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ON DIRECT AND INVERSE PROBLEMS FOR SYSTEMS OF ODD-ORDER QUASILINEAR EVOLUTION EQUATIONS

O.S. Balashov, A.V. Faminskii

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Abstract. Direct and inverse initial-boundary problems on a bounded interval for systems of odd-order quasilinear evolution equations with general nonlinearities are considered. In the case of inverse problems conditions of integral overdetermination are introduced and right-hand sides of equations of special types are chosen as controls. Results on well-posedness of such problems are established. Assumptions on smallness of the input data or smallness of a time interval are required.

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1 Introduction. Notation. Description of main results

Consider the following system of odd-order quasilinear equations

$$u_t - (-1)^l (a_{2l+1}\partial_x^{2l+1} u + a_{2l}\partial_x^{2l} u) - \sum_{j=0}^{l-1} (-1)^j \partial_x^j \big[a_{2j+1}(t,x)\partial_x^{j+1} u + a_{2j}(t,x)\partial_x^j u\big] + \sum_{j=0}^{l} (-1)^j \partial_x^j \big[g_j(t,x,u,\dots,\partial_x^{l-1} u)\big] = f(t,x), \quad l\in\mathbb{N}, \tag{1.1}$$

posed on an interval $I=(0,R)$ ($R>0$ is arbitrary). Here $u=u(t,x)=(u_1,\dots,u_n)^T$, $n\in\mathbb{N}$, is the unknown vector-function, $f=(f_1,\dots,f_n)^T$, $g_j=(g_{j1},\dots,g_{jn})^T$ are also vector-functions, $a_{2l+1}=diag(a_{(2l+1)i})$, $a_{2l}=diag(a_{(2l)i})$, $i=1,\dots,n$, are constant diagonal $n\times n$ matrices, $a_j(t,x)=\big(a_{jim}(t,x)\big)$, $i,m=1,\dots,n$, for $j=0,\dots,2l-1$, are $n\times n$ matrices.

In a rectangle $Q_T=(0,T)\times I$ for certain $T>0$ consider an initial-boundary value problem for system (1.1) with the initial condition

$$u(0,x)=u_0(x), \quad x\in[0,R], \tag{1.2}$$

and the boundary conditions

$$\partial_x^j u(t,0)=\mu_j(t),\ j=0,\dots,l-1, \quad \partial_x^j u(t,R)=\nu_j(t),\ j=0,\dots,l, \quad t\in[0,T], \tag{1.3}$$

where $u_0=(u_{01},\dots,u_{0n})^T$, $\mu_j=(\mu_{j1},\dots,\mu_{jn})^T$, $\nu_j=(\nu_{j1},\dots,\nu_{jn})^T$.

Besides this direct problem consider the following inverse problem: let for any $i = 1, \dots, n$ the function f_i be represented in the form

$$f_i(t,x) \equiv h_{0i}(t,x) + \sum_{k=1}^{m_i} F_{ki}(t) h_{ki}(t,x) \tag{1.4}$$

for a certain non-negative integer number m_i (if $m_i = 0$ then $f_i = h_{0i}$), where the functions h_{ki} are given and the functions F_{ki} are unknown. Then problem (1.1)–(1.3) is supplemented with overdetermination conditions in an integral form: if $m_i > 0$ for certain i, then

$$\int_I u_i(t,x)\omega_{ki}(x)\,dx = \varphi_{ki}(t), \quad t \in [0,T], \quad k = 1, \dots, m_i, \tag{1.5}$$

for certain given functions ω_{ki} and φ_{ki}. In particular, for certain i the overdetermination conditions on u_i can be absent, but in the case of the inverse problem we always assume that

$$M = \sum_{i=1}^{n} m_i > 0. \tag{1.6}$$

Then the aim is to find the functions F_{ki} such that the corresponding solution u to problem (1.1)–(1.3) satisfies conditions (1.5).

In the case of a single equation $n = 1$ equations of type (1.1) were considered in [9] (direct problem) and [10] (inverse problems). In particular, in these articles one can found examples of physical models, which can be described by such equations: the Korteweg–de Vries (KdV) and Kawahara equations with generalizations, the Kortewes–de Vries–Burgers and Benney-Lin equations, the Kaup–Kupershmidt equation and others (see also [1], [14]). However, besides the single equations, systems of odd-order quasilinear evolution equations also arise in real physical situations. Among such systems on can mention the Majda–Biello system (see [17])

$$\begin{cases} u_t + u_{xxx} + vv_x = 0, \\ v_t + \alpha v_{xxx} + (uv)_x = 0, \quad \alpha > 0, \end{cases}$$

and more general systems of KdV-type equations with coupled nonlinearities ([5]).

The KdV-type Boussinesq system ([6, 23, 25])

$$\begin{cases} u_t + v_x + v_{xxx} + (uv)_x = 0, \\ v_t + u_x + u_{xxx} + vv_x = 0 \end{cases}$$

and the coupled system of two KdV equations, derived in [13] and studied in [3, 4, 7, 15, 18, 19, 20, 21, 22] (also with more general nonlinearities)

$$\begin{cases} u_t + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 vv_x + a_2(uv)_x = 0, \\ b_1 v_t + rv_x + vv_x + b_2 a_3 u_{xxx} + v_{xxx} + b_2 a_2 uu_x = 0, \quad b_1 > 0, b_2 > 0, \end{cases}$$

are not directly written in form (1.1), but can be transformed to it by a linear change of unknown functions (see [3, 6, 23]).

In paper [9] initial-boundary value problem (1.1)–(1.3) was considered in the scalar case and a result on global well-posedness in the class of weak solutions under small input data was established. For simplicity it was assumed there that $\mu_j(t) = \nu_j(t) \equiv 0$ for $j \le l-1$. Note that the general case of (1.3) can be reduced to the homogeneous one by the simple substitution $v(t,x) = u(t,x) - \psi(t,x)$, where the sufficiently smooth function ψ satisfies (1.3) for $j \le l-1$, while the form of equation (1.1) is invariant under the corresponding transformation.

In the present paper a result on global well-posedness of problem (1.1)–(1.3) itself is obtained in the class of weak solutions under small input data. Note that in the aforementioned articles in the case of systems such a problem was not studied. The assumptions on system (1.1) are similar to the ones in [9, 10] in the case of single equations.

The significance of integral overdermination conditions in inverse problems is discussed in [24]. The study of inverse problems for the KdV-type equation with integral overdetermination was started in [8]. In paper [10] for problem (1.1)–(1.3) in the scalar case two inverse problems with one integral overdetemination condition of type (1.5) were considered. In the first one the right-hand side of the equation of a type similar to (1.4) was chosen as the control, in the second one — the boundary data ν_l. Results on well-posedness either for small input data or small time interval were established. In paper [12] an initial-boundary value problem on a bounded interval for the higher order nonlinear Schrödinger equation

$$iu_t + au_{xx} + ibu_x + iu_{xxx} + \lambda|u|^p u + i\beta\bigl(|u|^p u\bigr)_x + i\gamma\bigl(|u|^p\bigr)_x u = 0$$

(u is a complex-valued function) with initial and boundary conditions similar to (1.2), (1.3) was considered and three inverse problems were studied. The first two of them were similar to the problems considered in [10] with similar results. In the third problem two overdetermination conditions of (1.5) type were introduced and both the right-hand side of the equation and the boundary function were chosen as controls. The results were similar to the first two cases.

Note that the inverse problem with two integral overdetermination conditions for the Korteweg–de Vries type equation

$$u_t + u_{xxx} + uu_x + \alpha(t)u = F(t)g(t)$$

in the periodic case, where the functions α and F were unknown, was considered in [16] and the existence and uniqueness results were obtained for a small time interval.

In paper [21] an inverse problem on a bounded interval with the terminal overdetermination condition

$$u(T,x) = u_T(x)$$

for a given function u_T (such problems are called controllability ones) was studied for the aforementioned coupled system of two KdV equations. Results on existence of solutions under small input data were established.

In the present paper, results on well-posedness of inverse problem (1.1)–(1.6) are obtained either for small input data or small time interval. Note that since the amount of integral overdetermination conditions is arbitrary, the result is new even in the case of one equation.

Solutions of the considered problems are constructed in the special function space $\bigl(X(Q_T)\bigr)^n$ of all vector-functions $u = (u_1,\dots,u_n)^T$ such that such that for every $i = 1,\dots,n$

$$u_i(t,x) \in X(Q_T) = C([0,T]; L_2(I)) \cap L_2(0,T; H^l(I)),$$

endowed with the norm

$$\|u\|_{(X(Q_T))^n} = \sum_{i=1}^n \Bigl(\sup_{t\in(0,T)} \|u_i(t,\cdot)\|_{L_2(I)} + \|\partial_x^l u_i\|_{L_2(Q_T)}\Bigr).$$

For $r>0$ let $\overline{X}_{rn}(Q_T)$ denote the closed ball $\{u \in \bigl(X(Q_T)\bigr)^n : \|u\|_{(X(Q_T))^n} \le r\}$.

Introduce the notion of a weak solution of problem (1.1)–(1.3).

