ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2024, Volume 15, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russia

YESMUKHANBET SAIDAKHMETOVICH SMAILOV

Doctor of physical and mathematical sciences, Professor Smailov Esmuhanbet Saidakhmetovich passed away on May 24, 2024, at the age of 78 years.

Esmuhanbet Saidakhmetovich was well known to the scientific community as a high qualified specialist in science and education, and an outstanding organizer. Fundamental scientific articles and textbooks written in various fields of the theory of functions of several variables and functional analysis, the theory of approximation of functions, embedding theorems, and harmonic analysis are a significant contribution to the development of mathematics.

E.S. Smailov was born on October 18, 1946, in the village of Kyzyl Kesik, Aksuat district, Semipalatinsk region. In 1963, he graduated from high school with a silver medal, and in the same year he entered the Faculty of Mechanics and Mathematics of the Kazakh State University (Almaty) named after Kirov (now named after Al-Farabi). In 1971 he graduated from graduate school at the Institute of Mathematics and Mechanics.

He defended his PhD thesis in 1973 (supervisor was K.Zh. Nauryzbaev) and defended his doctoral thesis “Fourier multipliers, embedding theorems and related topics” in 1997. In 1993 he was awarded the academic title of professor.

E.S. Smailov since 1972 worked at the Karaganda State University named after E.A. Buketov as an associate professor (1972-1978), the head of the department of mathematical analysis (1978-1986, 1990-2000), the dean of the Faculty of Mathematics (1983-1987) and was the director of the Institute of Applied Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan in Karaganda (2004 -2018).

Professor Smailov was one of the leading experts in the theory of functions and functional analysis and a major organizer of science in the Republic of Kazakhstan. He had a great influence on the formation of the Mathematical Faculty of the Karaganda State University named after E.A. Buketov and he made a significant contribution to the development of mathematics in Central Kazakhstan. Due to the efforts of Y.S. Smailov, in Karaganda an actively operating Mathematical School on the function theory was established, which is well known in Kazakhstan and abroad.

He published more than 150 scientific papers and 2 monographs. Under his scientific advice, 4 doctoral and 10 candidate theses were defended.

In 1999 the American Biographical Institute declared professor Smailov “Man of the Year” and published his biography in the “Biographical encyclopedia of professional leaders of the Millennium”.

For his contribution to science and education, he was awarded the Order of “Kurmet” (=“Honour”).

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Esmuhanbet Saidakhmetovich Smailov.

Short communications

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 15, Number 3 (2024), 88 – 93

KOLMOGOROV WIDTHS OF ANISOTROPIC FUNCTION CLASSES AND FINITE-DIMENSIONAL BALLS

A.A. Vasil'eva

Communicated by V.D. Stepanov

Key words: Kolmogorov widths, anisotropic norms, Sobolev classes, Nikol'skii classes

AMS Mathematics Subject Classification: 41A46.

Abstract. In this paper, we obtain order estimates for the Kolmogorov widths of anisotropic periodic Sobolev and Nikol'skii classes, as well as anisotropic finite-dimensional balls.

DOI: https://doi.org/10.32523/2077-9879-2024-15-3-88-93

Let $1 \le p_1, \ldots, p_d \le \infty$, $\overline{p} = (p_1, \ldots, p_d)$, $\mathbb{T} = [0, 2\pi]$. The anisotropic norm $\|\cdot\|_{L_{\overline{p}}(\mathbb{T}^d)}$ is defined by

$$\|f\|_{L_{\overline{p}}(\mathbb{T}^d)} = \Bigl(\int_{\mathbb{T}} \ldots \Bigl(\int_{\mathbb{T}} \Bigl(\int_{\mathbb{T}} |f(x_1, x_2, \ldots, x_d)|^{p_1}\, dx_1\Bigr)^{p_2/p_1} dx_2\Bigr)^{p_3/p_2} \ldots dx_d\Bigr)^{1/p_d}$$

for finite p_j; if $p_j = \infty$ for some j, the corresponding jth integral norm is replaced by an essential supremum.

The space $L_{\overline{p}}(\mathbb{T}^d)$ consists of the equivalence classes of measurable functions satisfying $\|f\|_{L_{\overline{p}}(\mathbb{T}^d)} < \infty$.

