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YESMUKHANBET SAIDAKHMETOVICH SMAILOV

Doctor of physical and mathematical sciences, Professor Smailov Esmuhanbet Saidakhmetovich passed away on May 24, 2024, at the age of 78 years.

Esmuhanbet Saidakhmetovich was well known to the scientific community as a high qualified specialist in science and education, and an outstanding organizer. Fundamental scientific articles and textbooks written in various fields of the theory of functions of several variables and functional analysis, the theory of approximation of functions, embedding theorems, and harmonic analysis are a significant contribution to the development of mathematics.

E.S. Smailov was born on October 18, 1946, in the village of Kyzyl Kesik, Aksuat district, Semipalatinsk region. In 1963, he graduated from high school with a silver medal, and in the same year he entered the Faculty of Mechanics and Mathematics of the Kazakh State University (Almaty) named after Kirov (now named after Al-Farabi). In 1971 he graduated from graduate school at the Institute of Mathematics and Mechanics.

He defended his PhD thesis in 1973 (supervisor was K.Zh. Nauryzbaev) and defended his doctoral thesis “Fourier multipliers, embedding theorems and related topics” in 1997. In 1993 he was awarded the academic title of professor.

E.S. Smailov since 1972 worked at the Karaganda State University named after E.A. Buketov as an associate professor (1972-1978), the head of the department of mathematical analysis (1978-1986, 1990-2000), the dean of the Faculty of Mathematics (1983-1987) and was the director of the Institute of Applied Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan in Karaganda (2004 -2018).

Professor Smailov was one of the leading experts in the theory of functions and functional analysis and a major organizer of science in the Republic of Kazakhstan. He had a great influence on the formation of the Mathematical Faculty of the Karaganda State University named after E.A. Buketov and he made a significant contribution to the development of mathematics in Central Kazakhstan. Due to the efforts of Y.S. Smailov, in Karaganda an actively operating Mathematical School on the function theory was established, which is well known in Kazakhstan and abroad.

He published more than 150 scientific papers and 2 monographs. Under his scientific advice, 4 doctoral and 10 candidate theses were defended.

In 1999 the American Biographical Institute declared professor Smailov “Man of the Year” and published his biography in the “Biographical encyclopedia of professional leaders of the Millennium”.

For his contribution to science and education, he was awarded the Order of “Kurmet” (=“Honour”).

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Esmuhanbet Saidakhmetovich Smailov.

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WEAVING FRAMES LINKED WITH FRACTAL CONVOLUTIONS

G. Verma, A. Eberhard

Communicated by Dachun Yang

Key words: frames, weaving frames, bases, fractals, convolutions, perturbation.

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Abstract. Weaving frames have been introduced to deal with some problems in signal processing and wireless sensor networks. More recently, the notion of fractal operator and fractal convolutions have been linked with perturbation theory of Schauder bases and frames. However, the existing literature has established limited connections between the theory of fractals and frame expansions. In this paper we define weaving frames generated via fractal operators combined with fractal convolutions. The aim is to demonstrate how partial fractal convolutions are associated to Riesz bases, frames and the concept of weaving frames in a Hilbert space. The context deals with ones-sided convolutions i.e both left and right partial fractal convolution operators on Lebesgue space L^p $(1 \leq p \leq \infty)$. Some applications using partial fractal convolutions with null function have been obtained for the perturbation theory of bases and weaving frames.

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1 Introduction

Transmission of signals using frames, a redundant set of vectors in a Hilbert space are preferred over orthonormal or Reisz basis as they minimize the chance of losses and errors in the process of signal transmission. Due to the useful applications in the characterization of function spaces, signal processing, and many other fields of applications, the theory of frames has developed rather rapidly in recent years. One amongst many recent generalisation of frames given by Bemrose et al. in [2] is introduction of a new concept of *weaving frames* in separable Hilbert spaces. Because of some potential applications such as in wireless sensor networks and distributed signal processing, frames and weaving frames have attracted many researchers attention [1, 4, 9, 11]. Furthermore, some variations of woven frames were also considered [10, 12, 25, 26].