Definition 1. Let $u_0 \in \big(L_2(I)\big)^n$, $\mu_j, \nu_j \in \big(L_2(0,T)\big)^n$ $\forall j$, $f \in \big(L_1(Q_T)\big)^n$, $a_j \in \big(C(\overline{Q}_T)\big)^{n^2}$ $\forall j$. A function $u \in \big(X(Q_T)\big)^n$ is called a weak solution of problem (1.1)–(1.3) if $\partial_x^j u(t,0) \equiv \mu_j(t)$, $\partial_x^j u(t,R) \equiv \nu_j(t)$, $j = 0,\dots,l-1$, and for all test functions $\phi(t,x)$, such that $\phi \in \big(L_2(0,T;H^{l+1}(I))\big)^n$, $\phi_t \in \big(L_2(Q_T)\big)^n$, $\phi\big|_{t=T} \equiv 0$, $\partial_x^j\phi\big|_{x=0} = \partial_x^j\phi\big|_{x=R} \equiv 0$, $j = 0,\dots,l-1$, $\partial_x^l\phi\big|_{x=0} \equiv 0$, the functions $\big(g_j(t,x,u,\dots,\partial_x^{l-1}u), \partial_x^j\phi\big) \in L_1(Q_T)$, $j = 0,\dots,l$, and the following integral identity holds:

$$\iint_{Q_T} \Big[(u,\phi_t) - (a_{2l+1}\partial_x^l u, \partial_x^{l+1}\phi) + (a_{2l}\partial_x^l u, \partial_x^l\phi) + \sum_{j=0}^{l-1}\big((a_{2j+1}\partial_x^{j+1}u + a_{2j}\partial_x^j u), \partial_x^j\phi\big) - \sum_{j=0}^{l}\big(g_j(t,x,u,\dots,\partial_x^{l-1}u), \partial_x^j\phi\big) + (f,\phi)\Big]\,dxdt + \int_I (u_0, \phi\big|_{t=0})\,dx + \int_0^T (a_{2l+1}\nu_l, \partial_x^l\phi\big|_{x=R})\,dt = 0, \quad (1.7)$$

where $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^n$.

Let $\widehat{f}(\xi) \equiv \mathcal{F}[f](\xi)$ and $\mathcal{F}^{-1}[f](\xi)$ be the direct and inverse Fourier transforms of a function f, respectively . In particular, for $f \in \mathcal{S}(\mathbb{R})$

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)\,dx, \qquad \mathcal{F}^{-1}[f](x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x} f(\xi)\,d\xi.$$

For $s \in \mathbb{R}$ define the fractional order Sobolev space

$$H^s(\mathbb{R}) = \big\{f : \mathcal{F}^{-1}[(1+|\xi|^s)\widehat{f}(\xi)] \in L_2(\mathbb{R})\big\}$$

and for certain $T > 0$ let $H^s(0,T)$ be the space of restrictions on $(0,T)$ of functions from $H^s(\mathbb{R})$. To describe properties of boundary functions μ_j, ν_j we use the following function spaces. Let $m = l-1$ or $m = l$, define

$$\big(\mathcal{B}^m(0,T)\big)^n = \Big(\prod_{j=0}^{m} H^{(l-j)/(2l+1)}(0,T)\Big)^n,$$

endowed with the natural norm.

The coefficients of the linear part of the system further are always assumed to verify the following conditions:

$$a_{(2l+1)i} > 0, \quad a_{(2l)i} \le 0, \quad i = 1,\dots,n, \quad (1.8)$$

and for any $0 \le j \le l-1$, $i,m = 1,\dots n$

$$\partial_x^k a_{(2j+1)im} \in C(\overline{Q}_T),\ k = 0,\dots,j+1, \quad \partial_x^k a_{(2j)im} \in C(\overline{Q}_T),\ k = 0,\dots,j. \quad (1.9)$$

Let $y_m = (y_{m1},\dots,y_{mn})$ for $m = 0,\dots,l-1$. The functions $g_j(t,x,y_0,\dots,y_{l-1})$ for any $0 \le j \le l$ are always subjected to the following assumptions: for $i = 1,\dots,n$

$$g_{ji}, grad_{y_k} g_{ji} \in C(\overline{Q}_T \times \mathbb{R}^{ln}),\ j = 0,\dots,l-1, \quad g_{ji}(t,x,0,\dots,0) \equiv 0, \quad (1.10)$$

$$\big|grad_{y_k} g_{ji}(t,x,y_0,\dots,y_{l-1})\big| \le c\sum_{m=0}^{l-1}\big(|y_m|^{b_1(j,k,m)} + |y_m|^{b_2(j,k,m)}\big), k = 0,\dots,l-1, \quad \forall (t,x,y_0,\dots,y_{l-1}) \in Q_T \times \mathbb{R}^{ln}, \quad (1.11)$$

where $0 < b_1(j,k,m) \le b_2(j,k,m)$, $|y_m| = (y_m, y_m)^{1/2}$.

Regarding the functions ω_{ki} we always need the following conditions:

$$\omega \in H^{2l+1}(I), \quad \omega^{(m)}(0) = 0,\ m = 0,\dots,l, \quad \omega^{(m)}(R) = 0,\ m = 0,\dots,l-1, \tag{1.12}$$

for all ω_{ki} (where here ω stands for ω_{ki}).

Now we can pass to the main results and begin with the direct problem.

Theorem 1.1. *Let the coefficients a_j, $j = 0,\dots,2l+1$, satisfy conditions (1.8), (1.9). Let the functions g_j satisfy conditions (1.10), (1.11), where*

$$b_2(j,k,m) \le \frac{4l-2j-2k}{2m+1} \quad \forall\ j,k,m. \tag{1.13}$$

Let $u_0 \in \big(L_2(I)\big)^n$, $(\mu_0,\dots,\mu_{l-1}) \in \big(\mathcal{B}^{l-1}(0,T)\big)^n$, $(\nu_0,\dots,\nu_l) \in \big(\mathcal{B}^{l}(0,T)\big)^n$, $f \in \big(L_1(0,T;L_2(I))\big)^n$ for an arbitrary $T > 0$. Denote

$$c_0 = \|u_0\|_{(L_2(I))^n} + \|(\mu_0,\dots,\mu_{l-1})\|_{(\mathcal{B}^{l-1}(0,T))^n} + \|(\nu_0,\dots,\nu_l)\|_{(\mathcal{B}^{l}(0,T))^n} + \|f\|_{(L_1(0,T;L_2(I)))^n}. \tag{1.14}$$

Then there exists $\delta > 0$ such that under the assumption $c_0 \le \delta$ there exists a unique weak solution $u \in \big(X(Q_T)\big)^n$ of problem (1.1)–(1.3). Moreover, the map

$$\big(u_0, (\mu_0,\dots,\mu_{l-1}), (\nu_0,\dots,\nu_l), f\big) \to u \tag{1.15}$$

is Lipschitz continuous on the closed ball of the radius δ in the space $\big(L_2(I)\big)^n \times \big(\mathcal{B}^{l-1}(0,T)\big)^n \times \big(\mathcal{B}^{l}(0,T)\big)^n \times \big(L_1(0,T;L_2(I))\big)^n$ into the space $\big(X(Q_T)\big)^n$.

Theorem 1.2. *Let the hypotheses of Theorem 1.1 be satisfied except inequalities (1.13) which are substituted by the following ones:*

$$b_2(j,k,m) < \frac{4l-2j-2k}{2m+1} \quad \forall\ j,k,m. \tag{1.16}$$

Let c_0 is given by formula (1.14).

Then for a fixed arbitrary $\delta > 0$ there exists $T_0 > 0$ such that if $c_0 \le \delta$ and $T \in (0,T_0]$ there exists a unique weak solution $u \in \big(X(Q_T)\big)^n$ of problem (1.1)–(1.3). Moreover, the map (1.15) is Lipschitz continuous on the closed ball of the radius δ similarly to Theorem 1.1.

For the inverse problem the results are as follows.

Theorem 1.3. *Let the coefficients a_j, $j = 0,\dots,2l+1$, satisfy conditions (1.8), (1.9) and the functions g_j satisfy conditions (1.10), (1.11), (1.13). Let $u_0 \in \big(L_2(I)\big)^n$, $(\mu_0,\dots,\mu_{l-1}) \in \big(\mathcal{B}^{l-1}(0,T)\big)^n$, $(\nu_0,\dots,\nu_l) \in \big(\mathcal{B}^{l}(0,T)\big)^n$, $h_0 = (h_{01},\dots,h_{0n})^T \in \big(L_1(0,T;L_2(I))\big)^n$ for an arbitrary $T > 0$. Assume that condition (1.6) holds and for any $i = 1,\dots,n$, satisfying $m_i > 0$, for $k = 1,\dots m_i$ the functions ω_{ki} satisfy condition (1.12); $\varphi_{ki} \in W_1^1(0,T)$ and*

$$\varphi_{ki}(0) = \int_I u_{0i}(x)\omega_{ki}(x)\,dx; \tag{1.17}$$

$h_{ki} \in C([0,T];L_2(I))$ for $k = 1,\dots,m_i$. Let

$$\psi_{kji}(t) \equiv \int_I h_{ji}(t,x)\omega_{ki}(x)\,dx, \quad k,j = 1,\dots,m_i, \tag{1.18}$$

and assume that

$$\Delta_i(t) \equiv \det\bigl(\psi_{kji}(t)\bigr) \neq 0 \quad \forall\, t \in [0,T]. \tag{1.19}$$

Denote

$$c_0 = \|u_0\|_{(L_2(I))^n} + \|(\mu_0,\dots,\mu_{l-1})\|_{(\mathcal{B}^{l-1}(0,T))^n} + \|(\nu_0,\dots,\nu_l)\|_{(\mathcal{B}^{l}(0,T))^n} + \|h_0\|_{(L_1(0,T;L_2(I)))^n} + \sum_{i:m_i>0}\sum_{k=1}^{m_i} \|\varphi'_{ki}\|_{L_1(0,T)}. \tag{1.20}$$

Then there exists $\delta > 0$ *such that under the assumption* $c_0 \le \delta$ *there exist functions* $F_{ki} \in L_1(0,T)$, $i : m_i > 0$, $k = 1,\dots,m_i$, *and the corresponding weak solution* $u \in \bigl(X(Q_T)\bigr)^n$ *of problem* (1.1)–(1.3) *satisfying* (1.5), *where the function* f *is given by formula* (1.4). *Moreover, there exists* $r > 0$ *such that this solution* u *is unique in the ball* $\overline{X}_{rn}(Q_T)$ *with the corresponding unique functions* $F_{ki} \in L_1(0,T)$ *and the map*

$$\bigl(u_0, (\mu_0,\dots,\mu_{l-1}), (\nu_0,\dots,\nu_l), h_0, \{\varphi'_{ki}\}\bigr) \to (u, \{F_{ki}\}) \tag{1.21}$$

is Lipschitz continuous on the closed ball of the radius δ *in the space* $\bigl(L_2(I)\bigr)^n \times \bigl(\mathcal{B}^{l-1}(0,T)\bigr)^n \times \bigl(\mathcal{B}^{l}(0,T)\bigr)^n \times \bigl(L_1(0,T;L_2(I))\bigr)^n \times \bigl(L_1(0,T)\bigr)^M$ *into the space* $\bigl(X(Q_T)\bigr)^n \times \bigl(L_1(0,T)\bigr)^M$.

