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ORDER-SHARP ESTIMATES FOR DECREASING REARRANGEMENTS OF CONVOLUTIONS

E.G. Bakhtigareeva, M.L. Goldman

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Key words: convolutions, radial symmetrical functions, order-sharp estimates, decreasing rearrangements, symmetrical rearrangements, mean values of rearrangements.

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Abstract. In this paper, we study estimates for convolutions on some classes of measurable, positive and radial symmetrical functions. On this base we prove then order-sharp estimates for decreasing and symmetrical rearrangements of convolutions and for weighted mean values of rearrangements. These estimates give, in particular, a reversal of the well-known inequalities for convolutions proved by R. O'Neil.

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1 Introduction

In this paper, we consider estimates for decreasing rearrangements of convolutions. The books by S.G. Krein, Yu.I. Petunin and E.M. Semenov [12], C. Bennett and R. Sharpley [3] contain main definitions and basic facts related to this topic. The properties of the classical Bessel and Riesz potentials are described in the books by V.G. Maz'ya [13], S.M. Nikol'skii [14], E.M. Stein [17].

In Section 2 of the paper, we obtain two–sided estimates for convolutions for some classes of radial symmetrical functions. The case of functions that are positive on $\mathbb{R}^n$ is considered here. In Section 3 we consider the case, where one of the convolved function has support contained in the finite ball $B_R = \{x \in \mathbb{R}^n : |x| \le R\}$ for some $R \in (0, \infty)$. Such consideration will be useful for application of these results to generalized Bessel potentials. In that case the kernel of the convolution is splitted into two parts, and one part is supported in B_R.

We apply these estimates in Section 4 to obtaining two–sided estimates for symmetrical and decreasing rearrangements of convolutions. These estimates give, in particular, a reversal of the well-known inequality for convolutions proved by R. O'Neil [16]. They develop and refine the estimates obtained in our papers [5]–[6], [8]–[10]. We will use these results to justify pointwise and integral coverings for cones of decreasing rearrangements for generalized Bessel-Riesz potentials. As a result, exact descriptions of equivalent cones for cones of decreasing rearrangements of potentials will be obtained. They develop the results of our works [9], [10]. Note that E. Nursultanov and S. Tikhonov [15] obtained some further developments of O'Neil's results. For researches related to the topic, see [2, 4, 11].

In Section 5 we prove a lemma which may be useful in many considerations related to the subject of this paper. The proof of this lemma is related to the proofs of Theorems in Sections 2–4.

2 Two–sided estimates for convolutions. The case $R = \infty$

Let $\alpha \in (1, \infty)$, $R \in (0, \infty]$.

Definition 1. As $J_\alpha(\infty)$ we denote the class of all measurable functions $F : (0, \infty) \to (0, \infty)$, such that for all $\xi \in (0, \infty)$

$$\tau \in [\xi, 2\xi] \quad \text{implies} \quad \alpha^{-1} F(\xi) \leq F(\tau) \leq \alpha F(\xi). \tag{2.1}$$

Remark 1. Let $\alpha \in (1, \infty)$, $F \in J_\alpha(\infty)$, $m \in \mathbb{N}$, $\xi \in (0, \infty)$. Then, the following estimate holds

$$\eta \in [\xi, 2^m \xi] \Rightarrow \alpha^{-m} F(\xi) \leq F(\eta) \leq \alpha^m F(\xi). \tag{2.2}$$

Proof. Let us use the method of induction.

For $m = 1$ estimate (2.2) for $F \in J_\alpha(\infty)$ follows from the definition.

Assumption of induction: assume that estimate (2.2) holds for all numbers from 1 to m. Step of induction: let us prove that then it is true for the number $m + 1$.

For $\eta \in [\xi, 2^{m+1}\xi] = [\xi, 2^m \xi] \bigcup [2^m \xi, 2^{m+1}\xi]$ we have on $[\xi, 2^m \xi]$ estimate (2.2), and for $\eta \in [2^m \xi, 2^{m+1}\xi]$ the estimate holds for $F \in J_\alpha(\infty)$

$$\alpha^{-1} F(2^m \xi) \leq F(\eta) \leq \alpha F(2^m \xi).$$

For $\eta = 2^m \xi$, according to (2.2), $\alpha^{-m} F(\xi) \leq F(2^m \xi) \leq \alpha^m F(\xi)$, so that we obtain

$$\alpha^{-(m+1)} F(\xi) \leq F(\eta) \leq \alpha^{(m+1)} F(\xi), \quad \eta \in [2^m \xi, 2^{m+1}\xi].$$

Recall that $\alpha > 1$, so that (2.2) implies, in particular, that

$$\alpha^{-(m+1)} F(\xi) \leq F(\eta) \leq \alpha^{(m+1)} F(\xi), \quad \eta \in [\xi, 2^m \xi].$$

These estimates give the desired inequality:

$$\alpha^{-(m+1)} F(\xi) \leq F(\eta) \leq \alpha^{(m+1)} F(\xi), \quad \eta \in [\xi, 2^{m+1}\xi].$$

□

Definition 2. As $J_\alpha(R)$ with $R \in (0, \infty)$ we denote the class of all measurable functions $F : (0, \infty) \to [0, \infty)$, such that $F(\xi) > 0$, $\xi \in (0, R]$, $F(\xi) = 0$ for $\xi > R$ and

$$\xi \in (0, R),\, \tau \in [\xi, \min\{2\xi, R\}] \Rightarrow \alpha^{-1} F(\xi) \leq F(\tau) \leq \alpha F(\xi).$$

For a function $F \in J_\alpha(R)$, $R \in (0, \infty)$ we have an analogue of (2.2):

$$\xi \in (0, R), \tau \in [\xi, \min\{2^m \xi, R\}] \Rightarrow \alpha^{-m} F(\xi) \leq F(\tau) \leq \alpha^m F(\xi). \tag{2.3}$$

The following remark shows the link of two–sided estimates for the left and the right ends of the segment $[\xi, 2^m \xi]$.

Remark 2. 1. Let $\alpha \in (1, \infty)$. From (2.2) it follows easily that for $\beta = \alpha^2$

$$\tau \in [\xi, 2^m \xi] \Rightarrow \beta^{-m} F(2^m \xi) \leq F(\tau) \leq \beta^m F(2^m \xi). \tag{2.4}$$

2. Let $\beta \in (1, \infty)$. From (2.2) it follows easily that for $\alpha = \beta^2$

$$\tau \in [\xi, 2^m \xi] \Rightarrow \alpha^{-m} F(\xi) \leq F(\tau) \leq \alpha^m F(\xi). \tag{2.5}$$

The next remark shows the link of two–sided estimates for any two points of the segment $[\xi, 2^m\xi]$.

Remark 3. Let $\alpha \in (1,\infty)$, $m \in \mathbb{N}$, $F \in J_\alpha(\infty)$, so that estimate (2.2) holds. Then, it follows easily that for any two points $t, \tau \in [\xi, 2^m\xi]$ the following estimate holds:

$$\alpha^{-2m} F(t) \le F(\tau) \le \alpha^{2m} F(t).$$

Remark 4. Let $\alpha \in (1,\infty)$, $m \in \mathbb{N}$, $R \in (0,\infty)$, $F \in J_\alpha(R)$, so that estimate (2.3) holds. Then, it follows easily that for any $t, \tau \in [\xi, \min\{2^m\xi, R\}]$ the following estimate holds:

$$\alpha^{-2m} F(t) \le F(\tau) \le \alpha^{2m} F(t).$$

Theorem 2.1. *Let $\alpha, \beta \in (1,\infty)$; $F \in J_\alpha(\infty)$, $G \in J_\beta(\infty)$, $x \in \dot{\mathbb{R}}^n = \{x \in \mathbb{R}^n, x \neq 0\}$,*

$$f(x) = F(|x|),\ g(x) = G(|x|); \tag{2.6}$$

$$u(x) = (f * g)(x) = (g * f)(x) = \int\limits_{\mathbb{R}^n} f(y) g(x-y)\,dy; \tag{2.7}$$

$$\tilde{u}(x) = \int\limits_0^\infty \left[F(\tau)G(|x|+\tau) + F(|x|+\tau)G(\tau)\right]\tau^{n-1}d\tau. \tag{2.8}$$

Then, there exist constants $c_i = c_i(\alpha,\beta,n)$, $i = 1,2$, such that $0 < c_1 \le c_2 < \infty$ and

$$c_1 u(x) \le \tilde{u}(x) \le c_2 u(x), \quad x \in \dot{\mathbb{R}}^n. \tag{2.9}$$

Proof. 1. Let $S^{n-1} = \{\omega \in \mathbb{R}^n : |\omega| = 1\}$ be the unit sphere in $\mathbb{R}^n$, $C_n = \int\limits_{S^{n-1}} d\omega = 2\pi^{n/2}\Gamma(n/2)^{-1}$ be the integral over all angles in S^{n-1}.