We define the anisotropic Sobolev and Nikol'skii classes as in [30]. For different definitions of generalized smoothness, see, e.g., [4].

Given $r > 0$, $\alpha \in \mathbb{R}$, we set

$$F_r(x, \alpha) = 1 + 2\sum_{k=1}^{\infty} k^{-r} \cos(kx - \alpha\pi/2), \quad x \in \mathbb{R}.$$

Let $\overline{r} = (r_1, \ldots, r_d)$, $\overline{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $\overline{p} = (p_1, \ldots, p_d)$, $r_j > 0$, $\alpha_j \in \mathbb{R}$, $1 \le p_j \le \infty$, $j = 1, \ldots, d$.

Definition 1. The Sobolev class $W^{\overline{r}}_{\overline{p},\overline{\alpha}}(\mathbb{T}^d)$ consists of functions f on $\mathbb{T}^d$ such that for all $j \in \{1, \ldots, d\}$ the integral representation

$$f(x_1, \ldots, x_d) = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi_j(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_d) F_{r_j}(x_j - y, \alpha_j)\, dy$$

holds with $\|\varphi_j\|_{L_{\overline{p}}(\mathbb{T}^d)} \le 1$.

Let $h \in \mathbb{R}$, $f \in L_{\overline{p}}(\mathbb{T}^d)$. We continue f periodically to $\mathbb{R}^d$ and set

$$\Delta_h^{1,j} f(x_1, \ldots, x_d) = f(x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_d) -$$

$$-f(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d).$$

For $l \in \mathbb{N}$, $l \geq 2$, the operator $\Delta_h^{l,j}$ is defined by the equality $\Delta_h^{l,j} = \Delta_h^{1,j} \circ \Delta_h^{l-1,j}$.

Definition 2. The Nikol'skii class $H^{\overline{r}}_{\overline{p}}(\mathbb{T}^d)$ consists of all functions $f \in L_{\overline{p}}(\mathbb{T}^d)$ such that

$$\|f\|_{L_{\overline{p}}(\mathbb{T}^d)} \leq 1, \quad \|\Delta_h^{l_j,j} f\|_{L_{\overline{p}}(\mathbb{T}^d)} \leq |h|^{r_j}, \quad h \in \mathbb{R}, \quad 1 \leq j \leq d,$$

where $l_j = \lfloor r_j \rfloor + 1$.

It is well-known [30, Theorem 3.4.6] that $W^{\overline{r}}_{\overline{p},\overline{\alpha}}(\mathbb{T}^d) \subset C(\overline{r}) H^{\overline{r}}_{\overline{p}}(\mathbb{T}^d)$, where $C(\overline{r})$ is a positive number depending only on $\overline{r}$.

Definition 3. Let X be a normed space, $M \subset X$, $n \in \mathbb{Z}_+$. The Kolmogorov n-width of M in X is defined by

$$d_n(M, X) = \inf_{L \in \mathcal{L}_n(X)} \sup_{x \in M} \inf_{y \in L} \|x - y\|;$$

here, $\mathcal{L}_n(X)$ is the family of all subspaces in X of dimension at most n.

Estimates for the widths of Sobolev classes in L_q on one-dimensional domains were obtained in [19, 15, 20, 21, 5]. In [25, 26, 27, 28], the problem of estimating the widths of Sobolev and Nikol'skii classes in $L_q(\mathbb{T}^d)$ was studied (see Definitions 1 and 2 for $p_1 = \cdots = p_d = p$); in [7, 8, 29], a similar problem was considered for periodic Sobolev and Nikol'skii classes with dominating mixed smoothness. For details, see [11, 31, 30]. In [7], the anisotropic norms were also considered, but only for the following cases: 1) $1 < q_j \leq p_j < \infty$, $1 \leq j \leq d$, 2) $1 < p_j \leq q_j \leq 2$, $1 \leq j \leq d$, 3) $2 \leq p_j \leq q_j < \infty$, $1 \leq j \leq d$, 4) $1 < p_j \leq 2 \leq q_j < \infty$, $1 \leq j \leq d$, $p_1 = \cdots = p_d$. (Actually, from the proof we can see that $p_1 = \cdots = p_d$, $q_1 = \cdots = q_d$ in case 3).) Moreover, in cases 3), 4), estimates of the widths were obtained only for the "large smoothness". The case of "small smoothness" was considered in [8] for multivariate functions and isotropic norms. In [9, 33, 34], estimates for the widths of intersections of Sobolev classes were obtained.