Theorem 1.4. *Let the hypotheses of Theorem* 1.3 *be satisfied except inequalities* (1.13) *which are substituted by inequalities* (1.16). *Let* c_0 *be given by formula* (1.20). *Then two assertions are valid.*

1. For a fixed arbitrary $\delta > 0$ *there exists* $T_0 > 0$ *such that if* $c_0 \le \delta$ *and* $T \in (0, T_0]$, *there exist unique functions* $F_{ki} \in L_1(0,T)$, $i : m_i > 0$, $k = 1,\dots,m_i$, *and the corresponding unique weak solution* $u \in \bigl(X(Q_T)\bigr)^n$ *of problem* (1.1)–(1.3) *satisfying* (1.5), *where the function* f *is given by formula* (1.4).

2. For a fixed arbitrary $T > 0$ *there exists* $\delta > 0$ *such that under the assumption* $c_0 \le \delta$ *there exist unique functions* $F_{ki} \in L_1(0,T)$, $i : m_i > 0$, $k = 1,\dots,m_i$, *and the corresponding unique weak solution* $u \in \bigl(X(Q_T)\bigr)^n$ *of problem* (1.1)–(1.3) *satisfying* (1.5), *where the function* f *is given by formula* (1.4).

Moreover, map (1.21) *is Lipschitz continuous on the closed ball of the radius* δ *similarly to Theorem* 1.3.

Remark 1. Theorems 1.2 and 1.4 are valid for the aforementioned Majda–Biello system. In the case of such a system with more general nonlinearities

$$\begin{cases} u_t + u_{xxx} + \bigl(g_1(u,v)\bigr)_x = f_1, \\ v_t + \alpha v_{xxx} + \bigl(g_2(u,v)\bigr)_x = f_2, \quad \alpha > 0, \end{cases}$$

Theorems 1.1 and 1.3 are valid if

$$|\partial_{y_k} g_j(y_1,y_2)| \le c\bigl(|y_1|^{b_1} + |y_2|^{b_1} + |y_1|^{b_2} + |y_2|^{b_2}\bigr), \quad k,j = 1,2,$$

where $0 < b_1 \le b_2 \le 2$, for example, if $g_1(y_1,y_2) = c y_2^3$, $g_2(y_1,y_2) = c_1 y_1^2 y_2 + c_2 y_1 y_2^2$.

The paper is organized as follows. Section 2 contains certain auxiliary results on the corresponding linear initial-boundary value problem and interpolating inequalities. Section 3 is devoted to the direct problem, Section 4 — to the inverse one.

2 Preliminaries

Further we use the following interpolating inequality (see, for example, [2]): there exists a constant $c = c(R, l, p)$ such that for any $\varphi \in H^l(I)$, integer $m \in [0, l)$ and $p \in [2, +\infty]$

$$\|\varphi^{(m)}\|_{L_p(I)} \le c\|\varphi^{(l)}\|^{2s}_{L_2(I)}\|\varphi\|^{1-2s}_{L_2(I)} + c\|\varphi\|_{L_2(I)}, \quad s = s(p, l, m) = \frac{2m+1}{4l} - \frac{1}{2lp}. \tag{2.1}$$

On the basis of (2.1) in [10, Lemma 3.3] the following inequality was proved: let $j \in [0, l]$, $k, m \in [0, l-1]$, $b \in (0, (4l-2j-2k)/(2m+1)]$, then for any functions $v, w \in X(Q_T)$

$$\big\||\partial_x^m v|^b \partial_x^k w\big\|_{L_{2l/(2l-j)}(0,T;L_2(I))} \le c\big(T^{((4l-2j-2k)-(2m+1)b)/(4l)} + T^{(2l-j)/(2l)}\big)\|v\|^b_{X(Q_T)}\|w\|_{X(Q_T)}. \tag{2.2}$$

Besides nonlinear system (1.1) consider its linear analogue

$$u_t - (-1)^l(a_{2l+1}\partial_x^{2l+1}u + a_{2l}\partial_x^{2l}u) - \sum_{j=0}^{l-1}(-1)^j\partial_x^j\big[a_{2j+1}(t,x)\partial_x^{j+1}u + a_{2j}(t,x)\partial_x^j u\big] = f(t,x) + \sum_{j=0}^{l}(-1)^j\partial_x^j G_j(t,x), \tag{2.3}$$

$G_j = (G_{j1}, \dots, G_{jn})^T$. The notion of a weak solution to the corresponding initial-boundary value problem is similar to Definition 1. In particular, the corresponding integral identity (for the same test functions as in Definition 1) is as follows:

$$\iint_{Q_T}\Big[(u, \phi_t) - (a_{2l+1}\partial_x^l u, \partial_x^{l+1}\phi) + (a_{2l}\partial_x^l u, \partial_x^l\phi) + \sum_{j=0}^{l-1}\big((a_{2j+1}\partial_x^{j+1}u + a_{2j}\partial_x^j u), \partial_x^j\phi\big) + \big(f(t,x), \phi\big) + \sum_{j=0}^{l}\big(G_j(t,x), \partial_x^j\phi\big)\Big]\,dxdt + \int_I (u_0, \phi\big|_{t=0})\,dx + \int_0^T (a_{2l+1}\nu_l, \partial_x^l\phi\big|_{x=R})\,dt = 0. \tag{2.4}$$

First consider the case $a_j \equiv 0$ for $j \le 2l-1$. Then system (2.3) is obviously splitted into the set of separate equations and we can use the corresponding results from [11] and [9] for single equations.

Lemma 2.1. *Let the coefficients a_{2l+1} and a_{2l} satisfy condition* (1.8), *$a_j \equiv 0$ for $j \le 2l-1$, $u_0 \in \big(L_2(I)\big)^n$, $(\mu_0, \dots, \mu_{l-1}) \in \big(\mathcal{B}^{l-1}(0,T)\big)^n$, $(\nu_0, \dots, \nu_l) \in \big(\mathcal{B}^l(0,T)\big)^n$, $f = G_j \equiv 0$ $\forall j$.*

Then there exists a unique weak solution $u \in \big(X(Q_T)\big)^n$ of problem (2.3), (1.2), (1.3) *and for any $t \in (0, T]$*

$$\|u\|_{(X(Q_t))^n} \le c(T)\Big[\|u_0\|_{(L_2(I))^n} + \|(\mu_0, \dots, \mu_{l-1})\|_{(\mathcal{B}^{l-1}(0,t))^n} + \|(\nu_0, \dots, \nu_l)\|_{(\mathcal{B}^l(0,t))^n}\Big]. \tag{2.5}$$

Proof. This assertion succeeds from [11, Lemma 4.3]. □

Lemma 2.2. *Let the coefficients a_{2l+1} and a_{2l} satisfy condition* (1.8), *$a_j \equiv 0$ for $j \le 2l-1$, $u_0 \equiv 0$, $\mu_j \equiv 0$ for $j = 0, \dots, l-1$, $\nu_j \equiv 0$ for $j = 0, \dots, l$, $f \in \big(L_1(0,T;L_2(I))\big)^n$, $G_j \in \big(L_{2l/(2l-j)}(0,T;L_2(I))\big)^n$, $j = 0, \dots, l$.*

Then there exists a unique weak solution $u \in \big(X(Q_T)\big)^n$ *of problem* (2.3), (1.2), (1.3) *and for any* $t \in [0,T]$

$$\|u\|_{(X(Q_t))^n} \le c(T)\Big[\|f\|_{(L_1(0,t;L_2(I)))^n} + \sum_{j=0}^{l} \|G_j\|_{(L_{2l/(2l-j)}(0,t;L_2(I)))^n}\Big]; \tag{2.6}$$

moreover, for $i = 1,\dots,n$ *and* $\rho(x) \equiv 1 + x$

$$\int_I u_i^2(t,x)\rho(x)\,dx + \iint_{Q_t} \big((2l+1)a_{(2l+1)i} - 2a_{(2l)i}\rho(x)\big)\big(\partial_x^l u_i(\tau,x)\big)^2\,dxd\tau \le 2\iint_{Q_t} f_i u_i \rho\,dxd\tau + 2\sum_{j=0}^{l}\iint_{Q_t} G_{ji}(\partial_x^j u_i\rho + j\partial_x^{j-1}u_i)\,dxd\tau. \tag{2.7}$$

Proof. This assertion succeeds from [9, Lemma 4]. □

Theorem 2.1. *Let the coefficients* a_j *satisfy conditions* (1.8), (1.9), $u_0 \in \big(L_2(I)\big)^n$, $(\mu_0,\dots,\mu_{l-1}) \in \big(\mathcal{B}^{l-1}(0,T)\big)^n$, $(\nu_0,\dots,\nu_l) \in \big(\mathcal{B}^l(0,T)\big)^n$, $f \in \big(L_1(0,T;L_2(I))\big)^n$, $G_j \in \big(L_{2l/(2l-j)}(0,T;L_2(I))\big)^n$, $j = 0,\dots,l$.