For $x \in \dot{\mathbb{R}}^n$ we introduce the spherical system of coordinates with the center at the point 0 and the polar axis L_0 such that $x \in L_0$. In the spherical coordinates for $y \in \dot{\mathbb{R}}^n$ we have

$$y = (\tau,\omega),\ \tau = |y| > 0,\ \omega \in S^{n-1};$$

and we obtain that

$$\begin{aligned}\int\limits_{\dot{\mathbb{R}}^n} F(|y|)G(|x|+|y|)dy &= \int\limits_0^\infty F(\tau)G(|x|+\tau)\left(\int\limits_{S^{n-1}} d\omega\right)\tau^{n-1}d\tau \\ &= C_n \int\limits_0^\infty F(\tau)G(|x|+\tau)\tau^{n-1}d\tau.\end{aligned} \tag{2.10}$$

Let $\Omega = B(x, |x|/2)$ be the ball with the center x and the radius $r = |x|/2$. It follows from (2.6) and (2.7) that

$$u(x) = \int\limits_{\dot{\mathbb{R}}^n} F(|y|)G(|x-y|)dy = I_1 + I_2, \quad x \in \dot{\mathbb{R}}^n, \tag{2.11}$$

where

$$I_1 = \int\limits_{\dot{\mathbb{R}}^n \setminus \Omega} F(|y|)G(|x-y|)dy, \quad I_2 = \int\limits_{\Omega} F(|y|)G(|x-y|)dy. \tag{2.12}$$

For $y \in \dot{\mathbb{R}}^n \setminus \Omega$ we have $|x| \le 2|x-y|$, so

$$|y| = |y-x+x| \le |y-x| + |x| \le 3|y-x|.$$

Then,

$$|x-y| \le |x| + |y| \le 5|x-y| < 2^3|x-y|, \quad y \in \dot{\mathbb{R}}^n \setminus \Omega,$$

and for $G \in J_\beta(\infty)$ it follows from (2.2) with $m=3$, $\alpha=\beta$ that

$$\beta^{-3} \le G(|x|+|y|)/G(|x-y|) \le \beta^3, \quad y \in \dot{\mathbb{R}}^n \setminus \Omega.$$

It means that

$$\beta^{-3} I_1 \le \int\limits_{\dot{\mathbb{R}}^n \setminus \Omega} F(|y|)G(|x|+|y|)dy \le \beta^3 I_1. \tag{2.13}$$

The left–hand–side inequality in (2.13) shows that

$$I_1 \le \beta^3 \int\limits_{\dot{\mathbb{R}}^n} F(|y|)G(|x|+|y|)dy.$$

Therefore, analogously to (2.10) we obtain in the spherical coordinates

$$I_1 \le \beta^3 C_n \int\limits_0^\infty F(\tau)G(|x|+\tau)\tau^{n-1}d\tau. \tag{2.14}$$

Moreover, let K_Ω be a minimal cone with the cone apex at the origin, such that $\Omega \subset K_\Omega$. Denote

$$\Sigma_\Omega = \left\{\omega \in S^{n-1} : \omega \notin K_\Omega\right\},\ \sigma_n = \int\limits_{\Sigma_\Omega} d\omega;$$

$$\Delta_\Omega = \left\{\omega \in S^{n-1} : \omega \in K_\Omega\right\},\ \delta_n = \int\limits_{\Delta_\Omega} d\omega.$$

Our construction is such that the sets K_Ω, Σ_Ω, Δ_Ω are the same for all $x \in L_0$, they depend only on dimension n. Moreover, $\Sigma_\Omega \cap \Delta_\Omega = \{\emptyset\}$, $\Sigma_\Omega \cup \Delta_\Omega = S^{n-1}$. Then, $0 < \sigma_n, \delta_n$, $\sigma_n + \delta_n = \int\limits_{S^{n-1}} d\omega = C_n$, so that, in particular, $0 < \sigma_n < C_n$.

Note that $\Omega \subset K_\Omega \Rightarrow \mathbb{R}^n \setminus K_\Omega \subset \mathbb{R}^n \setminus \Omega$. Thus, the right–hand–side estimate in (2.13) implies

$$I_1 \ge \beta^{-3} \int\limits_{\mathbb{R}^n \setminus \Omega} F(|y|)G(|x|+|y|)dy \ge \beta^{-3} \int\limits_{\mathbb{R}^n \setminus K_\Omega} F(|y|)G(|x|+|y|)dy.$$

Like in (2.10), we obtain in the spherical coordinates that

$$\int\limits_{\mathbb{R}^n\setminus K_\Omega} F(|y|)G(|x|+|y|)dy = \int\limits_0^\infty F(\tau)G(|x|+\tau)\left(\int\limits_{\Sigma_\Omega} d\omega\right)\tau^{n-1}d\tau$$
$$= \sigma_n \int\limits_0^\infty F(\tau)G(|x|+\tau)\tau^{n-1}d\tau.$$

As a result,

$$I_1 \geq \beta^{-3}\sigma_n \int\limits_0^\infty F(\tau)G(|x|+\tau)\tau^{n-1}d\tau. \tag{2.15}$$

Estimates (2.14) and (2.15) give the two–sided inequality:

$$\beta^{-3}C_n^{-1}I_1 \leq \int\limits_0^\infty F(\tau)G(|x|+\tau)\tau^{n-1}d\tau \leq \beta^3\sigma_n^{-1}I_1. \tag{2.16}$$

2. We move on to the estimates for $I_2 = \int\limits_\Omega F(|y|)G(|x-y|)dy$. For $y \in \Omega$ we have

$$y \in \Omega \Rightarrow \begin{cases} |y| \leq |x| + |y-x|; \\ 3|y| \geq \frac{3}{2}|x| = |x| + \frac{1}{2}|x| \geq |x| + |y-x|. \end{cases}$$

Thus, $y \in \Omega \Rightarrow 2^{-2}(|x|+|y-x|) \leq |y| \leq |x| + |y-x|$.

For $F \in J_\alpha(\infty)$ it follows from here and from Remark 2 (see (2.4)) that

$$\alpha^{-2}F(|x|+|y-x|) \leq F(|y|) \leq \alpha^2 F(|x|+|y-x|), \quad y \in \Omega.$$

Therefore,

$$\alpha^{-2}I_2 \leq \int\limits_\Omega F(|x|+|y-x|)G(|y-x|)dy \leq \alpha^2 I_2.$$

We introduce the spherical system of coordinates with the center at the point x and the spherical radius $\lambda = |y-x|$. Then,

$$y \in \Omega,\, y \neq x \Leftrightarrow y - x = (\lambda, \omega),\, 0 < \lambda = |y-x| \leq |x|/2, \omega \in S^{n-1},$$

and we obtain the following equality with $C_n = \int\limits_{S^{n-1}} d\omega = 2\pi^{n/2}\Gamma(n/2)^{-1}$:

$$\int\limits_\Omega F(|x|+|y-x|)G(|y-x|)dy = C_n \int\limits_0^{|x|/2} F(|x|+\lambda)G(\lambda)\lambda^{n-1}d\lambda,$$

so

$$\alpha^{-2}C_n^{-1}I_2 \leq \int\limits_0^{|x|/2} F(|x|+\lambda)G(\lambda)\lambda^{n-1}d\lambda \leq \alpha^2 C_n^{-1}I_2. \tag{2.17}$$

3. For the further consideration it is convenient to use the following notation: let $A(x), B(x), C(x), D(x), E(x) \geq 0,\ x \in \dot{\mathbb{R}}^n$. We write $D(x) \cong E(x)$ if there exist constants $c_i = c_i(\alpha, \beta, n), i = 1, 2$, such that $0 < c_1 \leq c_2 < \infty$ and

$$c_1 D(x) \leq E(x) \leq c_2 D(x), x \in \dot{\mathbb{R}}^n. \tag{2.18}$$

Let us note that if

$$0 \leq C(x) \leq c_3 A(x), x \in \dot{\mathbb{R}}^n, \tag{2.19}$$

with $0 \leq c_3 = c_3(\alpha, \beta, n) < \infty$, then

$$A(x) + B(x) \cong A(x) + B(x) + C(x), x \in \dot{\mathbb{R}}^n. \tag{2.20}$$

Indeed, according to (2.19)

$$A(x) + B(x) \leq A(x) + B(x) + C(x) \leq (1 + c_3)(A(x) + B(x)), x \in \dot{\mathbb{R}}^n.$$

Let here (see estimates (2.16), (2.17))

$$A(x) := I_1 \cong \int_0^\infty F(\tau) G(|x| + \tau) \tau^{n-1} d\tau,$$

$$B(x) := I_2 \cong \int_0^{|x|/2} F(|x| + \tau) G(\tau) \tau^{n-1} d\tau,$$

$$C(x) := \int_{|x|/2}^\infty F(|x| + \tau) G(\tau) \tau^{n-1} d\tau.$$

For $\tau \geq |x|/2$ we have $|x| \leq 2\tau$, so that

$$\tau \leq |x| + \tau \leq 3\tau \Rightarrow |x| + \tau \in [\tau, 2^2\tau].$$

Therefore, for $F \in J_\alpha(\infty)$, $G \in J_\beta(\infty)$ we have estimates like in (2.2):

$$F(|x| + \tau) \leq \alpha^2 F(\tau), \quad G(\tau) \leq \beta^2 G(|x| + \tau),$$

so that

$$0 \leq C(x) \leq \alpha^2 \beta^2 \int_{|x|/2}^\infty F(\tau) G(|x| + \tau) \tau^{n-1} d\tau \leq \alpha^2 \beta^2 \int_0^\infty F(\tau) G(|x| + \tau) \tau^{n-1} d\tau,$$

that is

$$0 \leq C(x) \leq c_3 A(x), x \in \dot{\mathbb{R}}^n. \tag{2.21}$$

Let us consider

$$\tilde{u}(x) = \int_0^\infty \left[F(\tau) G(|x| + \tau) + F(|x| + \tau) G(\tau) \right] \tau^{n-1} d\tau \cong A(x) + B(x) + C(x).$$

Estimates (2.19) -(2.21) show that here

$$A(x) + B(x) + C(x) \cong A(x) + B(x).$$

Therefore,

$$\tilde{u}(x) \cong A(x) + B(x) = I_1 + I_2 = u(x).$$

This completes the proof of estimate (2.9).