In [1, 2], estimates for the widths of Nikol'skii–Besov–Amanov classes in Lorentz spaces with anisotropic norms were obtained; the parameters p_j, q_j satisfied the above conditions 1)–4) (see also [3]).

In the present paper, we obtain order estimates for the Kolmogorov widths of $W^{\overline{r}}_{\overline{p},\overline{\alpha}}(\mathbb{T}^d)$ and $H^{\overline{r}}_{\overline{p}}(\mathbb{T}^d)$ in $L_{\overline{q}}(\mathbb{T}^d)$ for $2 \leq q_j < \infty$, $j = 1, \ldots, d$. The parameters $p_j \in [1, +\infty]$ are arbitrary, except some limit cases (see the condition $\theta_{j_*} < \min_{j \in J \setminus \{j_*\}} \theta_j$ in Theorem 1 below). In addition, the estimate for the widths is obtained in the case $1 \leq p_j \leq q_j \leq 2$ for $1 \leq j \leq \nu$, $1 \leq q_j \leq p_j \leq \infty$ for $\nu + 1 \leq j \leq d$ (see Theorem 2).

For $2 \leq q < \infty$, $1 \leq p \leq \infty$, we set

$$\omega_{p,q} = \begin{cases} 0 & \text{for } p > q, \\ \frac{1/p - 1/q}{1/2 - 1/q} & \text{for } 2 < p \leq q, \\ 1 & \text{for } 1 \leq p \leq 2. \end{cases} \tag{1}$$

Let $I \subset \{1, \ldots, d\}$ be a nonempty set, $\overline{p} = (p_1, \ldots, p_d)$. We define the number $\langle \overline{p} \rangle_I$ by the equation $\frac{1}{\langle \overline{p} \rangle_I} = \frac{1}{|I|} \sum\limits_{j \in I} \frac{1}{p_j}$. We also write $\langle \overline{p} \rangle := \langle \overline{p} \rangle_{\{1, \ldots, d\}}$. For $I = \varnothing$, we set $\langle \overline{p} \rangle_I = 1$.

Let σ be a permutation of d elements such that

$$\omega_{p_{\sigma(1)},q_{\sigma(1)}} \le \omega_{p_{\sigma(2)},q_{\sigma(2)}} \le \dots \le \omega_{p_{\sigma(d)},q_{\sigma(d)}}. \quad (2)$$

The numbers μ, $\nu \in \{0, \dots, d\}$ are defined by the equations

$$\{1, \dots, \mu\} = \{j: \ \omega_{p_{\sigma(j)},q_{\sigma(j)}} = 0\}, \ \{1, \dots, \nu\} = \{j: \ \omega_{p_{\sigma(j)},q_{\sigma(j)}} < 1\}. \quad (3)$$

By (5), the condition $j \in \{1, \dots, \mu\}$ is equivalent to the equation $p_{\sigma(j)} \ge q_{\sigma(j)}$ for $q_{\sigma(j)} > 2$, and to $p_{\sigma(j)} > q_{\sigma(j)}$, for $q_{\sigma(j)} = 2$; the condition $j \in \{1, \dots, \nu\}$ is equivalent to the equation $p_{\sigma(j)} > 2$.

Let

$$I(t, s) = \{\sigma(t), \sigma(t+1), \dots, \sigma(s-1), \sigma(s)\}, \quad 1 \le t \le s \le d.$$

For $\overline{a} = (a_1, \dots, a_d)$, $\overline{b} = (b_1, \dots, b_d)$, we set $\overline{a} \circ \overline{b} = (a_1b_1, \dots, a_db_d)$.

Now we give notation for order equalities. Let X, Y be sets, $f_1, f_2: \ X \times Y \to \mathbb{R}_+$. We write $f_1(x, y) \underset{y}{\asymp} f_2(x, y)$ if, for each $y \in Y$, there is $c(y) \ge 1$ such that $[c(y)]^{-1} f_2(x, y) \le f_1(x, y) \le c(y) f_2(x, y)$ for all $x \in X$.

