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WEIGHTED HARDY INEQUALITIES AND THEIR APPLICATIONS TO OSCILLATION THEORY OF HALF–LINEAR DIFFERENTIAL EQUATIONS

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This paper is dedicated to Professor Lars-Erik Persson on occasion of his 65th birthday

Communicated by A. Kufner

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Abstract. For the equation

$$\big(\rho(t)|y'(t)|^{p-2}y'(t)\big)' + v(t)|y(t)|^{p-2}y(t) = 0, \quad t \in (a,b)$$

where $1 < p < \infty$, we establish the properties of oscillation and nonoscillation.

1 Introduction

For about last 50 years the weighted Hardy inequalities have been intensively investigated. There is a lot of papers and books devoted to this problem. The history of this problem, results and their applications in various areas of Analysis can be found in the monographs by A. Kufner and L.-E. Persson [7], and A. Kufner, L. Maligranda and L.-E. Persson [6].

The main aim of this paper is to apply the results on the weighted Hardy inequalities to the qualitative theory of half-linear second order differential equations.

On the interval $I = (a,b)$, $-\infty \le a < b \le +\infty$, we consider the following second order differential equation:

$$\big(\rho(t)|y'(t)|^{p-2}y'(t)\big)' + v(t)|y(t)|^{p-2}y(t) = 0, \tag{1}$$

where $1 < p < \infty$, ρ and v are continuous functions on I. Moreover, $\rho(t) > 0$ for any $t \in I$.

When $p = 2$ equation (1) becomes the linear Sturm–Liouville equation

$$(\rho(t)y'(t))' + v(t)y(t) = 0,$$

the investigation of qualitative properties of which was started by J. Sturm in his paper as early as 1836.

When $p \neq 2$ equation (1) is called half-linear because the set of its solutions has the property of homogeneity but not additivity.

A function $y : I \to R$ is said to be a solution of (1), if $y(t)$ is continuously differentiable together with $\rho(t)|y'(t)|^{p-2}y'(t)$ and satisfies equation (1) on I.

Systematic investigation of the half-linear equation (1) was started in the works by D. Mirzov [10] in 1976 and A. Elbert [3] in 1979, where in particular the existence, uniqueness and continuability of the solution of the initial-value problem were proved. Basic facts and results for half-linear equations odtained by 2005 are exposed in the fine book by O. Dosly and P. Rehak [1].

A nontrivial solution of equation (1) is called *oscillatory* at $t = b$, if it has infinite number of zeros converging to b, otherwise it is called *nonoscillatory* at $t = b$. Similarly we can give the definition of oscillatory and nonoscillatory of a solution at $t = a$.

Equation (1) is called *oscillatory* (*nonoscillatory*), if all its nontrivial solutions are oscillatory (nonoscillatory).

For equation (1) the Sturm theorem on separation of zeroes is valid, thus equation (1) is oscillatory (nonoscillatory), if one of its nontrivial solution is oscillatory (nonoscillatory).

To investigate the oscillation properties of (1) it is proper to use the notions such as conjugacy and disconjugacy of the equation (1).

Equation (1) is called *disconjugate* on the interval $[\alpha, \beta] \subset I$, if of its any nontrivial solution has no more than one zero on $[\alpha, \beta]$, otherwise it is called *conjugate* on $[\alpha, \beta]$.

The fundamental result in the qualitative theory of half-linear equations of form (1) is the so-called "roundabout theorem" [1].

Roundabout theorem. *The following statements are equivalent:*
(i) Equation (1) is disconjugate on $[\alpha, \beta]$.
(ii) There exists a solution of equation (1) having no zeros in $[\alpha, \beta]$.
(iii) There exists a solution w *of the generalized Riccati equation*

$$w' + v(t) + (p-1)\rho^{1-p'}(t)|w(t)|^{p'} = 0$$

on the whole interval $[\alpha, \beta]$, *where* $p' = \frac{p}{p-1}$.
(iv) The functional

$$F(f, \alpha, \beta) = \int_{\alpha}^{\beta} (\rho(t)|f'|^p - v(t)|f|^p)\, dt$$

is positive for any nontrivial $f \in AC[\alpha, \beta]$, *such that* $f' \in L_p(\alpha, \beta)$ *and* $f(\alpha) = f(\beta) = 0$.