□

Corollary 2.1. *Under the assumptions of Theorem* 2.1 *the following two–sided estimate holds:*

$$u(x) \cong F(|x|) \int\limits_0^{|x|} G(\tau)\tau^{n-1}d\tau + G(|x|) \int\limits_0^{|x|} F(\tau)\tau^{n-1}d\tau + \int\limits_{|x|}^{\infty} F(\tau)G(\tau)\tau^{n-1}d\tau \tag{2.22}$$

with positive constants depending only on α, β, n *(as in* (2.18)*).*

Proof. Indeed, for functions $F \in J_\alpha(\infty)$, $G \in J_\beta(\infty)$ we have

$$F(|x| + \tau) \cong F(|x|), \quad G(|x| + \tau) \cong G(|x|), \quad \tau \in (0, |x|];$$

$$F(|x| + \tau) \cong F(\tau), \quad G(|x| + \tau) \cong G(\tau), \quad \tau > |x|;$$

and estimate (2.9) implies (2.22). □

Remark 5. Under notation (2.6)- (2.8) let functions F and G be nonnegative and decreasing. Then,

$$u(x) \geq 2^{-1}C_n\tilde{u}(x), x \in \dot{\mathbb{R}}^n, \quad C_n = 2\pi^{n/2}\Gamma(n/2)^{-1}. \tag{2.23}$$

Proof. For decreasing functions F and G we have:

$$|y - x| \leq |x| + |y| \Rightarrow F(|y - x|) \geq F(|x| + |y|), \; G(|y - x|) \geq G(|x| + |y|).$$

Then,

$$u(x) = \int\limits_{\mathbb{R}^n} F(|y|)G(|y - x|)dy \geq \int\limits_{\mathbb{R}^n} F(|y|)G(|x| + |y|)dy.$$

Thus, in the spherical coordinates we have

$$u(x) \geq C_n \int\limits_0^{\infty} F(\tau)G(|x| + \tau)\tau^{n-1}d\tau. \tag{2.24}$$

But $u = f * g = g * f$, so

$$u(x) = \int\limits_{\mathbb{R}^n} F(|x - y|)G(|y|)dy \geq \int\limits_{\mathbb{R}^n} F(|x| + |y)G(|y|)dy.$$

In the spherical coordinates we have

$$u(x) \geq C_n \int\limits_0^{\infty} F(|x| + \tau)G(\tau)\tau^{n-1}d\tau. \tag{2.25}$$

We add estimates (2.24), (2.25) and obtain that

$$2u(x) \geq C_n \int\limits_0^{\infty} \left[F(\tau)G(|x| + \tau) + F(|x| + \tau)G(\tau)\right] \tau^{n-1}d\tau = C_n\tilde{u}(x). \tag{2.26}$$

This implies estimate (2.23). □

Corollary 2.2. *Under the assumptions of Remark* 5 *the following estimate holds for the symmetrical rearrangement of convolution*

$$u^{\#}(\rho) \geq 2^{-1}C_n \int\limits_0^\infty [F(\tau)G(\rho+\tau) + F(\rho+\tau)G(\tau)]\, \tau^{n-1} d\tau,\ \rho \in (0,\infty). \tag{2.27}$$

Indeed, estimate (2.23) implies the related estimate for symmetrical rearrangements:

$$u^{\#}(\rho) \geq 2^{-1}C_n \tilde{u}^{\#}(\rho),\ \rho \in (0,\infty).$$

But, under the assumptions of Remark 5, function $\tilde{u}$ (2.8) is nonnegative, radial symmetrical and decreasing as the function of $\rho = |x|$. Therefore, its symmetrical rearrangement $u^{\#}$ coincides with the integral in the right–hand side of (2.27).

3 Two–sided estimates for convolutions. The case $R < \infty$

First, we formulate a useful technical result.

Lemma 3.1. *1. Let $G \in J_\beta(\infty)$, $\xi \in (0,\infty)$. Then,*

$$\int\limits_0^{\xi/2} G(\lambda)\lambda^{n-1} d\lambda \leq \int\limits_0^{\xi} G(\lambda)\lambda^{n-1} d\lambda \leq (1+2^n\beta^3) \int\limits_0^{\xi/2} G(\lambda)\lambda^{n-1} d\lambda. \tag{3.1}$$

2. Let $F \in J_\alpha(R)$, $\xi \in (0,R]$. Then,

$$\int\limits_0^{\xi/2} F(\lambda)\lambda^{n-1} d\lambda \leq \int\limits_0^{\xi} F(\lambda)\lambda^{n-1} d\lambda \leq (1+2^n\alpha^3) \int\limits_0^{\xi/2} F(\lambda)\lambda^{n-1} d\lambda. \tag{3.2}$$

Proof. We will prove (3.1) (for (3.2) the proof is analogous). For $G \in J_\beta(\infty)$ we have

$$G(\lambda) \geq 0 \Rightarrow \int\limits_0^{\xi/2} G(\lambda)\lambda^{n-1} d\lambda \leq \int\limits_0^{\xi} G(\lambda)\lambda^{n-1} d\lambda. \tag{3.3}$$

Thus, the left part in estimate (3.1) holds. Let us prove the right part in estimate (3.1). Note that for $G \in J_\beta(\infty)$, $\xi \in (0,\infty)$ we have inequalities

$$\beta^{-1}G(\xi/2) \leq G(\lambda) \leq \beta G(\xi/2),\ \lambda \in [\xi/2,\xi].$$

Therefore,

$$\beta^{-1}G(\xi/2) \int\limits_{\xi/2}^{\xi} \lambda^{n-1} d\lambda \leq \int\limits_{\xi/2}^{\xi} G(\lambda)\lambda^{n-1} d\lambda \leq \beta G(\xi/2) \int\limits_{\xi/2}^{\xi} \lambda^{n-1} d\lambda,$$

and we obtain by calculation of integrals

$$\beta^{-1}n^{-1}(1-2^{-n})\xi^n G(\xi/2) \leq \int\limits_{\xi/2}^{\xi} G(\lambda)\lambda^{n-1} d\lambda \leq \beta n^{-1}(1-2^{-n})\xi^n G(\xi/2). \tag{3.4}$$

Moreover, by application of Remark 3, we have for $G \in J_\beta(\infty)$

$$\beta^{-2}G(\xi/2) \leq G(\lambda) \leq \beta G(\xi/2), \quad \lambda \in [\xi/4, \xi/2]\,,$$

and, therefore,

$$\int_{\xi/4}^{\xi/2} G(\lambda)\lambda^{n-1}d\lambda \geq \beta^{-2}\left(\int_{\xi/4}^{\xi/2} \lambda^{n-1}d\lambda\right) G(\xi/2) = \beta^{-2}n^{-1}2^{-n}(1-2^{-n})\xi^n G(\xi/2).$$

Thus,

$$\int_{0}^{\xi/2} G(\lambda)\lambda^{n-1}d\lambda \geq \int_{\xi/4}^{\xi/2} G(\lambda)\lambda^{n-1}d\lambda \geq \beta^{-2}n^{-1}2^{-n}(1-2^{-n})\xi^n G(\xi/2).$$

Together with the right estimate in (3.4) this shows that

$$\int_{\xi/2}^{\xi} G(\lambda)\lambda^{n-1}d\lambda \leq 2^n\beta^3 \int_{0}^{\xi/2} G(\lambda)\lambda^{n-1}d\lambda,$$

and we obtain

$$\int_{0}^{\xi} G(\lambda)\lambda^{n-1}d\lambda = \int_{0}^{\xi/2} G(\lambda)\lambda^{n-1}d\lambda + \int_{\xi/2}^{\xi} G(\lambda)\lambda^{n-1}d\lambda \leq (1+2^n\beta^3) \int_{0}^{\xi/2} G(\lambda)\lambda^{n-1}d\lambda.$$

Thus, we arrive at the right estimate in (3.1). □

Corollary 3.1. *Let $0 < \rho < 1$, $m \in \mathbb{N}$ be such that $2^{-m} \leq \rho \leq 2^{-m+1}$. Then, the following estimates hold.*

1. *For $1 < \beta < \infty$, $G \in J_\beta(\infty)$, $\xi \in (0,\infty)$ we have*

$$\int_{0}^{\rho\xi} G(\lambda)\lambda^{n-1}d\lambda \leq \int_{0}^{\xi} G(\lambda)\lambda^{n-1}d\lambda \leq \left(1+2^n\beta^3\right)^m \int_{0}^{\rho\xi} G(\lambda)\lambda^{n-1}d\lambda. \tag{3.5}$$

2. *For $1 < \alpha < \infty$, $0 < R < \infty$, $F \in J_\alpha(R)$, $\xi \in (0,R]$ we have*

$$\int_{0}^{\rho\xi} F(\lambda)\lambda^{n-1}d\lambda \leq \int_{0}^{\xi} F(\lambda)\lambda^{n-1}d\lambda \leq \left(1+2^n\alpha^3\right)^m \int_{0}^{\rho\xi} F(\lambda)\lambda^{n-1}d\lambda. \tag{3.6}$$

Proof. We will prove (3.5) (for (3.6) the proof is analogous). The left estimate in (3.5) is evident. By induction we can easily prove that for $m \in \mathbb{N}$ the following estimate holds

$$\int_{0}^{\xi} G(\lambda)\lambda^{n-1}d\lambda \leq \left(1+2^n\beta^3\right)^m \int_{0}^{2^{-m}\xi} G(\lambda)\lambda^{n-1}d\lambda. \tag{3.7}$$