Theorem 1. *Let $d \in \mathbb{N}$, $r_j > 0$, $\alpha_j \in \mathbb{R}$, $1 \le p_j \le \infty$, $2 \le q_j < \infty$, $j = 1, \dots, d$. Suppose that*

$$1 + \frac{d-\mu}{\langle \overline{r} \circ \overline{q} \rangle_{I(\mu+1, d)}} - \frac{d-\mu}{\langle \overline{r} \circ \overline{p} \rangle_{I(\mu+1, d)}} > 0.$$

We set

$$\theta_t = \frac{1}{\frac{t}{\langle \overline{r} \rangle_{I(1, t)}} + \frac{2(d-t)}{\langle \overline{r} \circ \overline{q} \rangle_{I(t+1, d)}}} \left(1 + (d-t)\left(\frac{1}{\langle \overline{r} \circ \overline{q} \rangle_{I(t+1, d)}} - \frac{1}{\langle \overline{r} \circ \overline{p} \rangle_{I(t+1, d)}}\right)\right),$$

$\mu \le t \le \nu$; if $\nu < d$ and there is $j \in \{\nu+1, \dots, d\}$ such that $q_{\sigma(j)} > 2$, we also write

$$\theta_d = \frac{\langle \overline{r} \rangle}{d} \left(1 + \frac{d-\nu}{2\langle \overline{r} \rangle_{I(\nu+1, d)}} - \frac{d-\nu}{\langle \overline{r} \circ \overline{p} \rangle_{I(\nu+1, d)}}\right).$$

Let $J = \{\mu, \mu+1, \dots, \nu\} \cup \{d\}$ if $\nu < d$ and there is $j \in \{\nu+1, \dots, d\}$ such that $q_{\sigma(j)} > 2$; otherwise, we set $J = \{\mu, \mu+1, \dots, \nu\}$. Suppose that there is $j_* \in J$ such that

$$\theta_{j_*} < \min_{j \in J \setminus \{j_*\}} \theta_j.$$

Then

$$d_n(W^{\overline{r}}_{\overline{p},\overline{\alpha}}(\mathbb{T}^d), L_{\overline{q}}(\mathbb{T}^d)) \underset{\overline{p},\overline{q},\overline{r},d}{\asymp} d_n(H^{\overline{r}}_{\overline{p}}(\mathbb{T}^d), L_{\overline{q}}(\mathbb{T}^d)) \underset{\overline{p},\overline{q},\overline{r},d}{\asymp} n^{-\theta_{j_*}}.$$

Theorem 2. *Let $r_j > 0$, $\alpha_j \in \mathbb{R}$, $1 \le j \le d$, $\nu \in \{0, \dots, d\}$, $1 \le p_j \le q_j \le 2$ for $1 \le j \le \nu$, $1 \le q_j \le p_j \le \infty$ for $\nu+1 \le j \le d$. Suppose that*

$$\theta := \frac{\langle \overline{r} \rangle}{d} \left(1 + \frac{\nu}{\langle \overline{r} \circ \overline{q} \rangle_{\{1, \dots, \nu\}}} - \frac{\nu}{\langle \overline{r} \circ \overline{p} \rangle_{\{1, \dots, \nu\}}}\right) > 0.$$

Then

$$d_n(W^{\overline{r}}_{\overline{p},\overline{\alpha}}(\mathbb{T}^d), L_{\overline{q}}(\mathbb{T}^d)) \underset{\overline{p},\overline{q},\overline{r},d}{\asymp} d_n(H^{\overline{r}}_{\overline{p}}(\mathbb{T}^d), L_{\overline{q}}(\mathbb{T}^d)) \underset{\overline{p},\overline{q},\overline{r},d}{\asymp} n^{-\theta}.$$

In order to prove Theorems 1, 2, we obtain order estimates for the widths of finite-dimensional balls (see Theorem 3 and Proposition 1 below). After that we apply the standard discretization method following [30].

Given $N \in \mathbb{N}$, $1 \le s \le \infty$, $(x_i)_{i=1}^N \in \mathbb{R}^N$, we set $\|(x_i)_{i=1}^N\|_{l_s^N} = \left(\sum_{i=1}^N |x_i|^s\right)^{1/s}$ for $s < \infty$, $\|(x_i)_{i=1}^N\|_{l_s^N} = \max_{1\le i\le N} |x_i|$ for $s = \infty$.

