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NEW WEIGHTED HARDY-TYPE INEQUALITIES FOR MONOTONE FUNCTIONS

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Key words: integral operator, Hardy-type inequality, weight, non-increasing function, non-decreasing function.

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Abstract. The famous Hardy inequality was formulated in 1920 and finally proved in 1925. Since then, this inequality has been greatly developed. The first development was related to the consideration of more general weights. The next step was to use more general operators with different kernels instead of the Hardy operator. At present, there are many works devoted to Hardy-type inequalities with iterated operators. Motivated by important applications, all these generalizations of the Hardy inequality are studied not only on the cone of non-negative functions but also on the cone of monotone non-negative functions. In this paper, new Hardy-type inequalities are proved for operators with kernels that satisfy less restrictive conditions than those considered earlier. The presented inequalities have already been characterized for non-negative functions. In this paper, we continue this study but for monotone non-negative functions.

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1 Introduction

Let $I=(0,\infty)$, $1<p,q<\infty$ and $p'=\frac{p}{p-1}$. Suppose that v, u and $v^{1-p'}$ are positive functions locally integrable on I.

We consider the following Hardy-type inequality

$$\left(\int_0^\infty u(x)\left|\int_0^x K(x,t)f(t)dt\right|^q dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty v(x)|f(x)|^p dx\right)^{\frac{1}{p}}, \tag{1.1}$$

for all functions $f\in L_{p,v}(I)$, where $C>0$ is independent of f and $L_{p,v}(I)$ is the weighted Lebesgue space of all functions f, Lebesgue measurable on I, such that $\|f\|_{p,v}=\left(\int_0^\infty v(x)|f(x)|^p dx\right)^{\frac{1}{p}}<\infty$. Here

$$Kf(x)=\int_0^x K(x,t)f(t)dt,\ x>0, \tag{1.2}$$

is an integral operator with a non-negative kernel $K(x,t)$.

Inequality (1.1) has been completely characterized for the kernel $K(x,t)\equiv 1$ (for more details see [8, 9]) and the kernel $K(x,t)\equiv (x-t)^{\alpha-1}$, $\alpha>1$ (see [17, 18, 19, 20] and for more details see [6]).

In works [3] and [7, 10, 11, 12], inequality (1.1) was studied for kernels $K(x,t)$ satisfying the Oinarov condition stating that there exists a number $d \geq 1$ such that

$$d^{-1}(K(x,s)+K(s,t)) \leq K(x,t) \leq d(K(x,s)+K(s,t)) \tag{1.3}$$

for all $x,s,t : x \geq s \geq t > 0$. A further development of this problem was the introduction of the classes $\mathcal{O}_n^{\pm}$, $n \geq 0$, which are less restrictive for kernels $K(x,t)$ than the Oinarov condition. We will refer to $\mathcal{O}_n^{\pm}$, $n \geq 0$, as *the Oinarov classes (the definitions of these classes are given in Section 2)*. In paper [13], inequality (1.1) was studied in the case $1 < p \leq q < \infty$. The case $1 < q < p < \infty$ was considered in the paper [1], but for kernels belonging to the Oinarov classes $\mathcal{O}_1^{\pm}$. In the recent paper [14] the case $1 < q < p < \infty$ is also discussed, but now kernels are from $\mathcal{O}_2^{\pm}$. For operators with kernels from the classes $\mathcal{O}_1^{\pm}$ in paper [14] an alternative criterion for the validity of (1.1) is presented.

If, in addition, f is a monotone function, characterizations of the Hardy-type inequalities help to find boundedness of certain operators in Lorentz spaces. Moreover, the Hardy-type inequalities restricted to monotone functions are used for the weighted Marcinkiewicz interpolation results. For more applications, we refer to monograph [9, Chapter 8] (see also [16]).

Motivated by the applications, in this paper, we find necessary and sufficient conditions for the validity of inequality (1.1) for operator (1.2) with kernels from the Oinarov classes $\mathcal{O}_2^{\pm}$ on the cone of monotone functions in the case $1 < q < p < \infty$. The case $1 < p \leq q < \infty$ was discussed in paper [2] for kernels from $\mathcal{O}_n^{-}$, $n \geq 0$. We note that the case when kernels belong to the classes $\mathcal{O}_n^{+}$, $n \geq 0$, has been left in [2] as an open question. The presented paper covers the class $\mathcal{O}_2^{+}$. As soon as inequality (1.1) is established for kernels from the general classes $\mathcal{O}_n^{\pm}$, $n \geq 0$, on the cone of non-negative functions in the case $1 < q < p < \infty$, it can be established on the cone of monotone functions in the same way as here. Moreover, in paper [2], the authors also considered the conjugate operator $K^*f(x) = \int\limits_x^\infty K(t,x)f(t)dt$, $x > 0$, but kernels were from $\mathcal{O}_n^{+}$, $n \geq 0$. Since the conjugate operator K^*f needs a different approach than operator (1.2), so this is one more topic for a separate paper.