Then there exists a unique weak solution $u \in \big(X(Q_T)\big)^n$ *of problem* (2.3), (1.2), (1.3) *and for any* $t \in (0,T]$

$$\|u\|_{(X(Q_t))^n} \le c(T)\Big[\|u_0\|_{(L_2(I))^n} + \|(\mu_0,\dots,\mu_{l-1})\|_{(\mathcal{B}^{l-1}(0,t))^n} + \|(\nu_0,\dots,\nu_l)\|_{(\mathcal{B}^l(0,t))^n} + \|f\|_{(L_1(0,t;L_2(I)))^n} + \sum_{j=0}^{l}\|G_j\|_{(L_{2l/(2l-j)}(0,t;L_2(I)))^n}\Big]. \tag{2.8}$$

Proof. Denote by $w = (w_1,\dots,w_n)^T$ the solution of problem (2.3), (1.2), (1.3) constructed in Lemma 2.1 Let $U(t,x) \equiv u(t,x) - w(t,x)$. Consider an initial-boundary value problem for the function U:

$$U_t - (-1)^l(a_{2l+1}\partial_x^{2l+1}U + a_{2l}\partial_x^{2l}U) - \sum_{j=0}^{l-1}(-1)^j\partial_x^j\big[a_{2j+1}(t,x)\partial_x^{j+1}U + a_{2j}(t,x)\partial_x^j U\big] = f(t,x) + \sum_{j=0}^{l}(-1)^j\partial_x^j\widetilde{G}_j(t,x), \tag{2.9}$$

where $\widetilde{G}_l \equiv G_l$, while $\widetilde{G}_j \equiv G_j + a_{2j+1}\partial_x^{j+1}w + a_{2j}\partial_x^j w$ for $j < l$, and zero initial and boundary conditions (1.2), (1.3). Note that by virtue of (2.1) for $m = 0$ or $m = 1$, $j < l$ and $i = 1,\dots,n$

$$\|\partial_x^{j+m}w_i\|_{L_2(I)} \le c\|\partial_x^l w_i\|_{L_2(I)}^{(j+m)/l}\|w_i\|_{L_2(I)}^{(l-j-m)/l} + c\|w_i\|_{L_2(I)}.$$

Therefore, $\widetilde{G}_j \in \big(L_{2l/(2l-j)}(0,T;L_2(I))\big)^n$ with

$$\|\widetilde{G}_j\|_{(L_{2l/(2l-j)}(0,t;L_2(I)))^n} \le \|G_j\|_{(L_{2l/(2l-j)}(0,t;L_2(I)))^n} + c(T)\|w\|_{(X(Q_t))^n}. \tag{2.10}$$

In order to obtain the solution to the initial-value problem for system (2.9) we apply the contraction principle and first construct it on a small time interval $[0,t_0]$ as the fixed point of a map

$U = \Lambda V$, where for $V \in \big(X(Q_{t_0})\big)^n$ the function $U \in \big(X(Q_{t_0})\big)^n$ is a solution to an initial-boundary value problem for the system

$$U_t - (-1)^l(a_{2l+1}\partial_x^{2l+1}U + a_{2l}\partial_x^{2l}U) = \sum_{j=0}^{l-1}(-1)^j\partial_x^j\big[a_{2j+1}(t,x)\partial_x^{j+1}V + a_{2j}(t,x)\partial_x^jV\big] \\ + f(t,x) + \sum_{j=0}^{l}(-1)^j\partial_x^j\widetilde{G}_j(t,x), \quad (2.11)$$

with zero initial and boundary conditions (1.2), (1.3). Note that similarly to (2.10) the hypothesis of Lemma 2.2 is verified and such a map is defined for any $t_0 \in (0,T]$. Moreover, according to (2.6)

$$\|U\|_{(X(Q_{t_0}))^n} \le c(T)\Big[\|f\|_{(L_1(0,t_0;L_2(I)))^n} + \sum_{j=0}^{l}\|\widetilde{G}_j\|_{(L_{2l/(2l-j)}(0,t_0;L_2(I)))^n} \\ + \sum_{j=0}^{l-1}\big(\|\partial_x^{j+1}V\|_{(L_{2l/(2l-j)}(0,t_0;L_2(I)))^n} + (\|\partial_x^jV\|_{(L_{2l/(2l-j)}(0,t_0;L_2(I)))^n}\big)\Big]. \quad (2.12)$$

By virtue of (2.1) if $j+m \le 2l-1$ for $i = 1,\dots,n$

$$\|\partial_x^m V_i\|_{L_{2l/(2l-j)}(0,t_0;L_2(I))} \\ \le c\Big(\int_0^{t_0}\big(\|\partial_x^l V_i\|_{L_2(I)}^{2m/(2l-j)}\|V_i\|_{L_2(I)}^{2(l-m)/(2l-j)} + \|V_i\|_{L_2(I)}^{2l/(2l-j)}\big)\,dt\Big)^{(2l-j)/(2l)} \\ \le ct_0^{(2l-j-m)/(2l)}\|V_i\|_{C([0,t_0];L_2(I))}^{(l-m)/l}\|\partial_x^l V_i\|_{L_2(Q_{t_0})}^{m/l} + ct_0^{(2l-j)/(2l)}\|V_i\|_{C([0,t_0];L_2(I))} \\ \le c(T)t_0^{1/(2l)}\|V_i\|_{X(Q_{t_0})}. \quad (2.13)$$

Therefore, it follows from (2.12) that

$$\|U\|_{(X(Q_{t_0}))^n} \le c(T)\Big[\|f\|_{(L_1(0,t_0;L_2(I)))^n} + \sum_{j=0}^{l}\|\widetilde{G}_j\|_{(L_{2l/(2l-j)}(0,t_0;L_2(I)))^n} \\ + t_0^{1/(2l)}\|V\|_{(X(Q_{t_0}))^n}\Big]. \quad (2.14)$$

Similarly to (2.14) for $\widetilde{V} \in \big(X(Q_{t_0})\big)^n$, $\widetilde{U} = \Lambda\widetilde{V}$

$$\|U - \widetilde{U}\|_{(X(Q_{t_0}))^n} \le c(T)t_0^{1/(2l)}\|V - \widetilde{V}\|_{(X(Q_{t_0}))^n}. \quad (2.15)$$

Inequalities (2.14), (2.15) provide existence of a unique solution $U \in \big(X(Q_{t_0})\big)^n$ to the considered problem if, for example, $c(T)t_0^{1/(2l)} \le 1/2$. Then since the value of t_0 depends only on T step by step this solution can be extended to the whole time segment $[0,T]$, moreover,

$$\|U\|_{(X(Q_t))^n} \le c(T)\Big[\|f\|_{(L_1(0,t;L_2(I)))^n}\sum_{j=0}^{l}\|\widetilde{G}_j\|_{(L_{2l/(2l-j)}(0,t;L_2(I)))^n}\Big]. \quad (2.16)$$

Combining (2.5) (applied to the function w), (2.10) and (2.16), for $u \equiv U + w$ we complete the proof. □

Introduce certain additional notation. Let

$$u = S(u_0, (\mu_0, \dots, \mu_{l-1}), (\nu_0, \dots, \nu_l), f, (G_0, \dots, G_l))$$

be a weak solution of problem (2.3), (1.2), (1.3) from the space $\big(X(Q_T)\big)^n$ under the hypotheses of Theorem 2.1. Define also

$$W = (u_0, (\mu_0, \dots, \mu_{l-1}), (\nu_0, \dots, \nu_l)),$$
$$\widetilde{S}W = S(W, 0, (0, \dots, 0)), \quad \widetilde{S} : \big(L_2(I) \times \mathcal{B}^{l-1}(0,T) \times \mathcal{B}^{l}(0,T)\big)^n \to \big(X(Q_T)\big)^n,$$
$$S_0 f = S(0, (0, \dots, 0), (0, \dots, 0), f, (0, \dots, 0)), S_0 : \big(L_1(0,T; L_2(I))\big)^n \to \big(X(Q_T)\big)^n,$$

$$\widetilde{S}_j G_j = S(0, (0, \dots, 0), (0, \dots, 0), 0, (0, \dots, G_j, \dots, 0)),$$
$$S_j : \big(L_{2l/(2l-j)}(0,T; L_2(I))\big)^n \to \big(X(Q_T)\big)^n, \quad j = 0, \dots, l.$$

Let $\widetilde{W_1^1}(0,T) = \{\varphi \in W_1^1(0,T) : \varphi(0) = 0\}$. Obviously, the equivalent norm in this space is $\|\varphi'\|_{L_1(0,T)}$.

Let a function $\omega \in C(\bar{I})$. On the space of functions $u(t,x)$, lying in $L_1(I)$ for all $t \in [0,T]$, define a linear operator $Q(\omega)$ by a formula $(Q(\omega)u)(t) = q(t; u, \omega)$, where

$$q(t; u, \omega) \equiv \int_I u(t,x)\omega(x)\,dx, \quad t \in [0,T].$$

Lemma 2.3. *Let the hypotheses of Theorem 2.1 be satisfied. Let the function ω satisfy condition (1.12).*