Let $k_1, \dots, k_d \in \mathbb{N}$, $1 \le p_1, \dots, p_d \le \infty$. By $l_{p_1,\dots,p_d}^{k_1,\dots,k_d}$ we denote the space $\mathbb{R}^{k_1\dots k_d} = \{(x_{j_1,\dots,j_d})_{1\le j_s\le k_s,\,1\le s\le d} :\ x_{j_1,\dots,j_d} \in \mathbb{R}\}$ with norm defined by induction: for $d = 1$ it is $\|\cdot\|_{l_{p_1}^{k_1}}$; for $d \ge 2$,

$$\|(x_{j_1,\dots,j_d})_{1\le j_s\le k_s,\,1\le s\le d}\|_{l_{p_1,\dots,p_d}^{k_1,\dots,k_d}} = \left\| \left(\|(x_{j_1,\dots,j_{d-1},j_d})_{1\le j_s\le k_s,\,1\le s\le d-1}\|_{l_{p_1,\dots,p_{d-1}}^{k_1,\dots,k_{d-1}}} \right)_{j_d=1}^{k_d} \right\|_{l_{p_d}^{k_d}}.$$

By $B_{p_1,\dots,p_d}^{k_1,\dots,k_d}$ we denote the unit ball of the space $l_{p_1,\dots,p_d}^{k_1,\dots,k_d}$.

For $d = 1$, estimates for the widths of these balls were obtained in [23, 24, 18, 19, 13, 14, 12]. The case $d = 2$ was studied in [9, 10, 16, 17, 22, 32, 6]; for details, see, e.g., [35].

Theorem 3. *Let $d \in \mathbb{N}$, $k_1, \dots, k_d \in \mathbb{N}$, $n \in \mathbb{Z}_+$, $n \le \frac{k_1\dots k_d}{2}$, $2 \le q_j < \infty$, $1 \le p_j \le \infty$, $j = 1, \dots, d$. Let σ be a permutation of $\{1, \dots, d\}$ such that (5) holds. The numbers $\mu \in \{0, \dots, d\}$ and $\nu \in \{0, \dots, d\}$ are defined by (5). We denote $p_j^* = \max\{p_j, 2\}$, $1 \le j \le d$. Then*

$$d_n(B_{p_1,\dots,p_d}^{k_1,\dots,k_d}, l_{q_1,\dots,q_d}^{k_1,\dots,k_d}) \underset{\overline{q}}{\asymp} \Phi(k_1, \dots, k_d, n) :=$$

$$:= \prod_{j=1}^{\mu} k_{\sigma(j)}^{1/q_{\sigma(j)}-1/p_{\sigma(j)}} \cdot \min\Big\{1, \min_{\mu+1\le t\le d} \prod_{j=\mu+1}^{t-1} k_{\sigma(j)}^{1/q_{\sigma(j)}-1/p^*_{\sigma(j)}} \times$$

$$\times (n^{-1/2} k_{\sigma(1)}^{1/2} \dots k_{\sigma(t-1)}^{1/2} k_{\sigma(t)}^{1/q_{\sigma(t)}} \dots k_{\sigma(d)}^{1/q_{\sigma(d)}})^{\omega_{p_{\sigma(t)},q_{\sigma(t)}}} \Big\};$$

in addition, for $\nu < d$,

$$\Phi(k_1, \dots, k_d, n) = \prod_{j=1}^{\mu} k_{\sigma(j)}^{1/q_{\sigma(j)}-1/p_{\sigma(j)}} \cdot \min\Big\{1, \min_{\mu+1\le t\le \nu} \prod_{j=\mu+1}^{t-1} k_{\sigma(j)}^{1/q_{\sigma(j)}-1/p_{\sigma(j)}} \times$$

$$\times (n^{-1/2} k_{\sigma(1)}^{1/2} \dots k_{\sigma(t-1)}^{1/2} k_{\sigma(t)}^{1/q_{\sigma(t)}} \dots k_{\sigma(d)}^{1/q_{\sigma(d)}})^{\omega_{p_{\sigma(t)},q_{\sigma(t)}}},$$

$$\prod_{j=\mu+1}^{\nu} k_{\sigma(j)}^{1/q_{\sigma(j)}-1/p_{\sigma(j)}} \cdot n^{-1/2} k_{\sigma(1)}^{1/2} \dots k_{\sigma(\nu)}^{1/2} k_{\sigma(\nu+1)}^{1/q_{\sigma(\nu+1)}} \dots k_{\sigma(d)}^{1/q_{\sigma(d)}} \Big\}.$$

The proof generalizes the arguments from [13, 32].