This paper is organized as follows. Section 2 contains all the auxiliary statements required to prove the main results. In Section 3, the validity of inequality (1.1) is established on the cone of non-increasing functions for operator (1.2) with kernels from the Oinarov class $\mathcal{O}_2^{+}$. In Section 4, we present a similar result but for the operator (1.2) with kernels from the class $\mathcal{O}_2^{-}$. Section 5 is devoted to the case $1 < p \leq q < \infty$ when kernels belong to the class $\mathcal{O}_2^{+}$, which has not been considered in [2].

2 Auxiliary statements

Throughout the paper, the symbol $A \ll B$ means that $A \leq cB$ with some constant $c > 0$. The symbol $A \approx B$ stands for $A \ll B \ll A$. Moreover, $f \uparrow$ and $f \downarrow$ mean non-decreasing or non-increasing non-negative functions, respectively.

Let us give the definitions of the classes $\mathcal{O}_1^{\pm}$ and $\mathcal{O}_2^{\pm}$. Let $\Omega = \{(x,t) \in I \times I : x \geq t\}$.

Definition 1. A measurable function $K_1(\cdot,\cdot) \geq 0$ defined on the set Ω belongs to the class $\mathcal{O}_1^{+}$, if it does not decrease in the first argument and there exists a non-negative function $K_{1,0}(\cdot,\cdot)$ measurable on Ω and a number $d_1 \geq 1$ such that

$$d_1^{-1}(K_{1,0}(x,s)+K_1(s,t)) \leq K_1(x,t) \leq d_1(K_{1,0}(x,s)+K_1(s,t)) \tag{2.1}$$

for all $x,s,t : x \geq s \geq t > 0$.

Definition 2. A measurable function $K_1(\cdot,\cdot) \geq 0$ defined on the set Ω belongs to the class $\mathcal{O}_1^-$, if it does not increase in the second argument and there exists a non-negative function $K_{0,1}(\cdot,\cdot)$ measurable on Ω and a number $\bar{d}_1 \geq 1$ such that

$$\bar{d}_1^{-1}\left(K_1(x,s)+K_{0,1}(s,t)\right) \leq K_1(x,t) \leq \bar{d}_1\left(K_1(x,s)+K_{0,1}(s,t)\right)$$

for all $x,s,t : x \geq s \geq t > 0$.

Definition 3. A measurable function $K_2(\cdot,\cdot) \geq 0$ defined on the set Ω belongs to the class $\mathcal{O}_2^+$, if it does not decrease in the first argument and there exist non-negative functions $K_{2,0}(\cdot,\cdot)$, $K_{2,1}(\cdot,\cdot)$ and $K_1(\cdot,\cdot)$ measurable on Ω and a number $d_2 \geq 1$ such that $K_1(\cdot,\cdot) \in \mathcal{O}_1^+$ and

$$d_2^{-1}\left(K_{2,0}(x,s)+K_{2,1}(x,s)K_1(s,t)+K_2(s,t)\right) \leq K_2(x,t) \leq d_2\left(K_{2,0}(x,s)+K_{2,1}(x,s)K_1(s,t)+K_2(s,t)\right) \quad (2.2)$$

for all $x,s,t : x \geq s \geq t > 0$.

Definition 4. A measurable function $K_2(\cdot,\cdot) \geq 0$ defined on the set Ω belongs to the class $\mathcal{O}_2^-$, if it does not increase in the second argument and there exist non-negative functions $K_{0,2}(\cdot,\cdot)$, $K_{1,2}(\cdot,\cdot)$ and $K_1(\cdot,\cdot)$ measurable on Ω and a number $\bar{d}_2 \geq 1$ such that $K_1(\cdot,\cdot) \in \mathcal{O}_1^-$ and

$$\bar{d}_2^{-1}\left(K_2(x,s)+K_1(x,s)K_{1,2}(s,t)+K_{0,2}(s,t)\right) \leq K_2(x,t) \leq \bar{d}_2\left(K_2(x,s)+K_1(x,s)K_{1,2}(s,t)+K_{0,2}(s,t)\right) \quad (2.3)$$

for all $x,s,t : x \geq s \geq t > 0$.

Note that since the classes $\mathcal{O}_2^{\pm}$ are wider than the classes of operators satisfying condition (1.3), many recent publications have been devoted to them (see, e.g., [5, 14]). Examples of kernels that belong to the classes $\mathcal{O}_1^{\pm}$ and $\mathcal{O}_2^{\pm}$ can be found in [14].

To prove our main results we use the following theorems established in [14].

