5.2) and (5.3), all series converges absolutely):

$$\begin{aligned}
\exp_t\big(\Psi(w)\big) &= \sum_{n=0}^{\infty} \frac{t^n \Psi^n(w)}{n!},\\
c_m(t) &= \frac{1}{2\pi i} \int_{|w|=1} \frac{\sum_{n=0}^{\infty} \frac{t^n \Psi^n(w)}{n!}}{w^{m+1}}\, dw\\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{1}{2\pi i} \int_{|w|=1} \frac{\Psi^n(w)}{w^{m+1}}\, dw\\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} b_m^{(n)} = \sum_{n=m}^{\infty} \frac{t^n}{n!} b_m^{(n)}.
\end{aligned}$$

The series $\sum_{n=m}^{\infty} \frac{t^n}{n!} b_m^{(n)}$ converges quickly. Therefore, we can use the approximate formula

$$c_m(t) \approx \sum_{n=m}^{N} \frac{t^n}{n!} b_m^{(n)},$$

where N is a large number; in our numerical examples we take $N = 20$. Numerical experiments show that for $t = 1$ the both algorithms give practically the same result.

6 A discrete model of a transmission line

We consider the circuit shown in Figure 5 consisting of $n = 150$ sections. It is a discrete transmission line model. We take the following parameters: $C = C_0/n$, $L = L_0/n$, $R = R_0/n$, $G = G_0/n$ (specific values of the constants C_0, L_0, R_0, G_0 are given in Figures 6-8). We use the state variable formulation [33] of the circuit to derive its equations in the form $\dot{x}(t) = Ax(t) + f(t)$ with the matrix A of the size 300×300. The chosen directions of voltages and currents are shown in Figure 5.

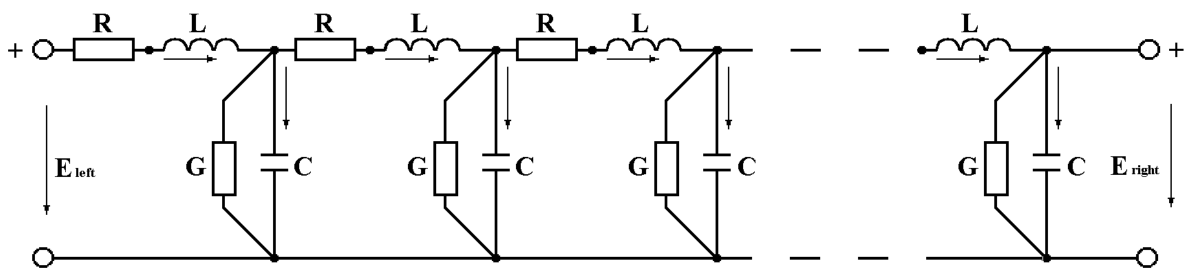


Figure 5: A discrete model of a transmission line

Let us assume that an independent voltage source $E_{\text{left}}(t)$ is connected to the left side, while the right side is open (the right contacts are disconnected). We take as unknowns the vector U_C of voltages across the inductors and the vector I_L of currents through the capacitors. Skipping dull calculations, we present the final differential equation that describe the considered circuit:

$$\begin{pmatrix} \dot{U}_C(t) \\ \dot{I}_L(t) \end{pmatrix} = -\begin{pmatrix} \frac{R_0}{C_0}\mathbf{1} & \frac{n}{C_0}(N - \mathbf{1}) \\ \frac{n}{L_0}(\mathbf{1} - N^T) & \frac{G_0}{L_0}\mathbf{1} \end{pmatrix} \begin{pmatrix} U_C(t) \\ I_L(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \dots \\ -\frac{n}{L_0}E_{left}(t) \\ \dots \\ 0 \end{pmatrix},$$

where the nonzero coordinate $-\frac{n}{L_0}E_{left}(t)$ in the free term corresponds to the first coordinate of I_L, $\mathbf{1}$ is the identity matrix of the size $n \times n$, and

$$N = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Thus the block matrix A has the form

$$A = -\begin{pmatrix} \frac{R_0}{C_0}\mathbf{1} & \frac{n}{C_0}(N - \mathbf{1}) \\ \frac{n}{L_0}(\mathbf{1} - N^T) & \frac{G_0}{L_0}\mathbf{1} \end{pmatrix}. \tag{6.1}$$