Then for the function $u = (u_1 \dots, u_n)^T = S(u_0, (\mu_0, \dots, \mu_{l-1}), (\nu_0, \dots, \nu_l), f, (G_0, \dots, G_l))$ the functions $q(\cdot; u_i, \omega) = Q(\omega)u_i \in W_1^1(0,T)$, i=1,…,n, and for almost every $t \in (0,T)$

$$\begin{aligned} q'(t; u_i, \omega) = r(t; u_i, \omega) &\equiv \nu_{li}(t) a_{(2l+1)i}\omega^{(l)}(R) \\ &+ \sum_{k=0}^{l-1} (-1)^{l+k} \Big[\nu_{ki}(t) \big(a_{(2l+1)i}\omega^{(2l-k)}(R) - a_{(2l)i}\omega^{(2l-k-1)}(R)\big) \\ &\qquad - \mu_{ki}(t) \big(a_{(2l+1)i}\omega^{(2l-k)}(0) - a_{(2l)i}\omega^{(2l-k-1)}(0)\big)\Big] \\ &+ \sum_{m=1}^{n}\sum_{j=0}^{l-1}\sum_{k=0}^{j-1} (-1)^{j+k} \Big[\nu_{km}(t) \big((a_{(2j+1)im}\omega^{(j)})^{(j-k)}(R) - (a_{(2j)im}\omega^{(j)})^{(j-k-1)}(R)\big) \\ &\qquad - \mu_{km}(t) \big((a_{(2j+1)im}\omega^{(j)})^{(j-k)}(0) - (a_{(2j)im}\omega^{(j)})^{(j-k-1)}(0)\big)\Big] \\ &+ (-1)^{l+1} \int_I u_i(t,x) \big(a_{(2l+1)i}\omega^{(2l+1)} - a_{(2l)i}\omega^{(2l)}\big)\,dx \\ &+ \sum_{m=1}^{n}\sum_{j=0}^{l-1} (-1)^{j+1} \int_I u_m(t,x) \big[(a_{(2j+1)im}\omega^{(j)})^{(j+1)} - (a_{(2j)im}\omega^{(j)})^{(j)}\big]\,dx \\ &+ \int_I f_i(t,x)\omega\,dx + \sum_{j=0}^{l} \int_I G_{ji}(t,x)\omega^{(j)}\,dx, \end{aligned} \tag{2.17}$$

$$\begin{aligned} \|q'(\cdot; u_i, \omega)\|_{L_1(0,T)} \le c(T)\Big[&\|u_0\|_{(L_2(I))^n} + \|(\mu_0, \dots, \mu_{l-1})\|_{(\mathcal{B}^{l-1}(0,T))^n} \\ &+ \|(\nu_0, \dots, \nu_l)\|_{(\mathcal{B}^{l}(0,T))^n} + \|f\|_{(L_1(0,T;L_2(I)))^n} + \sum_{j=0}^{l} \big(\|G_j\|_{(L_{2l/(2l-j)}(0,T;L_2(I)))^n}\big)\Big], \end{aligned} \tag{2.18}$$

where the constant c does not decrease in T.

Proof. For an arbitrary function $\psi \in C_0^\infty(0,T)$ let $\phi_i(t,x) \equiv \psi(t)\omega(x)$ for certain i, $\phi_m(x) \equiv 0$ when $m \neq i$. This function ϕ satisfies the assumption on a test function in Definition 1 and then equality (2.4) after integration by parts yields that

$$\int_0^T \psi'(t) q(t; u_i, \omega)\, dt = -\int_0^T \psi(t) r(t; u_i, \omega)\, dt. \tag{2.19}$$

Since $r \in L_1(0,T)$ it follows from (2.19) that there exists the weak derivative $q'(t; u_i, \omega) = r(t; u_i, \omega) \in L_1(0,T)$ and

$$\|q'\|_{L_1(0,T)} \le c\Big[\sum_{j=0}^{l-1} \|\mu_j\|_{(L_1(0,T))^n} + \sum_{j=0}^{l} \|\nu_j\|_{(L_1(0,T))^n} + \|f\|_{(L_1(0,T;L_2(I)))^n} + \sum_{j=0}^{l} \|G_j\|_{(L_1(0,T;L_1(I)))^n} + \|u\|_{(L_1(0,T;L_2(I)))^n}\Big].$$

Since $\|u\|_{(L_1(0,T;L_2(I))^n} \le T\|u\|_{(C([0,T];L_2(I)))^n} \le T\|u\|_{(X(Q_T))^n}$, application of inequality (2.8) completes the proof. □

3 The direct problem

Proof of the existence part of Theorem 1.1. On the space $\big(X(Q_T)\big)^n$ consider the map Θ

$$u = \Theta v \equiv \widetilde{S}W + S_0 f - \sum_{j=0}^{l} \widetilde{S}_j g_j(t,x,v,\ldots,\partial_x^{l-1}v). \tag{3.1}$$

Note that according to conditions (1.10), (1.11) for $i = 1,\ldots,n$

$$|g_{ji}(t,x,v,\ldots,\partial_x^{l-1}v)| \le c\sum_{k=0}^{l-1}\sum_{m=0}^{l-1}\big(|\partial_x^m v|^{b_1(j,k,m)} + |\partial_x^m v|^{b_2(j,k,m)}\big)|\partial_x^k v| \tag{3.2}$$

In particular, conditions (1.13) and inequality (2.2) yield that $g_{ji}(t,x,v,\ldots,\partial_x^{l-1}v) \in L_{2l/(2l-j)}(0,T;L_2(I))$, moreover,

$$\|g_j(t,x,v,\ldots,\partial_x^{l-1}v)\|_{(L_{2l/(2l-j)}(0,T;L_2(I)))^n} \le c\sum_{k=0}^{l-1}\sum_{m=0}^{l-1}\sum_{i=1}^{2}(T^{((4l-2j-2k)-(2m+1)b_i(j,k,m))/(4l)} + T^{(2l-j)/(2l)})\|v\|_{(X(Q_T))^n}^{b_i(j,k,m)+1}. \tag{3.3}$$

In particular, Theorem 2.1 ensures that the map Θ exists. Let

$$b_1 = \min_{j,k,m}(b_1(j,k,m)), \quad b_2 = \max_{j,k,m}(b_2(j,k,m)), \quad 0 < b_1 \le b_2, \tag{3.4}$$

then it follows from (3.3) that

$$\|g_j(t,x,v,\ldots,\partial_x^{l-1}v)\|_{(L_{2l/(2l-j)}(0,T;L_2(I)))^n} \le c(T)\left(\|v\|_{(X(Q_T))^n}^{b_1+1} + \|v\|_{(X(Q_T))^n}^{b_2+1}\right), \tag{3.5}$$

therefore, inequality (2.8) implies that

$$\|\Theta v\|_{(X(Q_T))^n} \le c(T)c_0 + c(T)\left(\|v\|_{(X(Q_T))^n}^{b_1+1} + \|v\|_{(X(Q_T))^n}^{b_2+1}\right). \tag{3.6}$$

Next, for any functions $v_1, v_2 \in \big(X(Q_T)\big)^n$

$$|g_{ji}(t,x,v_1,\dots,\partial_x^{l-1}v_1) - g_{ji}(t,x,v_2,\dots,\partial_x^{l-1}v_2)| \\ \le c\sum_{k=0}^{l-1}\sum_{m=0}^{l-1}\big(|\partial_x^m v_1|^{b_1(j,k,m)} + |\partial_x^m v_2|^{b_1(j,k,m)} + |\partial_x^m v_1|^{b_2(j,k,m)} + |\partial_x^m v_2|^{b_2(j,k,m)}\big) \\ \times |\partial_x^k(v_1-v_2)|, \quad (3.7)$$

therefore, similarly to (3.5)

$$\|g_j(t,x,v_1,\dots,\partial_x^{l-1}v_1) - g_j(t,x,v_2,\dots,\partial_x^{l-1}v_2)\|_{(L_{2l/(2l-j)}(0,T;L_2(I)))^n} \\ \le c(T)\Big(\|v_1\|^{b_1}_{(X(Q_T))^n} + \|v_2\|^{b_1}_{(X(Q_T))^n} + \|v_1\|^{b_2}_{(X(Q_T))^n} + \|v_2\|^{b_2}_{(X(Q_T))^n}\Big)\|v_1-v_2\|_{(X(Q_T))^n}. \quad (3.8)$$

and similarly to (3.6)

$$\|\Theta v_1 - \Theta v_2\|_{(X(Q_T))^n} \\ \le c(T)\Big(\|v_1\|^{b_1}_{(X(Q_T))^n} + \|v_2\|^{b_1}_{(X(Q_T))^n} + \|v_1\|^{b_2}_{(X(Q_T))^n} + \|v_2\|^{b_2}_{(X(Q_T))^n}\Big)\|v_1-v_2\|_{(X(Q_T))^n}. \quad (3.9)$$

Now, choose $r>0$ such that

$$r^{b_1} + r^{b_2} \le \frac{1}{4c(T)} \quad (3.10)$$

and then $\delta>0$ such that

$$\delta \le \frac{r}{2c(T)}. \quad (3.11)$$

Then it follows from (3.6) and (3.9) that on the ball $\overline{X}_{rn}(Q_T)$ the map Θ is a contraction. Its unique fixed point $u \in \big(X(Q_T)\big)^n$ is the desired solution. Moreover,

$$\|u\|_{(X(Q_T))^n} \le c(c_0). \quad (3.12)$$

□

Note that the above argument ensures uniqueness only in a certain ball. In order to establish uniqueness and continuous dependence in the whole space we apply another approach. Then the rest part of Theorem 1.1 succeeds from (3.12) and the theorem below.

Theorem 3.1. *Let the assumptions on the functions a_j and g_j from the hypotheses of Theorem 1.1 be satisfied. Let $u_0, \widetilde{u}_0 \in \big(L_2(I)\big)^n$, $(\mu_0,\dots,\mu_{l-1}), (\widetilde{\mu}_0,\dots\widetilde{\mu}_{l-1}) \in \big(\mathcal{B}^{l-1}(0,T)\big)^n$, $(\nu_0,\dots,\nu_l), (\widetilde{\nu}_0,\dots,\widetilde{\nu}_l) \in \big(\mathcal{B}^l(0,T)\big)^n$, $f, \widetilde{f} \in \big(L_1(0,T;L_2(I))\big)^n$ and let $u, \widetilde{u}$ be two weak solutions to corresponding problems* (1.1)–(1.3) *in the space $\big(X(Q_T)\big)^n$ with $\|u\|_{(X(Q_T))^n}, \|\widetilde{u}\|_{(X(Q_T))^n} \le K$ for a certain positive K.*

Then there exists a positive constant $c = c(T,K)$ such that

$$\|u-\widetilde{u}\|_{(X(Q_T))^n} \le c\big(\|u_0-\widetilde{u}_0\|_{(L_2(I))^n} + \|(\mu_0-\widetilde{\mu}_0,\dots,\mu_{l-1}-\widetilde{\mu}_{l-1})\|_{(\mathcal{B}^{l-1}(0,T))^n} \\ + \|(\nu_0-\widetilde{\nu}_0,\dots,\nu_l-\widetilde{\nu}_l)\|_{(\mathcal{B}^l(0,T))^n} + \|f-\widetilde{f}\|_{(L_1(0,T;L_2(I)))^n}\big). \quad (3.13)$$