Theorem A. *Let $1 < q < p < \infty$ and $K(\cdot,\cdot) \equiv K_2(\cdot,\cdot) \in \mathcal{O}_2^+$. Then inequality* (1.1) *holds if and only if $B_2 = \max\{B_{2,0}, B_{2,1}, B_{2,2}\} < \infty$. Moreover, $C \approx B_2$, where C is best constant in inequality* (1.1) *and*

$$B_{2,0} = \left(\int_0^\infty \left(\int_z^\infty K_{2,0}^q(x,z)u(x)dx\right)^{\frac{p}{p-q}} \left(\int_0^z v^{1-p'}(s)ds\right)^{\frac{p(q-1)}{p-q}} v^{1-p'}(z)dz\right)^{\frac{p-q}{pq}},$$

$$B_{2,1} = \left(\int_0^\infty \left(\int_z^\infty K_{2,1}^q(x,z)u(x)dx\right)^{\frac{p}{p-q}} \left(\int_0^z K_1^{p'}(z,s)v^{1-p'}(s)ds\right)^{\frac{p(q-1)}{p-q}} \times d\left(\int_0^z K_1^{p'}(z,t)v^{1-p'}(t)dt\right)\right)^{\frac{p-q}{pq}},$$

$$B_{2,2} = \left(\int_0^\infty \left(\int_z^\infty u(t)dt\right)^{\frac{p}{p-q}} \left(\int_0^z K_2^{p'}(z,s)v^{1-p'}(s)ds\right)^{\frac{p(q-1)}{p-q}} d\left(\int_0^z K_2^{p'}(z,s)v^{1-p'}(s)ds\right)\right)^{\frac{p-q}{pq}}.$$

Theorem B. *Let* $1 < q < p < \infty$ *and* $K(\cdot,\cdot) \equiv K_1(\cdot,\cdot) \in \mathcal{O}_1^-$. *Then inequality* (1.1) *holds if and only if* $\mathcal{B}_1 = \max\{\mathcal{B}_{0,1}, \mathcal{B}_{1,1}\} < \infty$. *Moreover,* $C \approx \mathcal{B}_1$, *where where* C *is the best constant in inequality* (1.1) *and*

$$\mathcal{B}_{0,1} = \left(\int_0^\infty \left(\int_0^t K_{0,1}^{p'}(t,x) v^{1-p'}(x) dx \right)^{\frac{q(p-1)}{p-q}} \left(\int_t^\infty u(s) ds \right)^{\frac{q}{p-q}} u(t) dt \right)^{\frac{p-q}{pq}},$$

$$\mathcal{B}_{1,1} = \left(\int_0^\infty \left(\int_0^t v^{1-p'}(x) dx \right)^{\frac{q(p-1)}{p-q}} \left(\int_t^\infty K_1^q(s,t) u(s) ds \right)^{\frac{q}{p-q}} d\left(-\int_t^\infty K_1^q(s,t) u(s) ds \right) \right)^{\frac{p-q}{pq}}.$$

Theorem C. *Let* $1 < q < p < \infty$ *and* $K(\cdot,\cdot) \equiv K_2(\cdot,\cdot) \in \mathcal{O}_2^-$. *Then inequality* (1.1) *holds if and only if* $\mathcal{B}_2 = \max\{\mathcal{B}_{0,2,}, \mathcal{B}_{1,2}, \mathcal{B}_{2,2}\} < \infty$. *Moreover,* $C \approx \mathcal{B}_2$, *where* C *is the best constant in inequality* (1.1) *and*

$$\mathcal{B}_{0,2} = \left(\int_0^\infty \left(\int_0^z K_{0,2}^{p'}(z,s) v^{1-p'}(s) ds \right)^{\frac{q(p-1)}{p-q}} \left(\int_z^\infty u(s) ds \right)^{\frac{q}{p-q}} u(z) dz \right)^{\frac{p-q}{pq}},$$

$$\mathcal{B}_{1,2} = \left(\int_0^\infty \left(\int_0^z K_{1,2}^{p'}(z,s) v^{1-p'}(s) ds \right)^{\frac{q(p-1)}{p-q}} \left(\int_z^\infty K_1^q(x,z) u(x) dx \right)^{\frac{q}{p-q}} \times d\left(-\int_z^\infty K_1^q(x,z) u(x) dx \right) \right)^{\frac{p-q}{pq}},$$

$$\mathcal{B}_{2,2} = \left(\int_0^\infty \left(\int_0^z v^{1-p'}(t) dt \right)^{\frac{p(q-1)}{p-q}} \left(\int_z^\infty K_2^q(x,z) u(x) dx \right)^{\frac{p}{p-q}} v^{1-p'}(z) dz \right)^{\frac{p-q}{pq}}.$$

In paper [15], there is a formula that gives the equivalence between inequality (1.1) for all non-increasing non-negative functions and a certain inequality, but for arbitrary non-negative functions. This equivalence is now called the Sawyer duality principle and has the form:

$$\sup_{0 \le f \downarrow} \frac{\int_0^\infty g(x) f(x) dx}{\left(\int_0^\infty v(x) f^p(x) dx \right)^{\frac{1}{p}}} \approx \left(\int_0^\infty v(x) \left(\frac{\int_0^x g(t) dt}{\int_0^x v(t) dt} \right)^{p'} dx \right)^{\frac{1}{p'}} + \frac{\int_0^\infty g(x) dx}{\left(\int_0^\infty v(x) dx \right)^{\frac{1}{p}}}. \tag{2.4}$$