"Roundabout theorem" gives two methods of investigation of equation (1). The first one is based on the equivalence of (i) and (iii) and is called "*Riccati technique*". The second one is based on the equivalence of (i) and (iv) and can be investigated by the *variational method*.

Main results in the qualitative theory of equation (1) were obtained by application of "Riccati technique".

Our aim is to develop the variational method of studying equation (1) by applying of achievements in the theory of weighted Hardy-type inequalities.

We would like to mention the pioneering work of M. Otelbaev [15] on application of Muckenhoupt's results on Hardy inequalities to the oscillation theory of the Sturm–Liouville equation.

Denote by $\overset{\circ}{AC}_p(I)$ the set of all compactly supported locally absolutely continuous functions f on I such that $f' \in L_p(I)$.

From the equivalence of (i) and (iv) the following statements can be deduced [12].

Lemma 1. *Equation (1) is disconjugate on I if and only if*

$$F(f,a,b) \equiv \int_a^b (\rho(t)|f'|^p - v(t)|f|^p)\,dt > 0 \tag{2}$$

for all nontrivial $f \in \overset{\circ}{AC}_p(I)$.

Lemma 2. *Equation (1) is nonoscillatory at $t = b$ ($t = a$) if and only if there exists $c \in I$ such that $F(f,c,b) > 0$ ($F(f,a,c) > 0$) for all nontrivial $f \in \overset{\circ}{AC}_p(c,b)$ ($f \in \overset{\circ}{AC}_p(a,c)$).*

Lemma 3. *Equation (1) is oscillatory at $t = b$ ($t = a$) if and only if for any $c \in I$ there exists $\widetilde{f} \in \overset{\circ}{AC}_p(c,b)$ ($\widetilde{f} \in \overset{\circ}{AC}_p(a,c)$) such that $F(\widetilde{f},c,b) \leq 0$ ($F(\widetilde{f},a,c) \leq 0$).*

2 Main results

In this section we suppose that $v(t) > 0$ for any $t \in I$. Then inequality (2) is equivalent to the Hardy inequality

$$\left(\int_a^b v(t)|f(t)|^p dt\right)^{\frac{1}{p}} < C\left(\int_a^b \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}}, \ f \in \overset{\circ}{AC}_p(I), \ f \not\equiv 0, \tag{3}$$

with $C = 1$.

Unfortunately, in the theory of weighted Hardy inequalities the exact values of the best constants C were found only in some particular cases. However, in many cases two-sided estimates for the best constants C were obtained.

For example, in [14, Theorem 8.8] it is proved that the best constants C in inequality (3) where "$<$" is replaced by "$\leq$" satisfies the following estimate

$$C \leq p^{\frac{1}{p}}(p')^{\frac{1}{p'}} A(a,b),$$

where

$$A(a,b) = \inf_{a<c<b} \max\{A_M(c,b), \overline{A}_M(a,c)\},$$

$$A_M(c,b)=\sup_{c<x<b}\left(\int_x^b v(t)dt\right)^{\frac{1}{p}}\left(\int_c^x \rho^{1-p'}(s)ds\right)^{\frac{1}{p'}},$$

$$\overline{A}_M(a,c)=\sup_{a<x<c}\left(\int_a^x v(t)dt\right)^{\frac{1}{p}}\left(\int_x^c \rho^{1-p'}(s)ds\right)^{\frac{1}{p'}}.$$

This together with Lemma 2 gives

Theorem 1. *If $A(a,b)<(\frac{1}{p})^{\frac{1}{p}}(\frac{1}{p'})^{\frac{1}{p'}}$, then equation (1) is disconjugate on the whole interval I.*

Next, we will consider the oscillation and nonoscillation problem for equation (1) only at $t=b$ because at $t=a$ the investigation is similar.

Let $c\in I$. We consider equation (1) on the interval (c,b).

Consider the case, when

$$\int_c^b v(t)dt<\infty \ \text{ and } \int_c^b \rho^{1-p'}(t)dt<\infty. \tag{4}$$

It is easy to see that in this case $\lim_{d\to b}A(d,b)=0$. Then there exists $c\in I$ such that $A(c,b)<(\frac{1}{p})^{\frac{1}{p}}(\frac{1}{p'})^{\frac{1}{p'}}$. Therefore, on the basis of Lemma 2 equation (1) is nonoscillatory.