This paper highlights the connection of weaving frames with fractal interpolation functions. Many authors have studies that the framework of fractal interpolation which makes it possible to enlarge and improve the classical methods of approximation theory. In previous papers, the authors defined fractal functions constructed by means of iterated function systems, see, e.g, [3, 19]. These maps are fractal perturbations of arbitrary continuous functions defined on compact intervals. Recently, in [18] the fractal convolution, an internal binary operation, has been treated as an operation between two functions, namely the germ function f and the base function b (aside from other elements such as partition and scale factors). Navascues [19, 20, 21] introduced fractal versions of functions, in spaces associated fractal operator and some related notions. The current literature [23, 24] contains some interesting developments in which new bases and frames are obtained from the old ones by using the algebraic operation of fractal convolution allowing construction of frames and bases consisting of self-referential functions.

In this piece of research, an interested reader will further notice the properties of a fractal operator and its relationship with perturbation of woven frames. The motivation is derived from an examination of one-sided fractal convolutions, which we call the left and right partial fractal convolutions, with a different perspective. To be particular, the objective is to extend the link between fractal convolutions with the perturbation theory of bases and frames for Lebesgue space L^p $(1 \le p \le \infty)$ encouraged by developments in [21, 28]. The current source of information is a sequel to study frames linked to fractal convolutions. Not only this, weaving properties of frames lay the primary foundation of the extension.

In Section 2, we collect some definitions, and recall known results that are available in the theory of frames [5, 6, 8] and for the concept of fractal interpolation functions [3, 18, 19], which will be used in following sections. Main results are contained in Section 3 and Section 4. Future plans are discussed in Section 5.

2 Background on frames and fractals

2.1 Preliminaries for frames

In this section, we provide an overview of the technical background on *frames* and related work relevant to our research.

Definition 1. Let $\mathcal{H}$ be a real (or complex) separable Hilbert space with inner product $\langle .,\rangle$. A countable sequence $\{f_k\} \subset \mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$, if there exist numbers $A,\ B > 0$ such that,

$$A\|f\|^2 \le \|\{\langle f, f_k\rangle\}\|_{\ell^2}^2 \le B\|f\|^2 \ \ \forall\ f \in \mathcal{H}.$$

The inequality is called the *frame inequality* of the frame.

The operator $V : \ell^2 \to \mathcal{H}$ defined as

$$V(\{c_k\}) = \sum_{k=1}^{\infty} c_k f_k,\ \ \{c_k\} \in \ell^2,$$

is called the *pre-frame operator* (or *synthesis operator*) and its adjoint operator $V^* : \mathcal{H} \to \ell^2$, which is called the *analysis operator* is given by

$$V^*(f) = \{\langle f, f_k\rangle\} \ \ \forall\ f \in \mathcal{H}.$$

By composing V and V^* we obtain the *frame operator* $S = VV^* : \mathcal{H} \to \mathcal{H}$ defined by

$$S(f) = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k,\ \forall\ f \in \mathcal{H}.$$

The frame operator S is a positive, self-adjoint and invertible operator on $\mathcal{H}$. This gives the *reconstruction formula* [6] for all $f \in \mathcal{H}$:

$$f = SS^{-1}f = \sum_{k=1}^{\infty} \langle S^{-1}f, f_k\rangle f_k \quad \left(= \sum_{k=1}^{\infty} \langle f, S^{-1}f_k\rangle f_k \right).$$

The scalars $\{\langle S^{-1}f, f_k\rangle\}$ are called *frame coefficients* of the vector $f \in \mathcal{H}$.