Equivalence (2.4) can be transformed into the following statement (see, e.g., [4]). The inequality

$$\left(\int_0^\infty u(x) (Kf(x))^q dx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty v(x) f^p(x) dx \right)^{\frac{1}{p}} \tag{2.5}$$

holds for a non-increasing function $f \geq 0$ if and only if the following two inequalities

$$\left(\int_0^\infty u\left(K\left(\int_x^\infty h\right)\right)^q\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty v^{1-p}V^p h^p\right)^{\frac{1}{p}}, \tag{2.6}$$

$$\left(\int_0^\infty u(K\mathbf{1})^q\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty v\right)^{\frac{1}{p}} \tag{2.7}$$

hold for any function $h \geq 0$ and $V(\infty) < \infty$, where $V(t) := \int_0^t v(x)dx$ and $\mathbf{1}$ is a function identically equal to 1 on I. From (2.4) it is obvious that in the case $V(\infty) = \infty$ for inequality (2.5) to hold we need only the validity of inequality (2.6).

3 Main result for the class $\mathcal{O}_2^+$

Assume that

$$M_1^\pm = \left(\int_0^\infty u(x)\left(\int_0^x K(x,t)dt\right)^q dx\right)^{\frac{1}{q}}\left(\int_0^\infty v(x)dx\right)^{-\frac{1}{p}},$$

$$M_2^\pm = \left(\int_0^\infty\left(\int_0^t\left(\int_0^x K(x,z)dz\right)^q u(x)dx\right)^{\frac{p}{p-q}}\left(\int_t^\infty V^{-p'}(s)v(s)ds\right)^{\frac{p(q-1)}{p-q}} V^{-p'}(t)v(t)dt\right)^{\frac{p-q}{pq}},$$

$$M_3^+ = \left(\int_0^\infty\left(\int_t^\infty K_{2,0}^q(x,z)u(x)dx\right)^{\frac{p}{p-q}}\left(\int_0^t s^{p'}V^{-p'}(s)v(s)ds\right)^{\frac{p(q-1)}{p-q}} t^{p'}V^{-p'}(t)v(t)dt\right)^{\frac{p-q}{pq}},$$

$$M_4^+ = \left(\int_0^\infty\left(\int_t^\infty K_{2,1}^q(x,z)u(x)dx\right)^{\frac{p}{p-q}}\left(\int_0^t\left(\int_0^s K_1(t,z)dz\right)^{p'} V^{-p'}(s)v(s)ds\right)^{\frac{p(q-1)}{p-q}}\right.$$
$$\left.\times\, d\left(\int_0^t\left(\int_0^s K_1(t,z)dz\right)^{p'} V^{-p'}(s)v(s)ds\right)\right)^{\frac{p-q}{pq}},$$

$$M_5^+ = \left(\int_0^\infty\left(\int_t^\infty u(x)dx\right)^{\frac{p}{p-q}}\left(\int_0^t\left(\int_0^s K(t,z)dz\right)^{p'} V^{-p'}(s)v(s)ds\right)^{\frac{p(q-1)}{p-q}}\right.$$
$$\left.\times\, d\left(\int_0^t\left(\int_0^s K(t,z)dz\right)^{p'} V^{-p'}(s)v(s)ds\right)\right)^{\frac{p-q}{pq}}.$$

$$M^+ = \max\{M_1^\pm, M_2^\pm, M_3^+, M_4^+, M_5^+\} \text{ and } \widetilde{M}^+ = \max\{M_2^\pm, M_3^+, M_4^+, M_5^+\}.$$

Our main result of this section reads as follows.

Theorem 3.1. *Let $1 < q < p < \infty$ and $K(\cdot,\cdot) \in \mathcal{O}_2^+$. Then inequality (1.1) holds for any non-increasing $f \geq 0$ if and only if $M^+ < \infty$ for $V(\infty) < \infty$ and $\widetilde{M}^+ < \infty$ for $V(\infty) = \infty$.*

Proof. Since $K\mathbf{1} = \int\limits_0^x K(x,t)dt$, inequality (2.7) has the form

$$\left(\int\limits_0^\infty u(x)\left(\int\limits_0^x K(x,t)dt\right)^q dx\right)^{\frac{1}{q}} \leq C\left(\int\limits_0^\infty v(x)dx\right)^{\frac{1}{p}},$$

which is equivalent to the condition $M_1^\pm < \infty$. As we mentioned above, in the case of $V(\infty) = \infty$, inequality (2.7) is not required, so the condition $M_1^\pm < \infty$ is also not required.