Indeed, for $m = 1$ it coincides with (3.1). Assumption of induction is that it holds for all numbers from 1 to m. Then, for the number $m + 1$ we have by application of (3.1) with $2^{-m}\xi$ instead of $\xi \in (0, \infty)$:

$$\int_0^{2^{-m}\xi} G(\lambda)\lambda^{n-1}d\lambda \le \left(1 + 2^n\beta^3\right) \int_0^{2^{-(m+1)}\xi} G(\lambda)\lambda^{n-1}d\lambda.$$

Therefore, application of (3.7) shows that

$$\int_0^{\xi} G(\lambda)\lambda^{n-1}d\lambda \le \left(1 + 2^n\beta^3\right)^{m+1} \int_0^{2^{-(m+1)}\xi} G(\lambda)\lambda^{n-1}d\lambda.$$

Thus, (3.7) holds for any $m \in \mathbb{N}$. Therefore, for $2^{-m} \le \rho \le 2^{-m+1}$ we have

$$\int_0^{\xi} G(\lambda)\lambda^{n-1}d\lambda \le \left(1 + 2^n\beta^3\right)^m \int_0^{2^{-m}\xi} G(\lambda)\lambda^{n-1}d\lambda \le \left(1 + 2^n\beta^3\right)^m \int_0^{\rho\xi} G(\lambda)\lambda^{n-1}d\lambda.$$

This is the right estimate in (3.5). □

Theorem 3.1. *Let*

$$\alpha, \beta \in (1, \infty),\ R \in (0, \infty),\ F \in J_\alpha(R),\ G \in J_\beta(\infty); \tag{3.8}$$

$$f(x) = F(|x|), g(x) = G(|x|), x \in \dot{\mathbb{R}}^n; \tag{3.9}$$

$$u(x) = (f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dy = \int_{\mathbb{R}^n} f(y)g(x - y)dy,\ x \in \dot{\mathbb{R}}^n. \tag{3.10}$$

For $x \in \dot{\mathbb{R}}^n$ we define $\tilde{u}(x)$ by the following formulas:

1. *If $|x| < 2R/3$, then*

$$\tilde{u}(x) = \int_0^{R-|x|} F(|x| + \lambda)G(\lambda)\lambda^{n-1}d\lambda + \int_0^{R} F(\lambda)G(|x| + \lambda)\lambda^{n-1}d\lambda. \tag{3.11}$$

2. *If $2R/3 \le |x| \le 4R/3$, then*

$$\tilde{u}(x) = F(R)\int_0^{R} G(\lambda)\lambda^{n-1}d\lambda + G(R)\int_0^{R} F(\lambda)\lambda^{n-1}d\lambda. \tag{3.12}$$

3. *If $4R/3 < |x| < \infty$, then*

$$\tilde{u}(x) = G(|x|)\int_0^{R} F(\lambda)\lambda^{n-1}d\lambda. \tag{3.13}$$

Then, there exist constants $c_i = c_i(\alpha, \beta, n)$, $i = 1, 2$, $0 < c_1 \le c_2 < \infty$, such that

$$c_1 u(x) \le \tilde{u}(x) \le c_2 u(x),\ x \in \dot{\mathbb{R}}^n. \tag{3.14}$$

Proof. 1. We consider the case $|x| < 2R/3$. In this case the proof is similar to the proof of Theorem 2.1. Let $\Omega = B(x, |x|/2)$ be the ball with the center $x \in \dot{\mathbb{R}}^n$ and the radius $r = |x|/2$. Note that for $0 < |x| < 2R/3$ we have $\Omega \subset B_R = B(0, R)$. Let $\dot{B}_R = B(0, R) \setminus \{0\}$.

We will take into account that $F(|y|) = 0$ for $|y| > R$ and obtain

$$u(x) = \int\limits_{B_R} F(|y|)G(|x-y|)dy = u_1 + u_2, \quad x \in \dot{\mathbb{R}}^n, \tag{3.15}$$

where

$$u_1 = \int\limits_{\dot{B}_R\setminus\Omega} F(|y|)G(|x-y|)dy, \quad u_2 = \int\limits_{\Omega} F(|y|)G(|x-y|)dy. \tag{3.16}$$

For $y \in \dot{B}_R \setminus \Omega$ we have $|x| \le 2|x-y|$, so

$$|y| = |y - x + x| \le |y - x| + |x| \le 3|y - x|.$$

Then,

$$|x-y| \le |x| + |y| \le 5|x-y|, \quad y \in \dot{B}_R \setminus \Omega,$$

that is

$$|x-y|/\left(|x|+|y|\right) \in \left[5^{-1}, 1\right] \subset \left[2^{-3}, 1\right],$$

and for $G \in J_\beta(\infty)$, $y \in \dot{B}_R \setminus \Omega$ we obtain from (2.2) (with $m = 3$, $\xi = 2^{-3}$) that

$$|x-y|/\left(|x|+|y|\right) \in \left[\xi, 2^3\xi\right] \Rightarrow \beta^{-3} \le G(|x-y|)/G(|x|+|y|) \le \beta^3. \tag{3.17}$$

It follows from (3.17) that

$$\beta^{-3}u_1 \le \int\limits_{\dot{B}_R\setminus\Omega} F(|y|)G(|x|+|y|)dy \le \beta^3 u_1. \tag{3.18}$$

The left–hand–side inequality in (3.18) shows that

$$u_1 \le \beta^3 \int\limits_{B_R} F(|y|)G(|x|+|y|)dy.$$

For $x \in \dot{\mathbb{R}}^n$ we introduce the spherical system of coordinates with the center at the point 0 and the polar axis L_0 such that $x \in L_0$. In the spherical coordinates for $y \in \dot{B}_R$ we have

$$y = (\tau, \omega), \; 0 < \tau = |y| \le R, \; \omega \in S^{n-1}.$$

Analogously to (2.10), we obtain that

$$u_1 \le \beta^3 C_n \int\limits_0^R F(\tau)G(|x|+\tau)\tau^{n-1}d\tau. \tag{3.19}$$

Here $C_n = 2\pi^{n/2}\Gamma(n/2)^{-1}$. As in Theorem 2.1, we introduce the minimal cone K_Ω with the cone apex at the origin, such that $\Omega \subset K_\Omega$, and define

$$\Sigma_\Omega = \left\{\omega \in S^{n-1} : \omega \notin K_\Omega\right\}, \; \sigma_n = \int\limits_{\Sigma_\Omega} d\omega; \;\; \Delta_\Omega = \left\{\omega \in S^{n-1} : \omega \in K_\Omega\right\}, \; \delta_n = \int\limits_{\Delta_\Omega} d\omega.$$

We have $y = |y|\omega \in B_R \setminus K_\Omega$ for $\omega \in \Sigma_\Omega$ and for any $0 < |y| \le R$. Note that our construction is such that the cone K_Ω and σ_n, δ_n do not depend on $x \in L_0$ with $0 < |x| < 2R/3$,they depend only on the dimension n. Moreover, $\Sigma_\Omega \cap \Delta_\Omega = \{\emptyset\}$, $\Sigma_\Omega \cup \Delta_\Omega = S^{n-1}$. Then, $0 < \sigma_n, \delta_n$; $\sigma_n + \delta_n = \int\limits_{S^{n-1}} d\omega = C_n$, so that, in particular, $0 < \delta_n < C_n$. The right–hand–side estimate in (3.18) shows that

$$u_1 \ge \beta^{-3} \int\limits_{B_R \setminus K_\Omega} F(|y|)G(|x| + |y|)dy.$$

As in (2.10) we obtain in the spherical coordinates that

$$\int\limits_{B_R \setminus K_\Omega} F(|y|)G(|x| + |y|)dy = \sigma_n \int\limits_0^R F(\tau)G(|x| + \tau)\tau^{n-1}d\tau.$$

As a result,

$$u_1 \ge \beta^{-3}\sigma_n \int\limits_0^R F(\tau)G(|x| + \tau)\tau^{n-1}d\tau. \tag{3.20}$$

Estimates (3.19) and (3.20) give the two–sided inequality:

$$\beta^{-3}C_n^{-1}u_1 \le \int\limits_0^R F(\tau)G(|x| + \tau)\tau^{n-1}d\tau \le \beta^3\sigma_n^{-1}u_1. \tag{3.21}$$

We move on to the estimates for $u_2 = \int\limits_\Omega F(|y|)G(|x-y|)dy$. Note that

$$y \in \Omega \Rightarrow \begin{cases} |y| \le |x| + |x-y| \le \frac{3}{2}|x| \le R; \\ 3|y| \ge \frac{3}{2}|x| = |x| + \frac{1}{2}|x| \ge |x| + |x-y|. \end{cases}$$

Therefore, for $y \in \Omega$ we have

$$|y| \le |x| + |x-y| \le \min\left\{2^2|y|, R\right\}.$$

For $F \in J_\alpha(R)$ it follows from here and from (2.3) with $m = 2$ that

$$\alpha^{-2}F(|x| + |x-y|) \le F(|y|) \le \alpha^2 F(|x| + |x-y|), \quad y \in \Omega.$$

Therefore,

$$\alpha^{-2}u_2 \le \int\limits_\Omega F(|x| + |x-y|)G(|x-y|)dy \le \alpha^2 u_2.$$