Proof. Let $w \in \big(X(Q_T)\big)^n$ be a solution to the linear problem

$$w_t - (-1)^l(a_{2l+1}\partial_x^{2l+1}w + a_{2l}\partial_x^{2l}w) = 0, \quad (3.14)$$

$$w(0,x) = u_0(x) - \widetilde{u}_0(x), \tag{3.15}$$

$$\partial_x^j w(t,0) = \mu_j(t) - \widetilde{\mu}_j(t),\ j=0,\dots,l-1, \quad \partial_x^j w(t,R) = \nu_j(t) - \widetilde{\nu}_j(t),\ j=0,\dots,l. \tag{3.16}$$

Lemma 2.1 ensures that such a function exists and according to (2.5)

$$\|w\|_{(X(Q_T))^n} \le c(T)\big(\|u_0-\widetilde{u}_0\|_{(L_2(I))^n} + \|(\mu_0-\widetilde{\mu}_0,\dots,\mu_{l-1}-\widetilde{\mu}_{l-1})\|_{(\mathcal{B}^{l-1}(0,T))^n} \\ + \|(\nu_0-\widetilde{\nu}_0,\dots,\nu_l-\widetilde{\nu}_l)\|_{(\mathcal{B}^l(0,T))^n}\big). \tag{3.17}$$

Let $v(t,x) \equiv u(t,x) - \widetilde{u}(t,x) - w(t,x)$, Then $v \in \big(X(Q_T)\big)^n$ is a solution to the initial-boundary problem in Q_T for the system

$$v_t - (-1)^l(a_{2l+1}\partial_x^{2l+1}v + a_{2l}\partial_x^{2l}v) = (f-\widetilde{f}) \\ + \sum_{j=0}^{l-1}(-1)^j\partial_x^j\big[a_{2j+1}(t,x)\partial_x^{j+1}(u-\widetilde{u}) + a_{2j}(t,x)\partial_x^j(u-\widetilde{u})\big] \\ - \sum_{j=0}^{l}(-1)^j\partial_x^j\big[g_j(t,x,u,\dots,\partial_x^{l-1}u) - g_j(t,x,\widetilde{u},\dots,\partial_x^{l-1}\widetilde{u})\big] \tag{3.18}$$

with zero initial and boundary conditions of (1.2), (1.3) type. Similarly to (2.11)–(2.13) $a_{2j+1}(t,x)\partial_x^{j+1}u + a_{2j}(t,x)\partial_x^j u \in \big(L_{2l/(2l-j)}(0,T;L_2(I))\big)^n$, similarly to (3.2), (3.3) $g_j(t,x,u,\dots,\partial_x^{l-1}u) \in \big(L_{2l/(2l-j)}(0,T;L_2(I))\big)^n$. The same properties hold in the case of the function $\widetilde{u}$. Therefore, the hypothesis of Lemma 2.2 is satisfied and for $i=1,\dots,n$ according to (2.7)

$$\int_I v_i^2(t,x)\rho\,dx + \iint_{Q_t}\big((2l+1)a_{(2l+1)i} - 2a_{(2l)i}\rho\big)\big(\partial_x^l v_i(\tau,x)\big)^2\,dxd\tau \\ \le 2\iint_{Q_t}(f_i-\widetilde{f}_i)v_i\rho\,dxd\tau \\ + 2\sum_{m=1}^{n}\sum_{j=0}^{l-1}\iint_{Q_t}\Big(a_{(2j+1)im}(t,x)\partial_x^{j+1}(v_m+w_m) + a_{(2j)im}(t,x)\partial_x^j(v_m+w_m)\Big) \\ \times(\partial_x^j v_i\rho + j\partial_x^{j-1}v_i)\,dxd\tau \\ - 2\sum_{j=0}^{l}\iint_{Q_t}\Big(g_{ji}(t,x,u,\dots,\partial_x^{l-1}u) - g_{ji}(t,x,\widetilde{u},\dots,\partial_x^{l-1}\widetilde{u})\Big) \\ \times(\partial_x^j v_i\rho + j\partial_x^{j-1}v_i)\,dxd\tau. \tag{3.19}$$

Note that by virtue of (1.8) uniformly in i and x

$$(2l+1)a_{(2l+1)i} - 2a_{(2l)i}\rho(x) \ge \alpha_0 > 0. \tag{3.20}$$

It follows from (2.1) for $p=2$ that if $j \le l-1$

$$\iint_{Q_t}|\partial_x^{j+1}v_m|\cdot|\partial_x^j v_i|\,dxd\tau \le c\int_0^t\Big[\|\partial_x^l v\|_{(L_2(I))^n}^{(2l-1)/l}\|v\|_{(L_2(I))^n}^{1/l} + \|v\|_{(L_2(I))^n}^2\Big]\,d\tau \\ \le \varepsilon\iint_{Q_t}|\partial_x^l v|^2\,dxd\tau + c(\varepsilon)\iint_{Q_t}|v|^2\rho\,dxd\tau, \tag{3.21}$$

where $\varepsilon > 0$ can be chosen arbitrarily small;

$$\iint_{Q_t} |\partial_x^{j+1} w_m| \cdot |\partial_x^j v_i| \, dxd\tau \le \Bigl(\iint_{Q_t} (\partial_x^j v_i)^2 \, dxd\tau \iint_{Q_t} (\partial_x^{j+1} w_m)^2 \, dxd\tau \Bigr)^{1/2}$$
$$\le \varepsilon \iint_{Q_t} |\partial_x^l v|^2 \, dxd\tau + c(\varepsilon) \iint_{Q_t} |v|^2 \rho \, dxd\tau + c\|w\|^2_{(X(Q_T))^n}. \quad (3.22)$$

Next, similarly to (3.7)

$$|g_{ji}(t,x,u,\dots,\partial_x^{l-1}u) - g_{ji}(t,x,\widetilde{u},\dots,\partial_x^{l-1}\widetilde{u})|$$
$$\le c \sum_{k=0}^{l-1} \sum_{m=0}^{l-1} \bigl(|\partial_x^m u|^{b_1(j,k,m)} + |\partial_x^m \widetilde{u}|^{b_1(j,k,m)} + |\partial_x^m u|^{b_2(j,k,m)} + |\partial_x^m \widetilde{u}|^{b_2(j,k,m)} \bigr)$$
$$\times |\partial_x^k (v+w)|. \quad (3.23)$$

Note that, for example, for $j \le l$, $k,m \le l-1$ if $0 \le b \le (4l-2j-2k)/(2m+1)$

$$\int_I |\partial_x^m u|^b |\partial_x^k v| \cdot |\partial_x^j v| \, dx \le \sup_{x\in I} |\partial_x^m u|^b \Bigl(\int_I |\partial_x^k v|^2 \, dx \int_I |\partial_x^j v|^2 \, dx \Bigr)^{1/2}$$
$$\le c \sup_{x\in I} |\partial_x^m u|^b \Bigl[\Bigl(\int_I |\partial_x^l v|^2 \, dx \Bigr)^{(k+j)/(2l)} \Bigl(\int_I |v|^2 \, dx \Bigr)^{(2l-j-k)/(2l)} + \int_I |v|^2 \, dx \Bigr]$$
$$\le \varepsilon \int_I |\partial_x^l v|^2 \, dx + c(\varepsilon) \Bigl[\sup_{x\in I} |\partial_x^m u|^{2lb/(2l-j-k)} + \sup_{x\in I} |\partial_x^m u|^b \Bigr] \int_I |v|^2 \rho \, dx, \quad (3.24)$$

where

$$\int_0^T \sup_{x\in I} |\partial_x^m u|^{2lb/(2l-j-k)} \, dt$$
$$\le \sup_{t\in(0,T)} \Bigl(\int_I |u|^2 \, dx \Bigr)^{(2l-2m-1)b/(4l-2j-2k)} \int_0^T \Bigl(\int_I |\partial_x^l u|^2 \, dx \Bigr)^{(2m+1)b/(4l-2j-2k)} dt$$
$$\le c(T) \|u\|^{2lb/(2l-j-k)}_{(X(Q_T))^n} \, dt; \quad (3.25)$$

also split b into two parts: $b = b' + b''$, where $0 \le b' \le (2l-2j)/(2m+1)$, $0 \le b'' \le (2l-2k)/(2m+1)$, then similarly to (3.24)

$$\int_I |\partial_x^m u|^b |\partial_x^k w| \cdot |\partial_x^j v| \, dx \le \sup_{x\in I} |\partial_x^m u|^{b'+b''} \Bigl(\int_I |\partial_x^j v|^2 \, dx \int_I |\partial_x^k w|^2 \, dx \Bigr)^{1/2}$$
$$\le \varepsilon \int_I |\partial_x^l v|^2 \, dx + c(\varepsilon) \Bigl[\sup_{x\in I} |\partial_x^m u|^{2lb'/(l-j)} + \sup_{x\in I} |\partial_x^m u|^{2b'} \Bigr] \int_I |v|^2 \rho \, dx$$
$$+ c \int_I |\partial_x^l w|^2 \, dx + c \Bigl[\sup_{x\in I} |\partial_x^m u|^{2lb''/(l-k)} + \sup_{x\in I} |\partial_x^m u|^{2b''} \Bigr] \int_I |w|^2 \, dx, \quad (3.26)$$

where similarly to (3.25)

$$\int_0^T \sup_{x\in I} |\partial_x^m u|^{2lb'/(l-j)} \, dt, \int_0^T \sup_{x\in I} |\partial_x^m u|^{2lb''/(l-k)} \, dt \le c(T,K). \quad (3.27)$$

Gathering (3.20)–(3.27) we deduce from inequality (3.19) that

$$\int_I v_i^2(t,x) \rho \, dx + \alpha_0 \iint_{Q_t} (\partial_x^l v_i)^2 \, dxd\tau \le \frac{\alpha_0}{2n} \iint_{Q_t} |\partial_x^l v|^2 \, dxd\tau$$
$$+ \int_0^t \gamma(\tau) \int_I |v|^2 \rho \, dxd\tau + 2 \int_0^t \|f - \widetilde{f}\|_{(L_2(I))^n} \|v_i\|_{L_2(I)} \, d\tau + c(T,K) \|w\|^2_{(X(Q_T))^n}, \quad (3.28)$$

where $\|\gamma\|_{L_1(0,T)} \le c(T,K)$. Summing inequalities (3.28) with respect to i, using estimate (3.17) and applying Gronwall lemma we complete the proof. □

In this section it remains to prove Theorem 1.2.