7 Numerical experiments

Example 1. We compare the approximation of the function $z \mapsto e^z$ by the 10-th Faber polynomial generated by the cross with different parameters and the 10-th Taylor polynomial. We take a discrete model of transmission line (Figure 5) consisting of 150 sections with parameters C_0, L_0, R_0, and G_0 shown at the tops of Figures 6-8. We calculate the spectrum of corresponding matrix (6.1) and the parameters a, b, and c of the corresponding cross that contains the spectrum. We graph the level curves of the functions

$$F(z) = \left|e^z - \sum_{k=0}^{10} c_k \Phi_k(z)\right|, \qquad T(z) = \left|e^z - \sum_{k=0}^{10} \frac{e^c}{k!}(z-c)^k\right|. \tag{7.1}$$

The results are shown in Figures 6-8. We present two level curves: the inner level curve corresponds to the minimal value C_F at which the F level curve surrounds the spectrum $\sigma(A)$; the outer level curve corresponds to the minimal value C_T at which the T level curve surrounds the spectrum $\sigma(A)$. The points of $\sigma(A)$ are shown by dots.

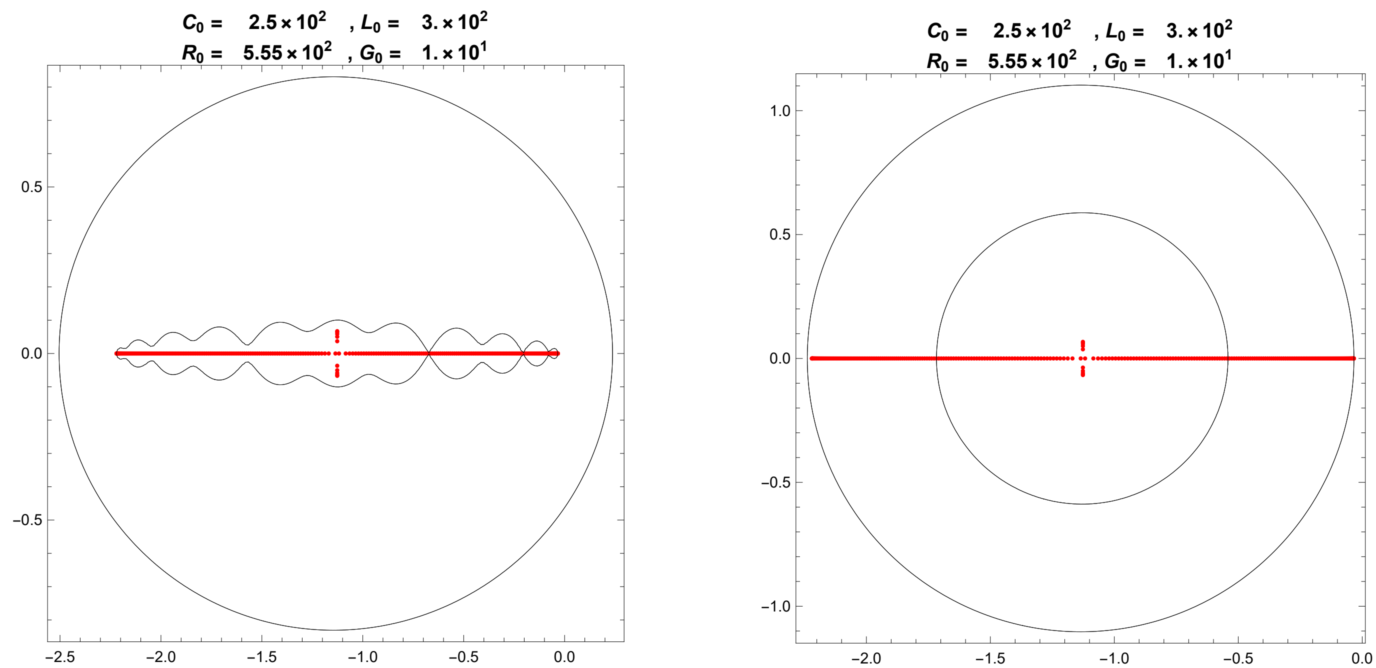


Figure 6: The eigenvalues of matrix (6.1) and the level curves of the functions F (left) and T (right) corresponding to the levels $C_F = 2.362 \cdot 10^{-11}$ and $C_T = 2.382 \cdot 10^{-8}$; $C_T/C_F = 1008.36$

Example 2. We consider the circuit with parameters shown in Figure 6 and the corresponding matrix A. We substitute A into the 10-th Faber polynomial and the 10-th Taylor polynomial, i. e. we calculate the matrices

$$E_F = \sum_{k=0}^{10} c_k \Phi_k(A), \qquad E_T = \sum_{k=0}^{10} \frac{e^c}{k!}(A - c\mathbf{1})^k.$$

We also calculate the precise matrix e^A using the `MatrixExp` command from 'Wolfram Mathematica' [34]. The comparison of accuracy gives

$$\|e^A - E_F\| = 4.7 \cdot 10^{-10}, \qquad \|e^A - E_T\| = 2.4 \cdot 10^{-8}.$$

For matrices, we use the norm induced by the Euclidean norm in $\mathbb{C}^{2n}$.