In Ω we introduce the spherical system of coordinates with the center at the point x and the spherical radius $\lambda = |y-x|$. Then,

$$y \in \Omega,\ y \ne x \Leftrightarrow y - x = (\lambda, \omega),\ \lambda = |y-x| = |x-y| \in (0, |x|/2],\ \omega \in S^{n-1},$$

and we obtain the equality

$$\int\limits_\Omega F(|x| + |x-y|)G(|x-y|)dy = C_n \int\limits_0^{|x|/2} F(|x| + \lambda)G(\lambda)\lambda^{n-1}d\lambda,$$

with $C_n = \left(\int\limits_{S^{n-1}} d\omega \right) = 2\pi^{n/2}\Gamma(n/2)^{-1}$. These estimates show that

$$\alpha^{-2}C_n^{-1}u_2 \le \int\limits_0^{|x|/2} F(|x|+\lambda)G(\lambda)\lambda^{n-1}d\lambda \le \alpha^{-2}C_n^{-1}u_2. \tag{3.22}$$

For the further consideration, let us recall the notation and properties (2.19)–(2.21). We consider here (see estimates (2.18)– (2.21))

$$A(x) := u_1 \cong \int\limits_0^R F(\lambda)G(|x|+\lambda)\lambda^{n-1}d\lambda, \tag{3.23}$$

$$B(x) := u_2 \cong \int\limits_0^{|x|/2} F(|x|+\lambda)G(\lambda)\lambda^{n-1}d\lambda, \tag{3.24}$$

$$C(x) := \int\limits_{|x|/2}^{R-|x|} F(|x|+\lambda)G(\lambda)\lambda^{n-1}d\lambda. \tag{3.25}$$

For $|x|/2 \le \lambda \le R-|x|$ we have $|x| \le 2\lambda$, so that

$$\lambda \le |x|+\lambda \le \min\{3\lambda, R\} \le \min\left\{2^2\lambda, R\right\}.$$

Now, for $F \in J_\alpha(R)$, $\alpha \in (1,\infty)$ we can apply estimate (2.3) with $\xi = \lambda$, $m = 2$. Then

$$F(|x|+\lambda) \le \alpha^2 F(\lambda).$$

For $G \in J_\beta(\infty)$, $\beta \in (1,\infty)$ we will apply analogue of Remark 3 with $\xi = \lambda$, $m = 2$ and β instead of α, and obtain:

$$G(\lambda) \le \beta^4 G(|x|+\lambda).$$

Therefore,

$$\begin{aligned} 0 \le C(x) &\le \alpha^2\beta^4 \int\limits_{|x|/2}^{R-|x|} F(\lambda)G(|x|+\lambda)\lambda^{n-1}d\lambda \\ &\le \alpha^2\beta^4 \int\limits_0^R F(\lambda)G(|x|+\lambda)\lambda^{n-1}d\lambda \le c_3 A(x). \end{aligned} \tag{3.26}$$

Let us consider $\tilde{u}(x)$ defined in (3.11) We see from (3.23) -(3.25) that

$$\tilde{u}(x) \cong A(x) + B(x) + C(x).$$

Estimates (3.20) –(3.21), (3.26) show that here

$$A(x) + B(x) + C(x) \cong A(x) + B(x).$$

Therefore,

$$\tilde{u}(x) \cong A(x) + B(x) = u_1 + u_2 = u(x).$$

This completes the proof of estimate (3.14) in the case $|x| < 2R/3$.

2. Now we consider the case $2R/3 \le |x| \le 4R/3$. Introduce the ball $\Omega_0 = B(x/2, |x|/4)$ with the center $x/2$ and the radius $r = |x|/4$. Note that $\Omega_0 \subset B_R = B(0,R)$. As in (3.15), (3.16) we have

$$u(x) = \int\limits_{B_R} F(|y|)G(|x-y|)dy = u_{1,0}(x) + u_{2,0}(x), \quad x \in \dot{\mathbb{R}}^n, \tag{3.27}$$

where

$$u_{1,0}(x) = \int\limits_{B_R \setminus \Omega_0} F(|y|)G(|x-y|)dy, \quad u_{2,0}(x) = \int\limits_{\Omega_0} F(|y|)G(|x-y|)dy.$$

For $y \in B_R \setminus \Omega_0$ we have $|x| < 4|x-y|$, $|y| \le |x| + |x-y| < 5|x-y|$; so

$$|x-y| \le |x| + |y| \le 4|x-y| + 5|x-y| = 9|x-y|.$$

For $G \in J_\beta(\infty)$ this implies that

$$G(|x-y|) \cong G(|x| + |y|), \quad y \in B_R \setminus \Omega_0,$$

and, therefore,

$$u_{1,0}(x) \cong \int\limits_{B_R \setminus \Omega_0} F(|y|)G(|x| + |y|)dy.$$

As in (3.16) - (3.21) we obtain from here that

$$u_{1,0} \cong \int\limits_0^R F(\tau)G(|x| + \tau)\tau^{n-1}d\tau. \tag{3.28}$$

But, for $2R/3 \le |x| \le 4R/3$, $0 < \tau \le R$ we have $2R/3 \le |x| + \tau \le 7R/3 < 2^2(2R/3)$, and for $G \in J_\beta(\infty)$ according to the analogue of Remark 3 with $\xi = 2R/3$, $m = 2$ and β instead of α we obtain $G(|x| + \tau) \cong G(R)$. Therefore,

$$u_{1,0}(x) \cong G(R) \int\limits_0^R F(\tau)\tau^{n-1}d\tau. \tag{3.29}$$

For $y \in \Omega_0$ we have $|x/2 - y| \le r = |x|/4$, so that $|y| \le |x|/2 + r = 3|x|/4 \le R$, $|y| \ge |x|/2 - r = |x|/4$. Thus, we have

$$|x|/4 \le |y| \le 3|x|/4; \quad 2R/3 \le |x| \le 4R/3.$$

For $F \in J_\alpha(R)$ it implies that

$$F(|y|) \cong F(|x|/4) \cong F(R), \quad y \in \Omega_0.$$

Therefore,

$$u_{2,0}(x) \cong F(R) \int\limits_{\Omega_0} G(|x-y|)dy.$$

In Ω_0 we introduce the spherical system of coordinates with the center at the point $x/2$ and the spherical radius $\lambda = |x-y|$. Then,

$$u_{2,0}(x) \cong F(R) \int\limits_0^{|x|/4} G(\lambda)\lambda^{n-1}d\lambda. \tag{3.30}$$

For $2R/3 \le |x| \le 4R/3$ we apply several times estimate (3.5) with related choose of $\xi \in \mathbb{R}_+$, and obtain that

$$\int_0^{|x|/4} G(\lambda)\lambda^{n-1}d\lambda \cong \int_0^R G(\lambda)\lambda^{n-1}d\lambda. \tag{3.31}$$

The constants in estimate (3.31) depend only on β, n (estimates of such type were proved in Lemma 3.1). Together with (3.27) and (3.29) this gives desired estimates (3.12), (3.14).

Remark 6. Under the assumptions of Theorem 3.1 let $2R/3 \le |x| \le R$. Then, we have the equivalence

$$\tilde{u}(x) \cong F(|x|)\int_0^{|x|} G(\lambda)\lambda^{n-1}d\lambda + G(|x|)\int_0^{|x|} F(\lambda)\lambda^{n-1}d\lambda. \tag{3.32}$$

To show this let us note that for $2R/3 \le |x| \le R$ and for functions $F \in J_\alpha(R)$, $G \in J_\beta(\infty)$ we have $F(R) \cong F(|x|)$, $G(R) \cong G(|x|)$. Moreover, an application of Corollary of Lemma 3.1 gives

$$\int_0^{|x|} G(\lambda)\lambda^{n-1}d\lambda \cong \int_0^R G(\lambda)\lambda^{n-1}d\lambda, \quad \int_0^{|x|} F(\lambda)\lambda^{n-1}d\lambda \cong \int_0^R F(\lambda)\lambda^{n-1}d\lambda.$$

This means that estimates (3.12), (3.14) *imply estimate* (3.32).

3. Consider the case $|x| > 4R/3$. We have the equality

$$u(x) = \int_{B_R} F(|y|)G(|x-y|)dy.$$

Note that $|y| \le R$, $|x| > 4R/3 \Rightarrow |x|/4 \le |x-y| \le 7|x|/4$, and for $G \in J_\beta(\infty)$ we obtain $G(|x-y|) \cong G(|x|)$, $y \in B_R$. Therefore,

$$u(x) \cong G(|x|)\int_{B_R} F(|y|)dy = C_n G(|x|)\int_0^R F(\tau)\tau^{n-1}d\tau.$$

□

4 Two–sided estimates for decreasing rearrangements of convolutions

4.1 Estimates for decreasing and symmetrical rearrangements

Here we consider estimates for decreasing and symmetrical rearrangements of convolutions. The books by S.G. Krein, Yu.I. Petunin and E.M. Semenov [12], C. Bennett and R. Sharpley [3] contain the main definitions and basic facts related to this topic. We recall some formulas.