Let us turn to inequality (2.6) for non-negative functions, the validity of which is necessary and sufficient for the validity of (2.5) for non-increasing functions for the both cases $V(\infty) < \infty$ and $V(\infty) = \infty$. Inequality (2.6) can be rewritten as follows:

$$\left(\int\limits_0^\infty u(x)\left(\int\limits_0^x K(x,t)\left(\int\limits_t^\infty h(s)ds\right)dt\right)^q dx\right)^{\frac{1}{q}} \leq C\left(\int\limits_0^\infty v^{1-p}(x)V^p(x)h^p(x)dx\right)^{\frac{1}{p}}. \tag{3.1}$$

Our aim is to characterize inequality (3.1) for any non-negative function $h \geq 0$. Let us transform the left-hand side S of (3.1). We split the inner integral in (3.1) and get

$$S \approx \left(\int\limits_0^\infty u(x)\left(\int\limits_0^x K(x,t)\left(\int\limits_t^x h(s)ds\right)dt\right)^q dx\right)^{\frac{1}{q}} + \left(\int\limits_0^\infty u(x)\left(\int\limits_0^x K(x,t)\left(\int\limits_x^\infty h(s)ds\right)dt\right)^q dx\right)^{\frac{1}{q}}. \tag{3.2}$$

The change of the order of integration in the first term of (3.2) gives

$$S \approx \left(\int\limits_0^\infty u(x)\left(\int\limits_0^x \left(\int\limits_0^s K(x,t)dt\right)h(s)ds\right)^q dx\right)^{\frac{1}{q}} + \left(\int\limits_0^\infty u(x)\left(\int\limits_0^x K(x,t)dt\right)^q \left(\int\limits_x^\infty h(s)ds\right)^q dx\right)^{\frac{1}{q}}.$$

Therefore, the validity of inequality (3.1) is equivalent to the validity of the following two inequalities:

$$\left(\int\limits_0^\infty u(x)\left(\int\limits_0^x \left(\int\limits_0^s K(x,t)dt\right)h(s)ds\right)^q dx\right)^{\frac{1}{q}} \leq C_1\left(\int\limits_0^\infty v^{1-p}(x)V^p(x)h^p(x)dx\right)^{\frac{1}{p}}, \tag{3.3}$$

$$\left(\int_0^\infty u(x)\left(\int_0^x K(x,t)dt\right)^q\left(\int_x^\infty h(s)ds\right)^q dx\right)^{\frac{1}{q}} \leq C_2\left(\int_0^\infty v^{1-p}(x)V^p(x)h^p(x)dx\right)^{\frac{1}{p}}. \tag{3.4}$$

The inequality (3.4) is the standard weighted Hardy inequality, which holds if and only if $M_2^{\pm} < \infty$ (see, e.g., [9]).

Inequality (3.3) can be rewritten in the form:

$$\left(\int_0^\infty u(x)\left(\int_0^x \overline{K}(x,s)s\,h(s)ds\right)^q dx\right)^{\frac{1}{q}} \leq C_1\left(\int_0^\infty v^{1-p}(x)V^p(x)h^p(x)dx\right)^{\frac{1}{p}}.$$

where $\overline{K}(x,s) = \frac{1}{s}\int_0^s K(x,t)dt$ with $K(x,t)$ from $\mathcal{O}_2^+$. Using relation (2.2), for $x \geq z \geq t$ we get

$$\overline{K}(x,s) \approx \frac{1}{s}\int_0^s (K_{2,0}(x,z) + K_{2,1}(x,z)K_1(z,t) + K(z,t))dt$$

$$= \frac{1}{s}K_{2,0}(x,z)\,s + K_{2,1}(x,z)\frac{1}{s}\int_0^s K_1(z,t)dt + \frac{1}{s}\int_0^s K(z,t)dt$$

$$= K_{2,0}(x,z) + K_{2,1}(x,z)\overline{K}_1(z,s) + \overline{K}(z,s), \tag{3.5}$$

where $\overline{K}_1(z,s) = \frac{1}{s}\int_0^s K_1(z,t)dt$. If we prove that $\overline{K}_1(z,s) \in \mathcal{O}_1^+$, we prove that $\overline{K}(x,s) \in \mathcal{O}_2^+$. By the definition $K_1(z,t) \in \mathcal{O}_1^+$, therefore from (2.1) for $z \geq \tau \geq t$ we have that $K_1(z,t) \approx K_{1,0}(z,\tau) + K_1(\tau,t)$. Hence,

$$\overline{K}_1(z,s) \approx \frac{1}{s}\int_0^s (K_{1,0}(z,\tau) + K_1(\tau,t))dt$$

$$= \frac{1}{s}K_{1,0}(z,\tau)\,s + \frac{1}{s}\int_0^s K_1(\tau,t)dt = K_{1,0}(z,\tau) + \overline{K}_1(\tau,s).$$

Then $\overline{K}_1(z,s)$ belongs to the class $\mathcal{O}_1^+$. Consequently, from (3.5) we obtain that $\overline{K}(x,s)$ belongs to the class $\mathcal{O}_2^+$. Thus, replacing $s\,h(s)$ by $g_1(s)$, by Theorem A inequality (3.3) holds for $g_1(s)$ if and only if $M_3^+ < \infty$, $M_4^+ < \infty$ and $M_5^+ < \infty$. □