Now, suppose that

$$\int_c^b \rho^{1-p'}(t)dt=\infty. \tag{5}$$

Denote by $W_p^1(\rho,I_c)$ the set of locally absolutely continuous on $I_c=(c,b)$ functions, for which the following semi-norm

$$\|f\|_{W_p^1(\rho,I_c)}=\left(\int_c^b \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}} \tag{6}$$

is finite.

Let

$$W_{p,L}^1(\rho,I_c)=\{f\in W_p^1(\rho,I_c):\lim_{t\to c+}f(t)=0\}.$$

The closure of the set $\overset{\circ}{AC}_p\,(I_c)$ with respect to semi–norm (6) we denote by $\overset{\circ}{W}{}_p^1(\rho,I_c)$. By [13, Lemma 1.6] we have

Lemma 4. *The equality $\overset{\circ}{W}{}_p^1(\rho,I_c)=W_{p,L}^1(\rho,I_c)$ holds if and only if condition (5) holds.*

When (5) holds, due to Lemma 4, instead of inequality (3) it is proper to consider the following weighted Hardy inequality

$$\left(\int_c^b v(t)|f(t)|^p dt\right)^{\frac{1}{p}} \le C\left(\int_c^b \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}}, \ f \in W^1_{p,L}(\rho, I_c). \tag{7}$$

By efforts of many mathematicians there were obtained necessary and sufficient conditions of the validity of (7). Moreover, there are numerous estimates for the best constant C in (7) of the form

$$k_p A_p(\rho, v, c, b) \le C \le K_p A_p(\rho, v, c, b), \tag{8}$$

where positive constants k_p and K_p depend only on p, and $A_p(\rho, v, c, b)$ does not increase with respect to $c \in I$.

Theorem 2. *If for the best constant C in (7) the estimate of form (8) holds, then the condition*

$$\lim_{c\to b-} A_p(\rho, v, c, b) \le \frac{1}{k_p}$$

is necessary and the condition

$$\lim_{c\to b-} A_p(\rho, v, c, b) < \frac{1}{K_p}$$

is sufficient for equation (1) to be nonoscillatory at $t = b$.

Proof. If equation (1) is nonoscillatory at $t = b$, then by Lemma 1.2 there exists $c \in I$ such that $F(f, c, b) > 0$ for all nontrivial $f \in \overset{\circ}{W}{}^1_p(\rho, I_c) \equiv W^1_{p,L}(p, I_c)$ that is equivalent to the validity of (7) with $C = 1$. Then by (8) we have $k_p A_p(\rho, v, c, b) \le 1$ hence $\lim\limits_{c\to b-} A_p(\rho, v, c, b) \le \dfrac{1}{k_p}$.

Conversely, let $\lim\limits_{c\to b-} A_p(\rho, v, c, b) < \frac{1}{K_p}$. Then there exists $c \in I$ and $A_p(\rho, v, c, b) < \frac{1}{K_p}$, i.e., $K_p A_p(\rho, v, c, b) < 1$. Therefore, due to (8) the best constant C in (7) satisfies $C < 1$ that means that $F(f, c, b) > 0$ for all nontrivial $f \in W^1_{p,L}(\rho, I_c) \equiv \overset{\circ}{W}{}^1_p(\rho, I_c)$ and in particular, for all nontrivial $f \in \overset{\circ}{AC}_p(I_c)$. Hence by Lemma 2 equation (1) is nonoscillatory at $t = b$. □

Theorem 3. *If for the best constant C in (7) the estimate of form (8) holds, then the condition $\lim\limits_{c\to b-} A_p(\rho, v, c, b) \ge \frac{1}{K_p}$ is necessary and the condition $\lim\limits_{c\to b-} A_p(\rho, v, c, b) > \frac{1}{k_p}$ is sufficient for equation (1) to be oscillatory at $t = b$.*

Proof. Let equation (1) be oscillatory at $t = b$. By Lemma 3 for any $c \in I$ there exists a nontrivial $\tilde{f} \in \overset{\circ}{AC}_p(c, b) \subset W^1_{p,L}(p, I_c)$ such that $F(\tilde{f}, c, b) \le 0$, i.e.,

$$\int_c^b \rho(t)|\tilde{f}'(t)|^p dt \le \int_c^b v(t)|f(t)|^p dt.$$

This gives that the best constant C (finite or not) in (7) is such that $C \geq 1$. Hence by (8) we have that $K_p A_p(\rho, v, c, b) \geq 1$ for all $c \in I$. Therefore, $\lim\limits_{c \to b-} A_p(\rho, v, c, b) \geq \frac{1}{K_p}$.