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function such that its distribution function

$$\lambda_h(y) = \mu_n\left\{x \in \dot{\mathbb{R}}^n : |h(x)| > y\right\}, \quad y \in [0,\infty),$$

is not identically equal to infinity. Then, $0 \le \lambda_h(y) \downarrow$ on $[0,\infty)$. The decreasing rearrangement of the function h is defined by the formula

$$h^*(\tau) = \inf\left\{y \in [0,\infty) : \lambda_h(y) \le \tau\right\}, \tau \in (0,\infty). \tag{4.1}$$

Note that $0 \le h^* \downarrow$ on $(0,\infty)$. The symmetrical rearrangement $h^\#$ is a radially symmetrical function related to the decreasing rearrangement by the formulas

$$h^\#(\rho) = h^*(V_n \rho^n), \quad h^*(\tau) = h^\#((\tau/V_n)^{1/n});\ \rho, \tau \in (0,\infty). \tag{4.2}$$

Here V_n is the volume of the unit ball in $\mathbb{R}^n$.

Moreover,

$$h(x) = H(|x|),\ 0 \le H \downarrow \text{on } (0,\infty) \Rightarrow h^\#(\rho) = H(\rho),\ \rho \in (0,\infty). \tag{4.3}$$

Theorem 4.1. *Under the assumptions of Theorem* 2.1 *let additionally* F, G *be decreasing. Then, there exist constants* $c_i = c_i(\alpha, \beta, n)$, $i = 1, 2$, *such that* $0 < c_1 \le c_2 < \infty$ *and for the symmetrical rearrangement of convolution* (2.7) *the following estimates hold*

$$c_1 u^\#(\rho) \le \int_0^\infty \left[F(\rho+\tau)G(\tau) + F(\tau)G(\rho+\tau)\right]\tau^{n-1}d\tau \le c_2 u^\#(\rho),\ \rho \in (0,\infty). \tag{4.4}$$

Moreover,

$$u^\#(\rho) \cong F(\rho)\int_0^\rho G(\tau)\tau^{n-1}d\tau + G(\rho)\int_0^\rho F(\tau)\tau^{n-1}d\tau + \int_\rho^\infty F(\tau)G(\tau)\tau^{n-1}d\tau \tag{4.5}$$

with understanding $\cong$ *as in* (2.18).

Proof. From (2.9) it follows that

$$c_1 u^\#(\rho) \le \tilde{u}^\#(\rho) \le c_2 u^\#(\rho),\ \rho \in (0,\infty).$$

Note that the function $\tilde{u}$ defined by (2.8) is radially symmetrical and decreases as a function of $\rho = |x|$. Thus, according to (4.3) it coincides with its symmetrical rearrangement, and we can apply definition (2.8) with $\rho = |x|$. By Theorem 2.1 this proves estimate (4.4).

Let us deduce (4.5) from (4.4). We have

$$\int_0^\infty F(\rho+\tau)G(\tau)\tau^{n-1}d\tau = \int_0^\rho F(\rho+\tau)G(\tau)\tau^{n-1}d\tau + \int_\rho^\infty F(\rho+\tau)G(\tau)\tau^{n-1}d\tau.$$

For $\tau \in [0,\rho]$ we have $\rho+\tau \in [\rho, 2\rho]$, so that for the function $F \in J_\alpha(\infty)$ there is the estimate:

$$\alpha^{-1}F(\rho+\tau) \le F(\rho) \le \alpha F(\rho+\tau).$$

Therefore,

$$\alpha^{-1}\int_0^\rho F(\rho+\tau)G(\tau)\tau^{n-1}d\tau \le F(\rho)\int_0^\rho G(\tau)\tau^{n-1}d\tau \le \alpha\int_0^\rho F(\rho+\tau)G(\tau)\tau^{n-1}d\tau.$$

For $\tau > \rho$ we have $\rho+\tau \in [\tau, 2\tau]$, so that for the function $F \in J_\alpha(\infty)$ there is the estimate:

$$\alpha^{-1}F(\rho+\tau) \le F(\tau) \le \alpha F(\rho+\tau).$$

Therefore,

$$\alpha^{-1}\int\limits_{\rho}^{\infty} F(\rho+\tau)G(\tau)\tau^{n-1}d\tau \le \int\limits_{\rho}^{\infty} F(\tau)G(\tau)\tau^{n-1}d\tau \le \alpha\int\limits_{\rho}^{\infty} F(\rho+\tau)G(\tau)\tau^{n-1}d\tau.$$

So, we have the two–sided estimate

$$\int\limits_{0}^{\infty} F(\rho+\tau)G(\tau)\tau^{n-1}d\tau \cong F(\rho)\int\limits_{0}^{\rho} G(\tau)\tau^{n-1}d\tau + \int\limits_{\rho}^{\infty} F(\tau)G(\tau)\tau^{n-1}d\tau.$$

Analogously, for $G \in J_\beta(\infty)$, we obtain

$$\int\limits_{0}^{\infty} F(\tau)G(\rho+\tau)\tau^{n-1}d\tau \cong G(\rho)\int\limits_{0}^{\rho} F(\tau)\tau^{n-1}d\tau + \int\limits_{\rho}^{\infty} F(\tau)G(\tau)\tau^{n-1}d\tau.$$

As a result,

$$\begin{aligned}&\int\limits_{0}^{\infty}\left[F(\rho+\tau)G(\tau)+F(\tau)G(\rho+\tau)\right]\tau^{n-1}d\tau\\ &\cong F(\rho)\int\limits_{0}^{\rho} G(\tau)\tau^{n-1}d\tau + G(\rho)\int\limits_{0}^{\rho} F(\tau)\tau^{n-1}d\tau + \int\limits_{\rho}^{\infty} F(\tau)G(\tau)\tau^{n-1}d\tau.\end{aligned}$$

We put this estimate into (4.4) and obtain (4.5). □

Remark 7. Note that the right–hand–side inequality in (4.4) follows immediately from Remark 5 and Corollary 2.2 (see estimate (2.27)) without restrictions $F \in J_\alpha$, $G \in J_\beta$.

Corollary 4.1. *Under the assumptions of Theorem* 4.1 *we define*

$$\varphi(\lambda) = F\left((\lambda/V_n)^{1/n}\right),\ \psi(\lambda) = G\left((\lambda/V_n)^{1/n}\right),\ \lambda \in (0,\infty). \tag{4.6}$$

Then, the following estimate holds for the decreasing rearrangement of the convolution u:

$$u^*(t) \cong \varphi(t)\int\limits_{0}^{t}\psi(\lambda)d\lambda + \psi(t)\int\limits_{0}^{t}\varphi(\lambda)d\lambda + \int\limits_{t}^{\infty}\varphi(\lambda)\psi(\lambda)d\lambda,\ t\in(0,\infty), \tag{4.7}$$

with understanding $\cong$ *as in* (2.18).

Proof. We introduce the new variable $\lambda = V_n\tau^n$ for integrals in (4.5). Then,

$$\tau = (\lambda/V_n)^{1/n},\ \tau^{n-1}d\tau = d\lambda/(nV_n),$$

and we obtain from (4.5)–(4.6)

$$u^{\#}(\rho) \cong F(\rho)\int\limits_{0}^{V_n\rho^n}\psi(\lambda)d\lambda + G(\rho)\int\limits_{0}^{V_n\rho^n}\varphi(\lambda)d\lambda + \int\limits_{V_n\rho^n}^{\infty}\varphi(\lambda)\psi(\lambda)d\lambda.$$

We put here $\rho = (t/V_n)^{1/n}$ and take into account notation (4.6) and the equality: $u^{\#}\left((t/V_n)^{1/n}\right) = u^*(t)$ (see (4.2)). Thus, we come to (4.7). □

Corollary 4.2. *Under the assumptions of Theorem* 4.1 *the following estimate holds for the decreasing rearrangement of the convolution:*

$$u^*(t) \cong f^*(t) \int_0^t g^*(\lambda)d\lambda + g^*(t) \int_0^t f^*(\lambda)d\lambda + \int_t^\infty f^*(\lambda)g^*(\lambda)d\lambda,\ t \in (0,\infty), \tag{4.8}$$

with understanding $\cong$ *as in* (2.18).

Proof. Indeed, formula (4.8) follows from (4.7) and from the equalities

$$f(x) = F(|x|),\ 0 \le F \downarrow \Rightarrow f^{\#}(\rho) = F(\rho) \Rightarrow f^*(t) = F\left((t/V_n)^{1/n}\right) = \varphi(t),$$

$$g(x) = G(|x|),\ 0 \le G \downarrow \Rightarrow g^{\#}(\rho) = G(\rho) \Rightarrow g^*(t) = G\left((t/V_n)^{1/n}\right) = \psi(t).$$

□

Remark 8. Note that under the assumptions of Theorem 4.1

$$2^{-n}t_1 \le t_2 \le t_1 \Rightarrow f^*(t_1) \le f^*(t_2) \le \alpha^2 f^*(t_1). \tag{4.9}$$

Indeed,

$$\frac{1}{2}\left(\frac{t_1}{V_n}\right)^{1/n} \le \left(\frac{t_2}{V_n}\right)^{1/n} \le \left(\frac{t_1}{V_n}\right)^{1/n},$$

and for $F \in J_\alpha(\infty)$ we have by application of Remark 3

$$\alpha^{-2}F\left(\left(\frac{t_1}{V_n}\right)^{1/n}\right) \le F\left(\left(\frac{t_2}{V_n}\right)^{1/n}\right) \le \alpha^2 F\left(\left(\frac{t_1}{V_n}\right)^{1/n}\right).$$

Moreover, the function F decreases and for $t_2 \le t_1$ in the left–hand–side of this estimate we can replace $\alpha^{-2} < 1$ by 1. Therefore, for the function $f^*(t) = F\left(\left(\frac{t}{V_n}\right)^{1/n}\right)$ we obtain (4.9).