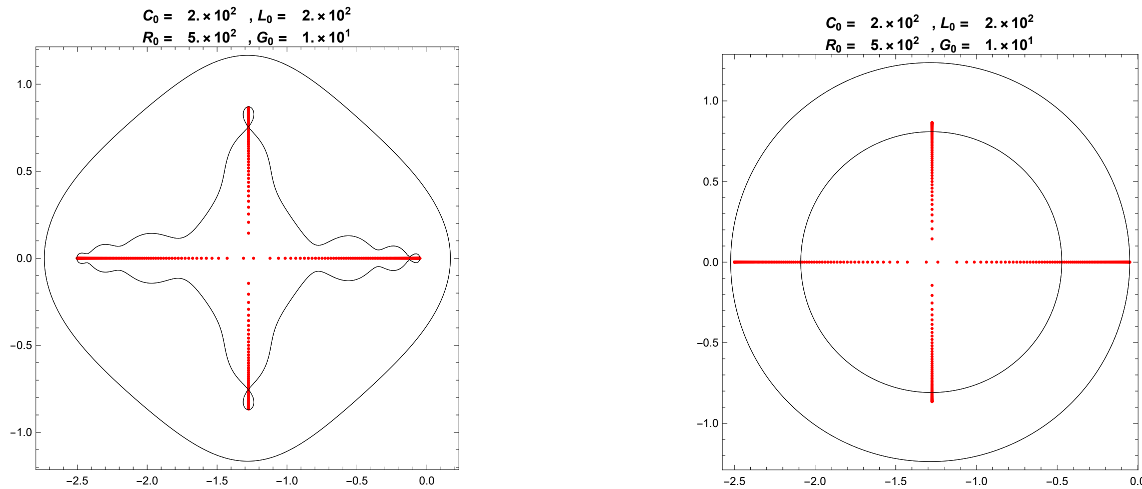


Figure 7: The eigenvalues of matrix (6.1) and the level curves of the functions F (left) and T (right) corresponding to the levels $C_F = 6.795 \cdot 10^{-10}$ and $C_T = 7.259 \cdot 10^{-8}$; $C_T/C_F = 106.8$

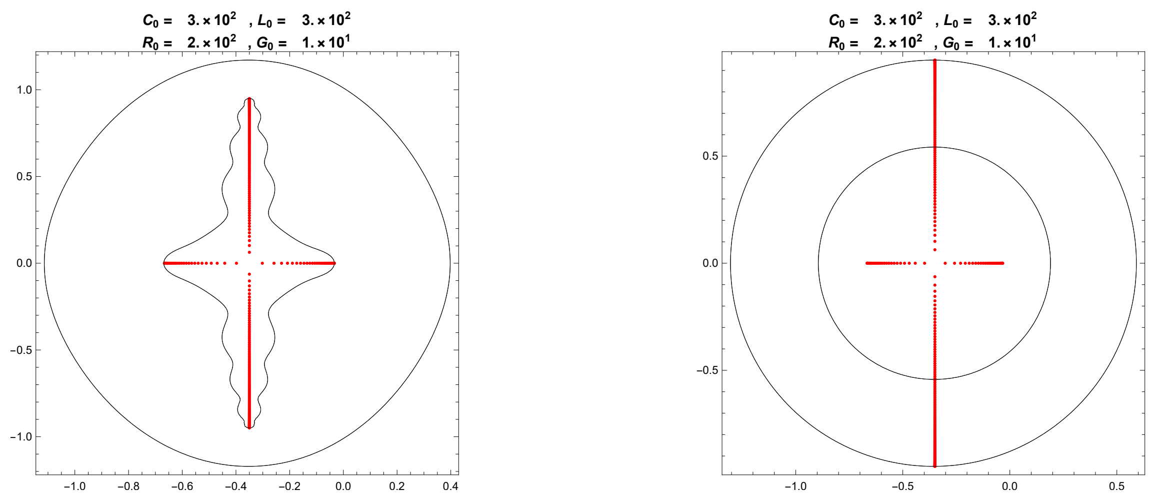


Figure 8: The eigenvalues of matrix (6.1) and the level curves of the functions F (left) and T (right) corresponding to the levels $C_F = 2.114 \cdot 10^{-11}$ and $C_T = 9.84 \cdot 10^{-9}$; $C_T/C_F = 465.484$

So, we have seen that the Faber polynomials can give higher accuracy than the Taylor ones of the same order. Of course, the calculation of the Faber polynomials takes more time. But this loss of time is insignificant compared to subsequent matrix operations.

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