Proposition 1. *Let $\nu \in \{0, \dots, d\}$, $1 \le p_j \le q_j \le 2$ for $1 \le j \le \nu$, $1 \le q_j \le p_j \le \infty$ for $\nu + 1 \le j \le d$, $n \le \frac{k_1\dots k_d}{2}$. Then*

$$d_n(B_{p_1,\dots,p_d}^{k_1,\dots,k_d}, l_{q_1,\dots,q_d}^{k_1,\dots,k_d}) \asymp k_{\nu+1}^{1/q_{\nu+1}-1/p_{\nu+1}} \dots k_d^{1/q_d-1/p_d}.$$

This estimate is a simple corollary of Malykhin's and Rjutin's result [22, Theorem 1] on estimates of the widths of a product of multi-dimensional octahedra.

Acknowledgments

In conclusion, the author expresses her sincere gratitude to V.N. Temlyakov and G.A. Akishev for useful discussion and references.

References

[1] G.A. Akishev, *Estimates for Kolmogorov widths of the Nikol'skii–Besov–Amanov classes in the Lorentz space.* Proc. Steklov Inst. Math. (Suppl.). 296, suppl. 1 (2017), 1–12.

[2] G. Akishev, *Estimates for the Kolmogorov width of classes of small smoothness in Lorentz space.* XXIV International Conference "Mathematics. Economics. Education". IX International symposium "Fourier series and their applications". International Conference on stochastic methods. Materials. Rostov-na-Donu, 2016. P. 99 (in Russian).

[3] G. Akishev, *Estimates of M-term approximations of functions of several variables in the Lorentz space by a constructive method*, Eurasian Math. J., 15 (2024), no. 2, 8–32.

[4] S. Artamonov, K. Runovski, H.-J. Schmeisser, *Methods of trigonometric approximation and generalized smoothness. II.* Eurasian Math. J. 13 (2022), no. 4, 18–43.

[5] S.B. Babadzhanov, V.M. Tikhomirov, *Diameters of a function class in an L_p-space* ($p \geq 1$). Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk. 11 (1967), no. 2, 24–30.

[6] S. Dirksen, T. Ullrich, *Gelfand numbers related to structured sparsity and Besov space embeddings with small mixed smoothness.* J. Compl. 48 (2018), 69–102.

[7] E.M. Galeev, *Kolmogorov widths in the space $\widetilde{L}_q$ of the classes $\widetilde{W}_p^{\overline{\alpha}}$ and $\widetilde{H}_p^{\overline{\alpha}}$ of periodic functions of several variables.* Math. USSR-Izv. 27 (1986), no. 2, 219–237.

[8] E.M. Galeev, *Estimates for widths, in the sense of Kolmogorov, of classes of periodic functions of several variables with small-order smoothness.* Vestnik Moskov. Univ. Ser. I Mat. Mekh. (1987), no. 1, 26–30 (in Russian).

[9] E.M. Galeev, *Kolmogorov widths of classes of periodic functions of one and several variables.* Math. USSR-Izv. 36 (1991), no. 2, 435–448.

[10] E.M. Galeev, *Kolmogorov n-width of some finite-dimensional sets in a mixed measure.* Math. Notes. 58 (1995), no. 1, 774–778.

[11] E.M. Galeev, *Widths of functional classes and finite-dimensional sets.* Vladikavkaz. Mat. Zh. 13 (2011), no. 2, 3–14.

[12] A.Yu. Garnaev, E.D. Gluskin, *On widths of a Euclidean ball.* Dokl. Akad. Nauk SSSR. 277 (1984), no. 5 , 1048–1052 [Sov. Math. Dokl. 30 (1984), 200–204]

[13] E.D. Gluskin, *On some finite-dimensional problems of the theory of diameters.* Vestn. Leningr. Univ. 13 (1981), no. 3, 5–10 (in Russian).