Analogously,

$$2^{-n}t_1 \le t_2 \le t_1 \Rightarrow g^*(t_1) \le g^*(t_2) \le \beta^2 g^*(t_1). \tag{4.10}$$

Corollary 4.3. *Under the assumptions of Theorem* 4.1 *for* $\xi \in (0,\infty)$ *the following estimates hold for the decreasing rearrangement of a function* f *(see* (2.7)*):*

$$\xi \le \eta \le 2\xi \Rightarrow f^*(2\xi) \le f^*(\eta) \le \alpha^2 f^*(2\xi); \tag{4.11}$$

$$\xi \le \eta \le 2\xi \Rightarrow g^*(2\xi) \le g^*(\eta) \le \beta^2 g^*(2\xi); \tag{4.12}$$

Proof. Indeed, we put $t_1 = 2\xi$ in (4.9) and obtain

$$\xi \le \eta \le 2\xi \Leftrightarrow 2^{-1}t_1 \le \eta \le t_1 \Rightarrow 2^{-n}t_1 \le \eta \le t_1 \Rightarrow f^*(2\xi) \le f^*(\eta) \le \alpha^2 f^*(2\xi).$$

Analogously, we obtain (4.12) from (4.10). □

Corollary 4.4. *Under the assumptions of Theorem* 4.1 *the following estimate holds for the decreasing rearrangement of the convolution* u*:*

$$u^*(t) \cong \int_0^\infty \left[f^*(t+\lambda)g^*(\lambda) + f^*(\lambda)g^*(t+\lambda)\right] d\lambda,\ t \in (0,\infty), \tag{4.13}$$

with understanding $\cong$ *as in* (2.18).

Proof. We must show that estimate (4.13) is equivalent to (4.8). We have the equality

$$I := \int_0^\infty [f^*(t+\lambda)g^*(\lambda) + f^*(\lambda)g^*(t+\lambda)]\,d\lambda = \int_0^t [\ldots]\,d\lambda + \int_t^\infty [\ldots]\,d\lambda.$$

Note that, according to (4.11), (4.12), the following estimates hold:

$$0 < \lambda \le t \Rightarrow t < t+\lambda \le 2t \Rightarrow f^*(t+\lambda) \cong f^*(t);\ g^*(t+\lambda) \cong g^*(t);$$

$$\lambda > t \Rightarrow \lambda < t+\lambda \le 2\lambda \Rightarrow f^*(t+\lambda) \cong f^*(\lambda);\ g^*(t+\lambda) \cong g^*(\lambda).$$

Therefore,

$$\int_0^t [f^*(t+\lambda)g^*(\lambda) + f^*(\lambda)g^*(t+\lambda)]\,d\lambda \cong f^*(t)\int_0^t g^*(\lambda)d\lambda + g^*(t)\int_0^t f^*(\lambda)d\lambda;$$

$$\int_t^\infty [f^*(t+\lambda)g^*(\lambda) + f^*(\lambda)g^*(t+\lambda)]\,d\lambda \cong \int_t^\infty f^*(\lambda)g^*(\lambda)d\lambda.$$

This shows that

$$I \cong f^*(t)\int_0^t g^*(\lambda)d\lambda + g^*(t)\int_0^t f^*(\lambda)d\lambda + \int_t^\infty f^*(\lambda)g^*(\lambda)d\lambda.$$

Now, we apply (4.8) and obtain (4.13). □

4.2 Estimates for integral mean values of rearrangements

We move on to estimating the integral mean value for the decreasing rearrangement of the convolution. Let

$$0 < \nu(\tau),\ \tau \in (0,\infty);\ 0 < V(t) := \int_0^t \nu(\tau)d\tau < \infty,\ t \in (0,\infty);$$

$$u_\nu^{**}(t) = \frac{1}{V(t)}\int_0^t u^*(\tau)\nu(\tau)d\tau,\ t \in (0,\infty). \tag{4.14}$$

Such variant of the mean value for the decreasing rearrangement was introduced in [1].

Theorem 4.2. *Under the assumptions of Theorem* 4.1 *the following estimate holds*

$$u_\nu^{**}(t) \cong I_1(t) + I_2(t) + I_3(t),\ t \in (0,\infty); \tag{4.15}$$

$$I_1(t) = V(t)^{-1}\int_0^t \left[f^*(\tau)\int_0^\tau g^*(\lambda)d\lambda + g^*(\tau)\int_0^\tau f^*(\lambda)d\lambda\right]\nu(\tau)d\tau; \tag{4.16}$$

$$I_2(t) = V(t)^{-1}\int_0^t f^*(\lambda)g^*(\lambda)V(\lambda)d\lambda; \quad I_3(t) = \int_t^\infty f^*(\lambda)g^*(\lambda)d\lambda. \tag{4.17}$$

Proof. By (4.13) we have

$$u_\nu^{**}(t) \cong \hat{I}_1(t) + \hat{I}_2(t) + \hat{I}_3(t),\ t \in (0,\infty); \tag{4.18}$$

$$\hat{I}_1(t) = V(t)^{-1} \int_0^t \left(\int_0^\tau \left[f^*(\tau+\lambda) g^*(\lambda) + f^*(\lambda) g^*(\tau+\lambda) \right] d\lambda \right) \nu(\tau) d\tau;$$

$$\hat{I}_2(t) = V(t)^{-1} \int_0^t \left(\int_\tau^t \left[f^*(\tau+\lambda) g^*(\lambda) + f^*(\lambda) g^*(\tau+\lambda) \right] d\lambda \right) \nu(\tau) d\tau;$$

$$\hat{I}_3(t) = V(t)^{-1} \int_0^t \left(\int_t^\infty \left[f^*(\tau+\lambda) g^*(\lambda) + f^*(\lambda) g^*(\tau+\lambda) \right] d\lambda \right) \nu(\tau) d\tau.$$

Let us recall inequalities (4.11), (4.12). Thus, we have estimates

$$0 < \lambda \le \tau \Rightarrow \tau < \tau + \lambda \le 2\tau \Rightarrow f^*(\tau+\lambda) \cong f^*(\tau);\ g^*(\tau+\lambda) \cong g^*(\tau);$$

$$\lambda > \tau \Rightarrow \lambda < \tau + \lambda \le 2\lambda \Rightarrow f^*(\tau+\lambda) \cong f^*(\lambda);\ g^*(\tau+\lambda) \cong g^*(\lambda).$$

Therefore, for $t \in (0,\infty)$

$$\hat{I}_1(t) \cong V(t)^{-1} \int_0^t \left[f^*(\tau) \int_0^\tau g^*(\lambda) d\lambda + g^*(\tau) \int_0^\tau f^*(\lambda) d\lambda \right] \nu(\tau) d\tau = I_1(t);$$

$$\hat{I}_2(t) \cong V(t)^{-1} \int_0^t \left[\int_\tau^t f^*(\lambda) g^*(\lambda) d\lambda \right] \nu(\tau) d\tau = V(t)^{-1} \int_0^t f^*(\lambda) g^*(\lambda) \int_0^\lambda \nu(\tau) d\tau d\lambda = I_2(t);$$

$$\hat{I}_3(t) \cong V(t)^{-1} \int_0^t \left[\int_t^\infty f^*(\lambda) g^*(\lambda) d\lambda \right] \nu(\tau) d\tau = V(t)^{-1} \int_t^\infty f^*(\lambda) g^*(\lambda) d\lambda \int_0^t \nu(\tau) d\tau = I_3(t).$$

Thus, (4.18) implies (4.15) - (4.17). □

In some special cases we can simplify the general answer.

Remark 9. Under the assumptions of Theorem 4.1 we assume additionally that there exists a constant $c_0 \in (0,\infty)$, such that

$$\nu(\tau)\tau \ge c_0 V(\tau),\ \tau \in (0,\infty). \tag{4.19}$$

Then,

$$u_\nu^{**}(t) \cong I_1(t) + I_3(t),\ t \in (0,\infty). \tag{4.20}$$

Moreover, here

$$I_1(t) \ge 2c_0 V(t)^{-1} \int_0^t f^*(\tau) g^*(\tau) V(\tau) d\tau. \tag{4.21}$$

Proof. We put estimate (4.19) into (4.16) and obtain

$$I_1(t) \geq 2c_0 V(t)^{-1} \int_0^t \left[f^*(\tau)\frac{1}{\tau}\int_0^\tau g^*(\lambda)d\lambda + g^*(\tau)\frac{1}{\tau}\int_0^\tau f^*(\lambda)d\lambda \right] V(\tau)d\tau.$$

Functions f^*, g^* decrease, so that we have inequalities

$$\frac{1}{\tau}\int_0^\tau g^*(\lambda)d\lambda \geq g^*(\tau),\ \frac{1}{\tau}\int_0^\tau f^*(\lambda)d\lambda \geq f^*(\tau).$$

Therefore,

$$I_1(t) \geq 2c_0 V(t)^{-1} \int_0^t \left[f^*(\tau)g^*(\tau)\right] V(\tau)d\tau = 2c_0 I_2(t).$$

This means that in the right–hand side of (4.15) the second term is covered by the first one, and we come to estimates (4.20), (4.21).