Proof of Theorem 1.2. Overall, the proof repeats the proof of the existence part of Theorem 1.1. The desired solution is constructed as a fixed point of the map Θ from (3.1). In comparison with (3.3) here we obtain the following estimate: let

$$\sigma = \frac{\min\limits_{j,k,m}(4l-2j-2k-(2m+1)b_2(j,k,m))}{4l} \tag{3.29}$$

(note that $\sigma > 0$ because of (1.16)), then

$$\|g_j(t,x,v,\dots,\partial_x^{l-1}v)\|_{(L_{2l/(2l-j)}(0,T;L_2(I)))^n} \le c(T)T^\sigma \sum_{k=0}^{l-1}\sum_{m=0}^{l-1}\sum_{i=1}^{2} \|v\|^{b_i(j,k,m)+1}_{(X(Q_T))^n}. \tag{3.30}$$

and similarly to (3.6), (3.9)

$$\|\Theta v\|_{(X(Q_T))^n} \le c(T)c_0 + c(T)T^\sigma\left(\|v\|^{b_1+1}_{(X(Q_T))^n} + \|v\|^{b_2+1}_{(X(Q_T))^n}\right). \tag{3.31}$$

$$\begin{aligned} &\|\Theta v_1 - \Theta v_2\|_{(X(Q_T))^n} \\ &\qquad \le c(T)T^\sigma\left(\|v_1\|^{b_1}_{(X(Q_T))^n} + \|v_2\|^{b_1}_{(X(Q_T))^n} + \|v_1\|^{b_2}_{(X(Q_T))^n} + \|v_2\|^{b_2}_{(X(Q_T))^n}\right) \\ &\qquad\qquad \times \|v_1 - v_2\|_{(X(Q_T))^n}. \end{aligned} \tag{3.32}$$

Now for a fixed δ choose $T_0 > 0$ such that

$$4c(T_0)T_0^\sigma\left((2c(T_0)\delta)^{b_1} + (2c(T_0)\delta)^{b_2}\right) \le 1 \tag{3.33}$$

(it is possible since $c(T)$ does not decrease in T) and then for every $T \in (0,T_0]$ choose an arbitrary r such that

$$r \ge 2c(T)\delta, \quad 4c(T)T^\sigma(r^{b_1} + r^{b_2}) \le 1 \tag{3.34}$$

(this set is not empty because of (3.33)). Then the map Θ is a contraction on the ball $\overline{X}_{rn}(Q_T)$.

In order to prove uniqueness in the whole space note that for an arbitrarily large r the value of T_0 can be chosen sufficiently small such that the solution of the considered problem $u \in (X(Q_{T_0})^n)$ is the unique fixed point of the contraction Θ in $\overline{X}_{rn}(Q_{T_0})$. □

4 The inverse problem

We start with the linear case. The following lemma is the crucial part of the study.

Lemma 4.1. *Let the assumptions on the functions a_j from the hypotheses of Theorem* 1.3 *be satisfied. Let condition* (1.6) *be valid and for any $i = 1,\dots,n$, satisfying $m_i > 0$, for $k = 1,\dots,m_i$ the functions ω_{ki} satisfy condition* (1.12), *$\varphi_{ki} \in \widetilde{W}_1^1(0,T)$, $h_{ki} \in C([0,T];L_2(I))$ and for the corresponding functions ψ_{kji} conditions* (1.19) *be satisfied.*

Then there exists a unique set of M functions

$$\begin{aligned} F &= \{F_{ki}(t), i: m_i > 0, k = 1,\dots,m_i\} \\ &\qquad = \Gamma\{\varphi_{ki}, i: m_i > 0, k = 1,\dots,m_i\} \in (L_1(0,T))^M \end{aligned}$$

such that for $f = (f_1, \dots, f_n)^T \equiv HF$, *where for any* $i = 1, \dots, n$ *the function* $f_i(t,x)$ *is presented by formula* (1.4), *where* $h_{0i} \equiv 0$ *(*$f_i \equiv 0$ *if* $m_i = 0$*), the corresponding function*

$$u = S_0 f = (S_0 \circ H)F, \tag{4.1}$$

satisfies all conditions (1.5). *Moreover, the linear operator* $\Gamma : \big(\widetilde{W}_1^1(0,T)\big)^M \to \big(L_1(0,T)\big)^M$ *is bounded and its norm does not decrease in* T.

Proof. First of all note that by virtue of (1.18), (1.19)

$$|\Delta_i(t)| \geq \Delta_0 > 0, \ |\psi_{kji}(t)| \leq \psi_0, \quad t \in [0,T]. \tag{4.2}$$

On the space $\big(L_1(0,T)\big)^M$ introduce M linear operators $\Lambda_{ki} = Q(\omega_{ki}) \circ S_0 \circ H$. Let $\Lambda = \{\Lambda_{ki}\}$. Then since $HF \in \big(L_1(0,T;L_2(I))\big)^n$ by Theorem 2.1 and Lemma 2.1 the operator Λ acts from the space $\big(L_1(0,T)\big)^M$ into the space $\big(\widetilde{W}_1^1(0,T)\big)^M$ and is bounded.

Note that the set of equalities $\varphi_{ki} = \Lambda_{ki} F$, $i : m_i > 0, k = 1, \dots, m_i$, for $F \in \big(L_1(0,T)\big)^M$ obviously means that the set of functions F is the desired one.

Let for i verifying $m_i > 0$

$$\begin{aligned}\widetilde{r}(t;u_i,\omega_{ki}) \equiv (-1)^{l+1} \int_I u_i(t,x) \left(a_{(2l+1)i}\omega_{ki}^{(2l+1)} - a_{2l}\omega_{ki}^{(2l)}\right) dx \\ + \sum_{m=1}^{n} \sum_{j=0}^{l-1} (-1)^{j+1} \int_I u_m(t,x) \left[(a_{(2j+1)im}\omega_{ki}^{(j)})^{(j+1)} - (a_{(2j)im}\omega_{ki}^{(j)})^{(j)}\right] dx,\end{aligned} \tag{4.3}$$

where $u = (u_1, \dots, u_n)^T = (S_0 \circ H)F$. Then from (2.17) it follows that for $q(t;u_i,\omega_{ki}) = \big(\Lambda_{ki}F\big)(t)$

$$q'(t;u_i,\omega_{ki}) = \widetilde{r}(t;u_i,\omega_{ki}) + \sum_{j=1}^{m_i} F_{ji}(t)\psi_{kji}(t), \tag{4.4}$$

where the functions ψ_{kji} are given by formula (1.18). Let

$$y_{ki}(t) \equiv q'(t;u_i,\omega_{ki}) - \widetilde{r}(t;u_i,\omega_{ki}), \quad k = 1, \dots, m_i. \tag{4.5}$$

and $\widetilde{\Delta}_{ki}(t)$ be the determinant of the $m_i \times m_i$-matrix, where in comparison with the matrix $\big(\psi_{kji}(t)\big)$ the k-th column is substituted by the column $\big(y_{1i}(t), \dots, y_{m_i i}(t)\big)^T$. Then (4.4) implies

$$F_{ki}(t) = \frac{\widetilde{\Delta}_{ki}(t)}{\Delta_i(t)}, \quad k = 1, \dots, m_i. \tag{4.6}$$

Let

$$z_{ki}(t) \equiv \varphi'_{ki}(t) - \widetilde{r}(t;u_i,\omega_{ki}), \quad k = 1, \dots, m_i, \tag{4.7}$$

and $\Delta_{ki}(t)$ be the determinant of the $m_i \times m_i$-matrix, where in comparison with $\widetilde{\Delta}_{ki}(t)$ the k-th column $\big(y_{1i}(t), \dots, y_{m_i i}(t)\big)^T$ is substituted by the column $\big(z_{1i}(t), \dots, z_{m_i i}(t)\big)^T$.

Introduce operators $A_{ki} : L_1(0,T) \to L_1(0,T)$ by

$$(A_{ki}F)(t) \equiv \frac{\Delta_{ki}(t)}{\Delta_i(t)} \tag{4.8}$$

and let $AF = \{A_{ki}F\}$, $A : \big(L_1(0,T)\big)^M \to \big(L_1(0,T)\big)^M$.

Note that $\varphi_{ki} = \Lambda_{ki} F$, for all $i : m_i > 0, k = 1, \dots, m_i$ if and only if $AF = F$.

Indeed, if $\varphi_{ki} = \Lambda_{ki} F$, then $\varphi'_{ki}(t) \equiv q'(t; u_i, \omega_{ki})$ for the function $q(t; u_i, \omega_{ki}) \equiv \big(\Lambda_{ki} F\big)(t)$ and equalities (4.5), (4.7) yield $\Delta_{ki}(t) \equiv \widetilde{\Delta}_{ki}(t)$. Hence, $AF = F$.

Vice versa, if $AF = F$, then $\Delta_{ki}(t) \equiv \widetilde{\Delta}_{ki}(t)$ and the condition $\Delta_i(t) \neq 0$ implies $z_{ki}(t) \equiv y_{ki}(t)$ and so $\varphi'_{ki}(t) \equiv q'(t; u_i, \omega_{ki})$. Since $\varphi_{ki}(0) = q(0; u_i, \omega_{ki}) = 0$, we have $q(t; u_i, \omega_{ki}) \equiv \varphi_{ki}(t)$.