4 Main result for the class $\mathcal{O}_2^-$

Assume that

$$M_3^- = \left(\int_0^\infty\left(\int_0^t K_{0,2}^{p'}(t,s)\,s^{p'}V^{-p'}(s)v(s)ds\right)^{\frac{q(p-1)}{p-q}}\left(\int_t^\infty u(x)dx\right)^{\frac{q}{p-q}} u(t)dt\right)^{\frac{p-q}{pq}} < \infty,$$

$$M_4^- = \left(\int_0^\infty \left(\int_0^t K_{1,2}^{p'}(t,s)\, s^{p'} V^{-p'}(s) v(s) ds \right)^{\frac{q(p-1)}{p-q}} \left(\int_t^\infty K_1^q(x,t) u(x) dx \right)^{\frac{q}{p-q}} \times d\left(-\int_t^\infty K_1^q(x,t) u(x) dx \right) \right)^{\frac{p-q}{pq}} < \infty,$$

$$M_5^- = \left(\int_0^\infty \left(\int_0^t s^{p'} V^{-p'}(s) v(s) ds \right)^{\frac{p(q-1)}{p-q}} \left(\int_t^\infty K^q(x,t) u(x) dx \right)^{\frac{p}{p-q}} t^{p'} V^{-p'}(t) v(t) dt \right)^{\frac{p-q}{pq}},$$

$$M_6^- = \left(\int_0^\infty \left(\int_0^t K_{0,1}^{p'}(t,s) V^{-p'}(s) v(s) \left(\int_0^s K_{1,2}(s,z) dz \right)^{p'} ds \right)^{\frac{q(p-1)}{p-q}} \left(\int_t^\infty u(x) dx \right)^{\frac{q}{p-q}} u(t) dt \right)^{\frac{p-q}{pq}},$$

$$M_7^- = \left(\int_0^\infty \left(\int_0^t V^{-p'}(s) v(s) \left(\int_0^s K_{1,2}(s,z) dz \right)^{p'} ds \right)^{\frac{q(p-1)}{p-q}} \left(\int_t^\infty K_1^q(x,t) u(x) dx \right)^{\frac{q}{p-q}} \times d\left(-\int_t^\infty K_1^q(x,t) u(x) dx \right) \right)^{\frac{p-q}{pq}} < \infty,$$

$$M_8^- = \left(\int_0^\infty \left(\int_0^t V^{-p'}(s) v(s) \left(\int_0^s K_{0,2}(s,z) dz \right)^{p'} ds \right)^{\frac{q(p-1)}{p-q}} \left(\int_t^\infty u(x) dx \right)^{\frac{q}{p-q}} u(t) dt \right)^{\frac{p-q}{pq}} < \infty,$$

$$M^- = \max\{M_1^\pm, M_2^\pm, M_3^-, M_4^-, M_5^-, M_6^-, M_7^-, M_8^-\},$$
$$\widetilde{M}^- = \max\{M_2^\pm, M_3^-, M_4^-, M_5^-, M_6^-, M_7^-, M_8^-\}.$$

Our main result of this section reads as follows.

Theorem 4.1. *Let $1 < q < p < \infty$ and $K(\cdot,\cdot) \in \mathcal{O}_2^-$. Then inequality (1.1) holds for any non-increasing $f \geq 0$ if and only if $M^- < \infty$ for $V(\infty) < \infty$ and $\widetilde{M}^- < \infty$ for $V(\infty) = \infty$.*

Proof. The beginning of the proof of Theorem 4.1 is the same as the beginning of the proof of Theorem 3.1, i.e., for the validity of (1.1) we need the condition $M_1^\pm < \infty$ for $V(\infty) < \infty$ and the condition $M_2^\pm < \infty$ for both $V(\infty) = \infty$ and $V(\infty) < \infty$.

Let us turn to inequality (3.3). Using relation (2.3) in inequality (3.3), it is equivalent to the inequality

$$\left(\int_0^\infty u(x) \left(\int_0^x \left(\int_0^s (K(x,s) + K_1(x,s) K_{1,2}(s,t) + K_{0,2}(s,t)) dt \right) h(s) ds \right)^q dx \right)^{\frac{1}{q}} \leq C_1 \left(\int_0^\infty v^{1-p}(x) V^p(x) h^p(x) dx \right)^{\frac{1}{p}}.$$

Thus, the validity of inequality (3.3) is equivalent to the validity of the following three inequalities:

$$\left(\int_0^\infty u(x)\left(\int_0^x K(x,s)\,s\,h(s)ds\right)^q dx\right)^{\frac{1}{q}} \le C_{11}\left(\int_0^\infty v^{1-p}(x)V^p(x)h^p(x)dx\right)^{\frac{1}{p}}, \tag{4.1}$$

$$\left(\int_0^\infty u(x)\left(\int_0^x K_1(x,s)\left(\int_0^s K_{1,2}(s,t)dt\right)h(s)ds\right)^q dx\right)^{\frac{1}{q}} \le C_{12}\left(\int_0^\infty v^{1-p}(x)V^p(x)h^p(x)dx\right)^{\frac{1}{p}}, \tag{4.2}$$

$$\left(\int_0^\infty u(x)\left(\int_0^x \left(\int_0^s K_{0,2}(s,t)dt\right)h(s)ds\right)^q dx\right)^{\frac{1}{q}} \le C_{13}\left(\int_0^\infty v^{1-p}(x)V^p(x)h^p(x)dx\right)^{\frac{1}{p}}. \tag{4.3}$$

If we replace $s\,h(s)$ by $g_1(s)$, then by Theorem C inequality (4.1) holds for $g_1(s)$ if and only if $M_3^- < \infty$, $M_4^- < \infty$ and $M_5^- < \infty$.