Conversely, if $\lim\limits_{c \to b-} A_p(\rho, v, c, b) > \frac{1}{k_p}$, then due to the fact that $A_p(\rho, v, c, b)$ is non-increasing with respect to $c \in I$ we have that $A_p(\rho, v, c, b) > \frac{1}{k_p}$ for all $c \in I$. Hence, from (8) the best constant C (finite or not) in (7) is such that $C > 1$ for all $c \in I$. Hence for any $c \in I$ there exists a nontrivial function $\hat{f} \in \mathring{W}_p^1(\rho, I_c) \equiv W_{p,L}^1(\rho, I_c)$ such that $F(\hat{f}, c, b) \leq 0$. However, from the density of $\mathring{AC}_p\,(I_c)$ in $\mathring{W}_p^1(\rho, I_c)$ for any $\hat{f}$ there exists $\tilde{f} \in \mathring{AC}_p\,(I_c)$ such that $F(\tilde{f}, c, b) \leq 0$. Therefore, by Lemma 3 equation (1) is oscillatory. $\square$

Let $a = 0$ and $b = +\infty$. Suppose that functions ρ and v are continuous on $[0, +\infty)$.

Consider the estimates of form (8) of Muckenhoupt [11] and Tomaselli–Persson–Stepanov [16, 17]

$$A_M(c, \infty) \leq C \leq p^{\frac{1}{p}} (p')^{\frac{1}{p'}} A_M(c, \infty), \tag{9}$$

where

$$A_M(c, \infty) = \sup_{x > c} \left(\int_x^\infty v(t)dt \right)^{\frac{1}{p}} \left(\int_c^x \rho^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

and

$$A_{TPS}(c, \infty) \leq C \leq p' A_{TPS}(c, \infty), \tag{10}$$

where

$$A_{TPS}(c, \infty) = \sup_{x > c} \left(\int_c^x \rho^{1-p'}(t)dt \right)^{-\frac{1}{p}} \left(\int_c^x v(t) \left(\int_c^t \rho^{1-p'}(s)ds \right)^p dt \right)^{\frac{1}{p}}.$$

In 1992 V.M. Manakov [9] showed that the coefficient $K_p = p^{\frac{1}{p}} (p')^{\frac{1}{p'}}$ in (9) is exact when the condition (5) holds.

By (9) and (10) on the basis of Theorems 2 and 3 we have

Theorem 4. *Suppose that condition (5) holds. If one of the following two conditions*

$$\lim_{z \to \infty} \sup_{x > z} \left(\int_z^x \rho^{1-p'}(s)ds \right)^{p-1} \int_x^\infty v(t)dt < \frac{1}{p} \left(\frac{p-1}{p} \right)^{p-1} \tag{11}$$

or

$$\lim_{z \to \infty} \sup_{x > z} \left(\int_z^x \rho^{1-p'}(s)ds \right)^{-1} \int_z^x v(t) \left(\int_z^t \rho^{1-p'}(s)ds \right)^p dt < \left(\frac{p-1}{p} \right)^p \tag{12}$$

holds, then equation (1) is nonoscillatory at $t = \infty$.

Theorem 5. *Suppose that condition (5) holds. If one of the following two conditions*

$$\lim_{z\to\infty}\sup_{x>z}\left(\int_z^x \rho^{1-p'}(s)ds\right)^{p-1}\int_x^\infty v(t)dt > 1 \tag{13}$$

or

$$\lim_{z\to\infty}\sup_{x>z}\left(\int_z^x \rho^{1-p'}(s)ds\right)^{-1}\int_z^x v(t)\left(\int_z^t \rho^{1-p'}(s)ds\right)^{p} dt > 1 \tag{14}$$

holds, then equation (1) is oscillatory at $t=\infty$.

From Theorem 5 it follows as corollary the Hille criterion [1].

Corollary 1. *If together with (5) the following condition*

$$\int_0^\infty v(t)dt = \infty \tag{15}$$

holds, then the equation (1) is oscillatory.

In the oscillation theory of equation (1) the conditions of types (11) – (14) are obtained by a modified method of "Riccati technique" [1, Theorems 3.1.2, 3.1.3, and 3.1.6].

At present numerous necessary and sufficient conditions for the validity of (7) are known, they can be found in the papers [4, 8].