Note that inequality (4.19) holds with the constant $c_0 = 1$ in the case of the increasing weight ν. □

Remark 10. The non-weighted case, where $\nu(\tau) \equiv 1$, is of special interest. Thus,

$$\nu(\tau) \equiv 1 \Rightarrow V(\tau) = \tau \Rightarrow u_\nu^{**}(t) = u^{**}(t) := \frac{1}{t}\int_0^t u^*(\tau)d\tau,\ t \in (0,\infty). \tag{4.22}$$

In this case we have the estimate

$$u^{**}(t) \cong t^{-1}\int_0^t f^*(\lambda)d\lambda \int_0^t g^*(\lambda)d\lambda + \int_t^\infty f^*(\lambda)g^*(\lambda)d\lambda. \tag{4.23}$$

Indeed, in the non–weighted case we have

$$\begin{aligned} I_1(t) &= t^{-1}\int_0^t \left[f^*(\tau)\int_0^\tau g^*(\lambda)d\lambda + g^*(\tau)\int_0^\tau f^*(\lambda)d\lambda \right] d\tau \\ &= t^{-1}\int_0^t \frac{d}{d\tau}\left[\int_0^\tau f^*(\lambda)d\lambda \int_0^\tau g^*(\lambda)d\lambda \right] d\tau = t^{-1}\int_0^t f^*(\lambda)d\lambda \int_0^t g^*(\lambda)d\lambda. \end{aligned} \tag{4.24}$$

We put this equality into (4.20), take into account equality (4.17) for $I_3(t)$ and obtain (4.23).

5 One useful lemma

The following lemma may be useful in many considerations related to the subject of this paper. The proof of this lemma is related to the proofs of Theorems in Sections 2–4.

Lemma 5.1. *Let functions $F, G \geq 0$ be measurable on $(0,\infty)$, let*

$$G \in J_\beta(\infty), \tag{5.1}$$

$$R \in (0,\infty],\ F \in J_\alpha(R), \tag{5.2}$$

Denote

$$D_\infty(\rho) = \int_0^\infty \left[F(\rho+\tau)G(\tau) + F(\tau)G(\rho+\tau)\right]\tau^{n-1}d\tau,\ \rho \in (0,\infty); \tag{5.3}$$

$$D_R(\rho) = \int_0^{R-\rho} F(\rho+\tau)G(\tau)\tau^{n-1}d\tau + \int_0^R F(\tau)G(\rho+\tau)\tau^{n-1}d\tau,\ R<\infty,\ \rho \in (0,R]; \tag{5.4}$$

$$D_R(\rho) = \int_0^R F(\tau)G(\rho+\tau)\tau^{n-1}d\tau,\ R<\infty,\ \rho > R. \tag{5.5}$$

1. Then, for $R=\infty$ we have the estimate:

$$D_\infty(\rho) \cong F(\rho)\int_0^\rho G(\tau)\tau^{n-1}d\tau + G(\rho)\int_0^\rho F(\tau)\tau^{n-1}d\tau + \int_\rho^\infty F(\tau)G(\tau)\tau^{n-1}d\tau,\ \rho \in (0,\infty). \tag{5.6}$$

2. For $R<\infty$ we have the estimates:

(a) if $\rho \in (0, R/2]$, then

$$D_R(\rho) \cong F(\rho)\int_0^\rho G(\tau)\tau^{n-1}d\tau + G(\rho)\int_0^\rho F(\tau)\tau^{n-1}d\tau + \int_\rho^R F(\tau)G(\tau)\tau^{n-1}d\tau; \tag{5.7}$$

(b) if $\rho \in (R/2, R]$, then

$$D_R(\rho) \cong F(\rho)\int_0^{R-\rho} G(\tau)\tau^{n-1}d\tau + G(\rho)\int_0^R F(\tau)\tau^{n-1}d\tau; \tag{5.8}$$

(c) if $\rho > R$, then

$$D_R(\rho) \cong G(\rho)\int_0^R F(\tau)\tau^{n-1}d\tau. \tag{5.9}$$

In these formulas $A \cong B$ means that for each formula there exist constants $0 < d_1 \leq d_2 < \infty$, depending only on α, β, such that $d_1 \leq A/B \leq d_2$.

Proof. 1. For $R=\infty$ we have

$$D_\infty(\rho) = A_1(\rho) + A_2(\rho); \tag{5.10}$$

$$A_1(\rho) = \int_0^\rho F(\rho+\tau)G(\tau)\tau^{n-1}d\tau + \int_\rho^\infty F(\rho+\tau)G(\tau)\tau^{n-1}d\tau,$$

$$A_2(\rho) = \int_0^\rho F(\tau)G(\rho+\tau)\tau^{n-1}d\tau + \int_\rho^\infty F(\tau)G(\rho+\tau)\tau^{n-1}d\tau.$$

For $0 \le \tau \le \rho$ we have $\rho+\tau \in [\rho, 2\rho] \Rightarrow \alpha^{-1}F(\rho) \le F(\rho+\tau) \le \alpha F(\rho)$.

For $\rho \le \tau$ we have $\rho+\tau \in [\tau, 2\tau] \Rightarrow \alpha^{-1}F(\tau) \le F(\rho+\tau) \le \alpha F(\tau)$, (see (5.2)). Therefore,

$$A_1(\rho) \cong F(\rho)\int_0^\rho G(\tau)\tau^{n-1}d\tau + \int_\rho^\infty F(\tau)G(\tau)\tau^{n-1}d\tau.$$

Analogously, for $0 \le \tau \le \rho$ we have $\beta^{-1}G(\rho) \le G(\rho+\tau) \le \beta G(\rho)$; for $\rho \le \tau$ we have $\beta^{-1}G(\tau) \le G(\rho+\tau) \le \beta G(\tau)$, (see (5.1)). Thus,

$$A_2(\rho) \cong G(\rho)\int_0^\rho F(\tau)\tau^{n-1}d\tau + \int_\rho^\infty F(\tau)G(\tau)\tau^{n-1}d\tau.$$

As a result, we come to estimate (5.6).

2. Let $R < \infty$, $\rho \in (0, R/2]$. Then, $\rho \le R-\rho$ and

$$D_R(\rho) = B_1(\rho) + B_2(\rho);$$

$$B_1(\rho) = \int_0^\rho F(\rho+\tau)G(\tau)\tau^{n-1}d\tau + \int_\rho^{R-\rho} F(\rho+\tau)G(\tau)\tau^{n-1}d\tau,$$

$$B_2(\rho) = \int_0^\rho F(\tau)G(\rho+\tau)\tau^{n-1}d\tau + \int_\rho^R F(\tau)G(\rho+\tau)\tau^{n-1}d\tau.$$

As in Step 1 we have

$$F(\rho+\tau) \cong F(\rho),\ G(\rho+\tau) \cong G(\rho), \quad 0 \le \tau \le \rho;$$

$$F(\rho+\tau) \cong F(\tau) \text{ for } \rho < \tau \le R-\rho,\ G(\rho+\tau) \cong G(\tau) \text{ for } \tau > \rho,$$

so that

$$B_1(\rho) \cong F(\rho)\int_0^\rho G(\tau)\tau^{n-1}d\tau + \int_\rho^{R-\rho} F(\tau)G(\tau)\tau^{n-1}d\tau; \tag{5.11}$$

$$B_2(\rho) \cong G(\rho)\int_0^\rho F(\tau)\tau^{n-1}d\tau + \int_\rho^R F(\tau)G(\tau)\tau^{n-1}d\tau. \tag{5.12}$$

We take into account that the second term in (5.11) is majored by the second term in (5.12) and obtain

$$D_R(\rho) = B_1(\rho) + B_2(\rho) \cong F(\rho)\int_0^\rho G(\tau)\tau^{n-1}d\tau + G(\rho)\int_0^\rho F(\tau)\tau^{n-1}d\tau + \int_\rho^R F(\tau)G(\tau)\tau^{n-1}d\tau.$$

It gives estimate (5.7).

3. Now, let $R \in (0,\infty)$, $\rho \in (R/2, R]$. Then, $R-\rho < R/2 < \rho$, and

$$D_R(\rho) = E_1(\rho) + E_2(\rho),$$

$$E_1(\rho) = \int_0^{R-\rho} F(\rho+\tau)G(\tau)\tau^{n-1}d\tau,\ E_2(\rho) = \int_0^{R} F(\tau)G(\rho+\tau)\tau^{n-1}d\tau.$$

For $0 \le \tau \le R-\rho$ we have $\rho < \rho+\tau \le R < 2\rho$, so that

$$F(\rho+\tau) \cong F(\rho) \Rightarrow \int_0^{R-\rho} F(\rho+\tau)G(\tau)\tau^{n-1}d\tau \cong F(\rho)\int_0^{R-\rho} G(\tau)\tau^{n-1}d\tau.$$

For $0 < \tau \le R$ we have $\rho < \rho+\tau \le \rho+R \le \rho+2\rho = 3\rho$, so that

$$G(\rho+\tau) \cong G(\rho) \Rightarrow \int_0^{R} F(\tau)G(\rho+\tau)\tau^{n-1}d\tau \cong G(\rho)\int_0^{R} F(\tau)\tau^{n-1}d\tau.$$

As a result,

$$D_R(\rho) = E_1(\rho) + E_2(\rho) \cong F(\rho)\int_0^{R-\rho} G(\tau)\tau^{n-1}d\tau + G(\rho)\int_0^{R} F(\tau)\tau^{n-1}d\tau.$$

4. It remains to consider the case $R \in (0,\infty)$, $\rho > R$. Then, $G(\rho+\tau) \cong G(\rho)$ for $0 < \tau \le R$, so that (see (5.5))

$$D_R(\rho) = \int_0^{R} F(\tau)G(\rho+\tau)\tau^{n-1}d\tau \cong G(\rho)\int_0^{R} F(\tau)\tau^{n-1}d\tau.$$

This estimate coincides with (5.9).

$\square$

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