If we replace $\left(\int_0^s K_{1,2}(s,t)dt\right)h(s)$ by $g_2(s)$, then by Theorem B inequality (4.2) holds for $g_2(s)$ if and only if $M_6^- < \infty$ and $M_7^- < \infty$.

If we replace $\left(\int_0^s K_{0,2}(s,t)dt\right)h(s)$ by $g_3(s)$, then (4.3) is the standard weighted Hardy inequality for $g_3(s)$, which holds if and only if $M_8^- < \infty$ (see, e.g., [9]).

□

Remark 1. Let us note that the proofs of Theorems 3.1 and 4.1 need different approaches because the kernel $\overline{K}(x,s) = \frac{1}{s}\int_0^s K(x,t)dt$ belongs to the class $\mathcal{O}_2^+$ if the kernel $K(x,t)$ belongs to the class $\mathcal{O}_2^+$ but it does not belong to the class $\mathcal{O}_2^-$ if the kernel $K(x,t)$ belongs to the class $\mathcal{O}_2^-$.

5 Supplementary results

In the paper [13], it was proved that if $1 < p \le q < \infty$ and $K(\cdot,\cdot) \in \mathcal{O}_2^+$, then inequality (1.1) holds for any $f \ge 0$ if and only if one of the following conditions

$$A_1^+ = \sup_{0<z<\infty}\left(\int_z^\infty u(x)\left(\int_0^z K^{p'}(x,s)v^{1-p'}(s)ds\right)^{\frac{q}{p'}} dx\right)^{\frac{1}{q}} < \infty,$$

$$A_2^+ = \sup_{0<z<\infty}\left(\int_0^z v^{1-p'}(s)\left(\int_z^\infty K^q(x,s)u(x)dx\right)^{\frac{p'}{q}} ds\right)^{\frac{1}{p'}} < \infty$$

holds, in addition, $C \approx A_1^+ \approx A_2^+$, where C is the best constant in inequality (1.1).

Using the above result and following the same steps as in the proof of Theorem 3.1, we can present the statement on the cone of non-increasing functions for the case $1 < p \le q < \infty$ when kernels $K(\cdot,\cdot)$ belong to the class $\mathcal{O}_2^+$, which was not considered in [2].

Theorem 5.1. *Let* $1 < p \leq q < \infty$ *and* $K(\cdot,\cdot) \in \mathcal{O}_2^+$. *Then inequality* (1.1) *holds for any non-increasing* $f \geq 0$ *if and only if one of the conditions* $max\{M_1^{\pm}, \mathcal{M}_2^+, \mathcal{M}_3^+\} < \infty$ *and* $max\{M_1^{\pm}, \mathcal{M}_2^+, \mathcal{M}_4^+\} < \infty$ *holds for* $V(\infty) < \infty$ *and one of the conditions* $max\{\mathcal{M}_2^+, \mathcal{M}_3^+\} < \infty$ *and* $max\{\mathcal{M}_2^+, \mathcal{M}_4^+\} < \infty$ *holds for* $V(\infty) = \infty$, *where*

$$\mathcal{M}_2^+ = \sup_{0<z<\infty} \left(\int_0^z \left(\int_0^x K(x,t)dt \right)^q u(x)dx \right)^{\frac{1}{q}} \left(\int_z^\infty V^{-p'}(s)v(s)ds \right)^{\frac{1}{p'}},$$

$$\mathcal{M}_3^+ = \sup_{0<z<\infty} \left(\int_z^\infty u(x) \left(\int_0^z \left(\int_0^s K(x,t)dt \right)^{p'} V^{-p'}(s)v(s)ds \right)^{\frac{q}{p'}} dx \right)^{\frac{1}{q}},$$

$$\mathcal{M}_4^+ = \sup_{0<z<\infty} \left(\int_0^z V^{-p'}(s)v(s) \left(\int_z^\infty \left(\int_0^s K(x,t)dt \right)^q u(x)dx \right)^{\frac{p'}{q}} ds \right)^{\frac{1}{p'}}.$$