However, in order to apply the estimate of form (7) to oscillation theory of equation (1) we need to have more exact estimates for C.

A. Wedestig [18] obtained the following estimate for the best constant C in (7)

$$\sup_{1<s<p}\left(\frac{p^p(s-1)}{p^p(s-1)+(p-s)^p}\right)^{\frac{1}{p}} A_W(s,c,\infty) \le$$

$$\le C \le \inf_{1<s<p}\left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} A_W(s,c,\infty), \tag{16}$$

where

$A_W(s,c,\infty) =$

$$= \sup_{x>c}\left(\int_c^x \rho^{1-p'}(t)dt\right)^{\frac{s-1}{p}}\left(\int_x^\infty v(\tau)\left(\int_c^\tau \rho^{1-p'}(t)dt\right)^{p-s} d\tau\right)^{\frac{1}{p}}.$$

By (16) we have

Theorem 6. *Suppose that condition (5) holds. If for some $s \in (1,p)$ we have that*

$$\limsup_{c\to\infty}\sup_{x>c}\left(\int_c^x \rho^{1-p'}(t)dt\right)^{s-1}\int_x^\infty v(\tau)\left(\int_c^\tau \rho^{1-p'}(t)dt\right)^{p-s}d\tau < \left(\frac{p-s}{p-1}\right)^{p-1}, \tag{17}$$

then equation (1) is nonoscillatory at $t=\infty$. However, if for some $s \in (1,p)$ we have that

$$\limsup_{c\to\infty}\sup_{x>c}\left(\int_c^x \rho^{1-p'}(t)dt\right)^{s-1}\int_x^\infty v(\tau)\left(\int_c^\tau \rho^{1-p'}(t)dt\right)^{p-s}d\tau > 1+\frac{\left(1-\frac{s}{p}\right)^p}{s-1}, \tag{18}$$

then equation (1) is oscillatory at $t=\infty$.

Proof. Indeed, if for some $\sigma \in (1,p)$ inequality (17) holds, then there exists $c>0$ such that $A_W^p(\sigma,c,\infty) < \left(\frac{p-\sigma}{p-1}\right)^{p-1}$. Hence $\inf\limits_{1<s<p}\left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} A_W(s,c,\infty) < 1$ and by (16) the best constant C in (7) is such that $C<1$ that implies to the validity of the inequality $F(f,c,\infty)>0$ for all nontrivial $f \in W^1_{p,L}(\rho,I_c) \equiv \overset{\circ}{W}{}^1_p(\rho,I_c)$. Therefore, equation (1) is nonoscillatory at $t=\infty$.

If for some $\sigma \in (1,p)$ condition (18) is valid, then because of the fact that $A_W(\sigma,c,\infty)$ is non-increasing with respect to $c>0$ we get thut $A_W^p(\sigma,c,\infty) > 1+\frac{(1-\frac{\sigma}{p})^p}{\sigma-1}$ for all $c>0$. This gives that

$$\sup_{1<s<p}\left(\frac{p^p(s-1)}{p^p(s-1)+(p-s)^p}\right)A_W(s,c,\infty)>1$$

for all $c>0$. Hence by (16) it follows that for all $c>0$ the best constant C in (7) is such that $C>1$. Therefore, equation (1) is oscillatory. $\square$

Now, let

$$\int_c^b v(t)dt=\infty \ \text{ and } \ \int_c^b \rho^{1-p'}(t)dt<\infty. \tag{19}$$

In the oscillation theory of equation (1) for $\rho>0$ and $v>0$ the "reciprocity principle" holds, which states that the equation (1) and the equation

$$\left(v^{1-p'}(t)|y'(t)|^{p'-2}y'(t)\right)'+\rho^{1-p'}(t)|y(t)|^{p'-2}y(t)=0 \tag{20}$$

are simultaneously either oscillatory or nonoscillatory.

By (19) and Lemma 4 we have that $\overset{\circ}{W}{}^1_{p'}(v^{1-p'},I_c)=W^1_{p',L}(v^{1-p'},I_c)$. Instead of (7) for equation (20) consider the inequality

$$\left(\int_c^b \rho^{1-p'}(t)|f(t)|^{p'}dt\right)^{\frac{1}{p'}} \le C\left(\int_c^b v^{1-p'}(t)|f'(t)|^{p'}dt\right)^{\frac{1}{p'}}, \tag{21}$$

for $f \in W^1_{p',L}(v^{1-p'}, I_c)$.