[14] E.D. Gluskin, *Norms of random matrices and diameters of finite-dimensional sets.* Math. USSR-Sb. 48 (1984), no. 1, 173–182.

[15] R.S. Ismagilov, *Diameters of sets in normed linear spaces and the approximation of functions by trigonometric polynomials.* Russian Math. Surveys, 29 (1974), no. 3, 169–186.

[16] A.D. Izaak, *Kolmogorov widths in finite-dimensional spaces with mixed norms.* Math. Notes. 55 (1994), no. 1, 30–36.

[17] A.D. Izaak, *Widths of Hölder–Nikol'skij classes and finite-dimensional subsets in spaces with mixed norm.* Math. Notes. 59 (1996), no. 3, 328–330.

[18] B.S. Kashin, *The diameters of octahedra.* Usp. Mat. Nauk. 30 (1975), no. 4, 251–252 (in Russian).

[19] B.S. Kashin, *The widths of certain finite-dimensional sets and classes of smooth functions.* Math. USSR-Izv. 11 (1977), no. 2, 317–333.

[20] B.S. Kashin, *Widths of Sobolev classes of small-order smoothness.* Moscow Univ. Math. Bull., 36 (1981), no. 5, 62–66.

[21] Yu.I. Makovoz, *On a method of estimation from below of diameters of sets in Banach spaces.* Math. USSR-Sb. 16 (1972), no. 1, 139–146.

[22] Yu.V. Malykhin, K.S. Ryutin, *The product of octahedra is badly approximated in the $l_{2,1}$-metric.* Math. Notes. 101 (2017), no. 1, 94–99.

[23] A. Pietsch, *s-numbers of operators in Banach space.* Studia Math. 51 (1974), 201–223.

[24] M.I. Stesin, *Aleksandrov diameters of finite-dimensional sets and of classes of smooth functions.* Dokl. Akad. Nauk SSSR. 220 (1975), no. 6, 1278–1281 [Soviet Math. Dokl.].

[25] V.N. Temlyakov, *On the approximation of periodic functions of several variables with bounded mixed difference.* Soviet Math. Dokl. 22 (1980), 131–135.

[26] V.N. Temlyakov, *Widths of some classes of functions of several variables.* Soviet Math Dokl. 26 (1982), 619–622.

[27] V.N. Temlyakov, *Approximation of periodic functions of several variables by trigonometric polynomials, and widths of some classes of functions.* Math. USSR-Izv. 27 (1986), no. 2, 285–322.

[28] V.N. Temlyakov, *Approximations of functions with bounded mixed derivative.* Proc. Steklov Inst. Math. 178 (1989), 1–121.

[29] V.N. Temlyakov, *Approximation of functions with a bounded mixed difference by trigonometric polynomials, and the widths of some classes of functions.* Math. USSR-Izv. 20 (1983), no. 1, 173–187.

[30] V. Temlyakov, *Multivariate approximation.* Cambridge Univ. Press, 2018. 534 pp.

[31] V.M. Tikhomirov, *Theory of approximations.* In: Current problems in mathematics. Fundamental directions. vol. 14. (Itogi Nauki i Tekhniki) (Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1987), pp. 103–260 [Encycl. Math. Sci. vol. 14, 1990, pp. 93–243].

[32] A.A. Vasil'eva, *Kolmogorov and linear widths of the weighted Besov classes with singularity at the origin.* J. Approx. Theory. 167 (2013), 1–41.

[33] A.A. Vasil'eva, *Kolmogorov widths of the intersection of a finite family of Sobolev classes.* Eurasian Math. J. 13 (2022), no. 4, 88–93.

[34] A.A. Vasil'eva, *Kolmogorov widths of an intersection of a finite family of Sobolev classes.* Izv. Math. 88 (2024), no. 1, 18–42.

[35] A.A. Vasil'eva, *Estimates for the Kolmogorov widths of an intersection of two balls in a mixed norm.* Sb. Math. 215 (2024), no. 1, 74–89.

Anastasia Andreevna Vasil'eva
Moscow Center for Fundamental and Applied Mathematics
Lomonosov Moscow State University
GSP-1, Leninskiye gory, 1, MSU, Main building, faculty of mechanics and mathematics,
119991 Moscow, Russian Federation
E-mail: vasilyeva_nastya@inbox.ru

Received: 05.04.2024