Remark 2. On the basis of the duality principle for a non-decreasing function $f \geq 0$:

$$\sup_{0\leq f\uparrow} \frac{\int_0^\infty g(x)f(x)dx}{\left(\int_0^\infty v(x)f^p(x)dx\right)^{\frac{1}{p}}} \approx \left(\int_0^\infty v(x) \left(\frac{\int_x^\infty g(t)dt}{\int_x^\infty v(t)dt} \right)^{p'} dx \right)^{\frac{1}{p'}} + \frac{\int_0^\infty g(x)dx}{\left(\int_0^\infty v(x)dx\right)^{\frac{1}{p}}},$$

where $g \geq 0$ is any function, we can characterize inequality (1.1) on the cone of non-decreasing functions for operator (1.2) with kernels from the Oinarov classes O_2^+ and O_2^-. However, we omit both statements and their proofs here, since they are similar. Let us only present as an example that the value $M_2^{\pm}$ turns to

$$\mathbb{M}_2^{\pm} = \left(\int_0^\infty \left(\int_t^\infty \left(\int_x^\infty K(z,x)dz \right)^q u(x)dx \right)^{\frac{p}{p-q}} \left(\int_0^t V_*^{-p'}(s)v(s)ds \right)^{\frac{p(q-1)}{p-q}} V_*^{-p'}(t)v(t)dt \right)^{\frac{p-q}{pq}},$$

where $V_*(t) := \int_t^\infty v(x)dx$. All other quantities in M^+ and M^- can be rewritten similarly.

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References

[1] L.S. Arendarenko, R. Oinarov, L.-E. Persson, *On the boundedness of some classes of integral operators in weighted Lebesgue spaces.* Eurasian Math. J. 3 (2012), no. 1, 5–17.

[2] L.S. Arendarenko, R. Oinarov, L.-E. Persson, *Some new Hardy-type integral inequalities on cones of monotone functions.* Oper. Theory Adv. Appl. 229 (2013), 77–89.

[3] S. Bloom, R. Kerman, *Weighted norm inequalities for operators of Hardy type.* Proc. Amer. Math. Soc. 113 (1991), no. 1, 135–141.

[4] A. Gogatishvili, V.D. Stepanov, *Reduction theorems for weighted integral inequalities on the cone of monotone functions.* Russian Math. Surveys. 68 (2013), no. 4, 597–664.

[5] A. Kalybay, *Boundedness of one class of integral operators from second order weighted Sobolev space to weighted Lebesgue space.* J. Funct. Spaces. Article ID 5257476 (2022), https://doi.org/10.1155/2022/5257476.

[6] A. Kalybay, R. Oinarov, *Boundedness of Riemann–Liouville operator from weighted Sobolev space to weighted Lebesgue space.* Eurasian Math. J. 12 (2021), no. 1, 39–48.

[7] A. Kalybay, S. Shalginbayeva, *Estimate of the best constant of discrete Hardy-type inequality with matrix operator satisfying the Oinarov condition.* Eurasian Math. J., 15 (2024), no. 2, 42–47.

[8] A. Kufner, L. Maligranda, L.-E. Persson, *The prehistory of the Hardy inequality.* Amer. Math. Monthly 113 (2006), no. 10, 715–732.

[9] A. Kufner, L. Maligranda, L.-E. Persson, *The Hardy inequality. About its history and some related results.* Pilsen, Vydavatelský Servis, 2007.

[10] K. Kuliev, *On estimates for norms of some integral operators with Oinarov's kernel.* Eurasian Math. J., 13 (2022), no. 3, 67–81

[11] R. Oinarov, *Weighted inequalities for one class of integral operators.* Dokl. Math. 44 (1992), no. 1, 291–293.

[12] R. Oinarov, *Two-sided norm estimates for certain classes of integral operators.* Proc. Steklov Inst. Math., 204 (1994), 205–214.

[13] R. Oinarov, *Boundedness and compactness of Volterra type integral operators.* Siberian Math. J. 48 (2007), no. 5, 884–896.

[14] R. Oinarov, A. Temirkhanova, A. Kalybay, *Boundedness of one class of integral operators from L_p to L_q for $1 < q < p < \infty$.* Ann. Funct. Anal. 14 (2023), no. 65.

[15] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces.* Studia Math. 96 (1990), no. 2, 145–158.

[16] A. Senouci, A. Zanou, *Some integral inequalities for quasimonotone functions in weighted variable exponent Lebesgue space with $0 < p(x) < 1$.* Eurasian Math. J. 11 (2020), no. 4, 58–65.

[17] V.D. Stepanov, *Weighted inequalities of Hardy type for higher-order derivatives and their applications.* Dokl. Math. 38 (1989), no. 2, 389–393.

[18] V.D. Stepanov, *Two-weighted estimates of Riemann–Liouville integrals.* Math. USSR-Izv. 36 (1991), no. 3, 669–681.

[19] V.D. Stepanov, *Weighted inequalities of Hardy type for Riemann-Liouville fractional integrals.* Siberian Math. J. 31 (1990), no. 3, 513–522.

[20] V.D. Stepanov, *Weighted norm inequalities of Hardy type for a class of integral operators.* J. London Math. Soc. 50 (1994), no. 1, 105–120.

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