From the estimate of form (8) for inequality (21) we get new conditions of oscillation and nonoscillation of equation (1) that are different from the conditions in Theorems 2 and 3.

Two-sided estimates for the best constant of (21) in the terms of Muckenhoupt and Tomaselli–Persson–Stepanov have the following form

$$\overline{A}_M(c,\infty) \le C \le p^{\frac{1}{p}}(p')^{\frac{1}{p'}}\overline{A}_M(c,\infty),$$

where

$$\overline{A}_M(c,\infty) = \sup_{x>c}\left(\int_c^x v(t)dt\right)^{\frac{1}{p}}\left(\int_x^\infty \rho^{1-p'}(s)ds\right)^{\frac{1}{p'}},$$

and

$$\overline{A}_{TPS}(c,\infty) \le C \le p\overline{A}_{TPS}(c,\infty),$$

where

$$\overline{A}_{TPS}(c,\infty) = \sup_{x>c}\left(\int_c^x v(t)dt\right)^{-\frac{1}{p'}}\left(\int_c^x \rho^{1-p'}(t)\left(\int_c^t v(s)ds\right)^{p'}dt\right)^{\frac{1}{p'}}.$$

Therefore, we can formulate the following

Theorem 7. *Suppose that condition (19) hold. If one of the following two conditions*

$$\lim_{z\to\infty}\overline{A}^{p'}_M(z,\infty) < \frac{1}{p'}\left(\frac{1}{p}\right)^{p'-1}$$

or

$$\lim_{z\to\infty}\overline{A}^{p'}_{TPS}(z,\infty) < \left(\frac{1}{p}\right)^{p'}$$

holds, then equation (1) is nonoscillatory.

However, if one of the following two conditions

$$\lim_{z\to\infty}\overline{A}^{p'}_M(z,\infty) > 1$$

or

$$\lim_{z\to\infty}\overline{A}^{p'}_{TPS}(z,\infty) > 1$$

holds, then equation (1) is oscillatory.

In this and in the previous theorems there are sufficiently large gaps between conditions for oscillation and nonoscillation. To have more exact conditions we need to have exact values of coefficients k_p and K_p. The pointed out gap would be “zero”, if we have exact values of the best constants in (7) and (21).

As an example let us consider the following equation

$$\left(t^{\alpha}|y'(t)|^{p-2}y'(t)\right)' + \gamma t^{-(p-\alpha)}|y(t)|^{p-2}y(t) = 0,\ \gamma > 0,\ t > 0. \tag{22}$$

At $\alpha \neq p-1$ we have that $\int\limits_1^{\infty} t^{\alpha(1-p')}dt = \infty$ or $\int\limits_1^{\infty} t^{-(p-\alpha)}dt = \infty$, i.e., one of the conditions (5) or (19) is holds. Therefore, if $\alpha < p-1$ inequality (7) for equation (22) has the form [5, Theorem 330]

$$\left(\int\limits_0^{\infty} \gamma t^{-(p-\alpha)}|f(t)|^p dt\right)^{\frac{1}{p}} < \frac{\gamma^{\frac{1}{p}} p}{|p-\alpha-1|}\left(\int\limits_0^{\infty} x^{\alpha}|f'(x)|^p dx\right)^{\frac{1}{p}}, \tag{23}$$

when $f \in W^1_{p,L}(I)$. However, if $\alpha > p-1$ the inequality (21) is equivalent to inequality (23) when $f \in W^1_{p,R}(I)$. In (23) the constant $\frac{\gamma^{\frac{1}{p}} p}{|p-\alpha-1|}$ is the best possible. Hence, if $\alpha \neq p-1$ equation (22) is disconjugate on $(0,\infty)$ if and only if $\gamma \leq \left(\frac{|p-\alpha-1|}{p}\right)^p$ or it is conjugate on $(0,\infty)$ if and only if $\gamma > \left(\frac{|p-\alpha-1|}{p}\right)^p$. Since the inequality (21) is valid on any interval (c,∞), $c > 0$, with the same constant, equation (20) is oscillatory if and only if $\gamma > \left(\frac{|p-\alpha-1|}{p}\right)^p$. This last statement is proved by another method in the book [1, Theorem 1.4.4].

Finally we would like to note that some spectral properties of the differential operator entering equation (1) were derived, again with the help of the Hardy-type inequalities, in the paper of P. Drabek and A. Kufner [2].

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