Next, we show that the operator A is a contraction under the choice of a special norm in the space $\big(L_1(0,T)\big)^M$.

Let $F_1, F_2 \in \big(L_1(0,T)\big)^M$, $u_m \equiv (S_0 \circ H) F_m$, $m = 1,\ 2$, and let $\Delta^*_{ki}(t)$ be the determinant of the $m_i \times m_i$-matrix, where in comparison with the matrix $\big(\psi_{kji}(t)\big)$ the k-th column is substituted by the column, where on the j-th line stands the element $\widetilde{r}(t; u_{1i}, \omega_{ji}) - \widetilde{r}(t; u_{2i}, \omega_{ji}) = \widetilde{r}(t; u_{1i} - u_{2i}, \omega_{ji})$. Then

$$\big(A_{ki} F_1\big)(t) - \big(A_{ki} F_2\big)(t) = -\frac{\Delta^*_{ki}(t)}{\Delta_i(t)}. \tag{4.9}$$

By (2.8) for $t \in [0,T]$

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{(L_2(I))^n} \le c(T) \sum_{i:m_i>0} \sum_{j=1}^{m_i} \|h_{ji}\|_{C([0,T];L_2(I))} \|F_{1ji} - F_{2ji}\|_{L_1(0,t)}. \tag{4.10}$$

Let $\gamma > 0$, then by virtue of (4.2), (4.3), (4.9) and (4.10)

$$\begin{aligned}
&\|e^{-\gamma t}(AF_1 - AF_2)\|_{(L_1(0,T))^M} \\
&\qquad \le \frac{c\big(\{\|\omega_{ji}\|_{H^{2l+1}(I)}\}, \psi_0\big)}{\Delta_0} \int_0^T e^{-\gamma t} \|u_1(t,\cdot) - u_2(t,\cdot)\|_{(L_2(I))^n}\, dt \\
&\qquad \le c\big(T, \big(\{\|\omega_{ji}\|_{H^{2l+1}(I)}\}, \psi_0, \{\|h_{ji}\|_{C([0,T];L_2(I))}\}\big) \\
&\qquad \times \int_0^T e^{-\gamma t} \int_0^t \sum_{i:m_i>0} \sum_{j=1}^{m_i} |F_{1ji}(\tau) - F_{2ji}(\tau)|\, d\tau dt \\
&= c \int_0^T \sum_{i:m_i>0} \sum_{j=1}^{m_i} |F_{1ji}(\tau) - F_{2ji}(\tau)| \int_\tau^T e^{-\gamma t}\, dt d\tau \le \frac{c}{\gamma} \|e^{-\gamma\tau}(F_1 - F_2)\|_{(L_1(0,T))^M}.
\end{aligned} \tag{4.11}$$

It remains to choose sufficiently large γ.

As a result, for any set of functions $\varphi_{ki} \in \big(\widetilde{W}_1^1(0,T)\big)^M$ there exists a unique set of functions $F \in \big(L_1(0,T)\big)^M$ satisfying $AF = F$, that is $\varphi_{ki} = \Lambda_{ki} F$. This means that the operator Λ is invertible and so the Banach theorem implies that the inverse operator $\Gamma = \Lambda^{-1} : \big(\widetilde{W}_1^1(0,T)\big)^M \to \big(L_1(0,T)\big)^M$ is continuous. In particular,

$$\|\Gamma\{\varphi_{ki}\}\|_{(L_1(0,T))^M} \le c(T) \|\{\varphi_{ki}\}\|_{(\widetilde{W}_1^1(0,T))^M}. \tag{4.12}$$

For an arbitrary $T_1 > T$ extend the functions φ_{ki} by the constant $\varphi_{ki}(T)$ to the interval (T, T_1). Then the analogue of inequality (4.12) on the interval $(0, T_1)$ for such a function evidently holds with $c(T) \le c(T_1)$. This means that the norm of the operator Γ is non-decreasing in T. □

The next result is the solution of the corresponding inverse problem for the full linear problem.

Theorem 4.1. *Let the function f be given by formula* (1.4) *and condition* (1.6) *be satisfied. Let the functions a_i, u_0, $(\mu_0, \dots, \mu_{l-1})$, $(\nu_0, \dots, \nu_l)$, h_0, φ_{ki}, ω_{ki}, h_{ki} satisfy the hypothesis of Theorem* 1.3 *and the functions G_j satisfy the hypothesis of Theorem* 2.1.

Then there exists a unique set of M functions

$$F = \{F_{ki}(t), i : m_i > 0, k = 1, \dots, m_i\} \in (L_1(0,T))^M$$

such that the corresponding unique weak solution $u \in \big(X(Q_T)\big)^n$ of problem (2.3), (1.2), (1.3) *satisfies all conditions* (1.5). *Moreover, the functions F and u are given by formulas*

$$F = \Gamma\Big\{\varphi_{ki} - Q(\omega_{ki})\big(\widetilde{S}W + S_0 h_0 + \sum_{j=0}^{l} \widetilde{S}_j G_j\big)_i\Big\}, \tag{4.13}$$

$$u = \widetilde{S}W + S_0 h_0 + \sum_{j=0}^{l} S_j G_j + (S_0 \circ H)F. \tag{4.14}$$

Proof. Set

$$v \equiv S(u_0, (\mu_0, \dots, \mu_{l-1}), (\nu_0, \dots, \nu_l), h_0, (G_0, \dots, G_l)) = \widetilde{S}W + S_0 h_0 + \sum_{j=0}^{l} \widetilde{S}_j G_j.$$

Lemma 2.1 implies $Q(\omega_{ki})v_i \in W_1^1(0,T)$. Moreover, by virtue of (1.17) $Q(\omega_{ki})v_i\big|_{t=0} = \varphi_{ki}(0)$. Set

$$\widetilde{\varphi}_{ki} \equiv \varphi_{ki} - Q(\omega_{ki})v_i,$$

then $\widetilde{\varphi}_{ki} \in \widetilde{W_1^1}(0,T)$. In turn, Lemma 4.1 implies that the functions $F \equiv \Gamma\{\widetilde{\varphi}_{ki}\}$ and $u \equiv v + (S_0 \circ H)F$ provide the desired result. Uniqueness also follows from Lemma 4.1. $\square$

Now we pass to the nonlinear equation.

Proof of Theorem 1.3. On the space $\big(X(Q_T)\big)^n$ consider a map Θ

$$u = \Theta v \equiv \widetilde{S}W + S_0 h_0 - \sum_{j=0}^{l} \widetilde{S}_j g_j(t, x, v, \dots, \partial_x^{l-1} v) + (S_0 \circ H)F, \tag{4.15}$$

$$F \equiv \Gamma\Big\{\varphi_{ki} - Q(\omega_{ki})\big(\widetilde{S}W + S_0 h_0 - \sum_{j=0}^{l} \widetilde{S}_j g_j(t, x, v, \dots, \partial_x^{l-1} v)\big)_i\Big\}. \tag{4.16}$$

Then estimate (3.5) and Theorem 4.1 applied to $G_j(t,x) \equiv g_j(t, x, v, \dots, \partial_x^{l-1} v)$ ensure that the map Θ exists.

Apply Lemmas 2.3 and 4.1, then the function F from (4.16) is estimated as follows:

$$\begin{aligned} \|F\|_{(L_1(0,T))^M} \le c(T)\Big[&\|u_0\|_{(L_2(I))^n} + \|(\mu_0, \dots, \mu_{l-1})\|_{(\mathcal{B}^{l-1}(0,T))^n} + \|(\nu_0, \dots, \nu_l)\|_{(\mathcal{B}^{l}(0,T))^n} \\ &+ \|h_0\|_{(L_1(0,T;L_2(I)))^n} + \|\{\varphi'_{ki}\}\|_{(L_1(0,T))^M} + \|v\|^{b_1+1}_{(X(Q_T))^n} + \|v\|^{b_2+1}_{(X(Q_T))^n}\Big]; \end{aligned} \tag{4.17}$$

therefore, since also

$$\|HF\|_{(L_1(0,T;L_2(I)))^n} \le \max_{i:m_i>0, k=1,\dots,m_i} \big(\|h_{ki}\|_{C([0,T];L_2(I))}\big) \|F\|_{(L_1(0,T))^M},$$

Theorem 2.1 provides for the map Θ estimate (3.6).

Next, for any functions $v_1, v_2 \in \big(X(Q_T)\big)^n$ since

$$\Theta v_1 - \Theta v_2 = -\sum_{j=0}^{l} \widetilde{S}_j \left[g_j(t,x,v_1,\dots,\partial_x^{l-1}v_1) - g_j(t,x,v_2,\dots,\partial_x^{l-1}v_2)\right]$$

$$+ (S_0 \circ H \circ \Gamma)\Big\{Q(\omega_{ki})\Big(\sum_{j=0}^{l} \widetilde{S}_j \left[g_j(t,x,v_1,\dots,\partial_x^{l-1}v_1) - g_j(t,x,v_2,\dots,\partial_x^{l-1}v_2)\right]\Big)_i\Big\}, \quad (4.18)$$

using (3.8) we derive estimate (3.9).

Now choose $r > 0$ and $\delta > 0$ as in (3.10), (3.11). Then it follows from (3.6) and (3.9) that on the ball $\overline{X}_{rn}(Q_T)$ the map Θ is a contraction. Its unique fixed point $u \in \big(X(Q_T)\big)^n$ is the desired solution. Moreover, Theorem 4.1 implies that the function F in (4.16) (for $v \equiv u$) is determined in a unique way.

Continuous dependence is obtained similarly to (3.6), (3.9). □

Proof of Theorem 1.4. In general, the proof repeats the previous argument. The desired solution is constructed as a fixed point of the map Θ from (4.15), (4.16). In comparison with (3.6), (3.9) here (also with the use of (4.18)) we obtain estimates (3.31) and (3.32), where σ is defined in (3.29).

The end of the proof is the same as in Theorem 1.2 (with the corresponding supplements as in Theorem 1.3). □

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