Definition 2 (Bessel sequence)**.** A family $\{\phi_m\} \subset \mathcal{H}$ is a *Bessel sequence for* $\mathcal{H}$ *if there exist a positive constant* K *such that,*

$$\sum_{m=0}^{\infty} |\langle f, \phi_m \rangle|^2 \leq K\|f\|^2, \ \forall \ f \in \mathcal{H}.$$

Definition 3 (Riesz sequence)**.** [5] A family $\{\phi_m\}$ in a separable Hilbert space $\mathcal{H}$ is a *Riesz sequence* for $\mathcal{H}$ if for all $\{c_m\} \in l^2(\mathbb{N}_0)$, there exists constants $0 < A \leq B < \infty$ such that,

$$A\sum_{m=0}^{\infty} |c_m|^2 \leq \sum_{m=0}^{\infty} \|c_m\phi_m\|^2 \leq B\sum_{m=0}^{\infty} |c_m|^2.$$

Definition 4 (Riesz basis)**.** [5] A collection of vectors $\{\phi_m\}$ in a Hilbert space $\mathcal{H}$ is a Riesz basis for $\mathcal{H}$ if it is the image of an orthonormal basis for H under an invertible linear transformation. In other words, if there is an orthonormal basis $\{e_k\}$ for $\mathcal{H}$ and an invertible transformation $\mathcal{T}$ such that $\mathcal{T}e_k = \phi_k$.

Let Ξ be a finite or countable index set.

Definition 5 (Weaving frames)**.** [2] Two frames $\{\phi_i\}_{i\in\Xi}$ and $\{\psi_i\}_{i\in\Xi}$ in a separable Hilbert space $\mathcal{H}$ are said to be woven, if there are universal positive constants A and B such that for every subset $\sigma \subset \Xi$, the family $\{\phi_i\}_{i\in\sigma} \cup \{\psi_i\}_{i\in\sigma^c}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds A and B, respectively.

Definition 6. [2] A family of frames $\{\phi_{ij}\}_{j=1,i\in\Xi}^{M}$ in $\mathcal{H}$ is said to be woven if there are universal constants A and B so that for every partition $\{\sigma_j\}_{j=1}^{M}$ of Ξ, the family $\{\phi_{ij}\}_{j=1,i\in\sigma_j}^{M}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds A and B, respectively. Each such family $\{\phi_{ij}\}_{j=1,i\in\sigma_j}^{M}$ is called a weaving.

Remark 1. If a family of frames $\{\phi_{ij}\}_{j=1,i\in\Xi}^{M}$ is a *Bessel sequence* for $\mathcal{H}$ with bound B_j, then every weaving is a *Bessel sequence* with the Bessel bound $\sum_{j=1}^{M} B_j$.

Remark 2. If $\{\phi_i\}_{i\in\Xi}$ is a Riesz basis with Riesz bounds A, B and π is a permutation of Ξ, then for every $\sigma \subset \Xi$, the family $\{\phi_i\}_{i\in\sigma} \bigcup \{\phi_{\pi(i)}\}_{i\in\sigma^c}$ is a frame sequence with bounds A and $2B$. However, $\{\phi_i\}_{i\in\Xi}$ and $\{\phi_{\pi(i)}\}_{i\in\Xi}$ are woven if and only if $\pi = \Xi_d$.

2.2 Preliminaries for fractal functions

In this section, we provide an overview of the technical background and related work relevant to fractal interpolation functions (FIFs). The notion of FIF can be used to associate a parameterized family of fractal functions with a prescribed function that belongs to a standard function space. We observe that the Read-Bajraktarević operator leads the way to address the technical details concerning this in the setting of $\mathcal{L}^p$-spaces, see [18, 20, 28].

Let $\Delta := \{x_0, x_1, \ldots, x_T\}$, where $T \in \mathbb{N}, T > 1$, $x_0 < x_1 \ldots, x_T$ and $I = [x_0, x_T]$. We denote by $\mathcal{L}^p(I)$, the Banach space of all real-valued Lebesgue integrable functions defined on I, equipped with $\mathcal{L}^p$-norm $\|.\|$ for $1 \leq p < \infty$. The *germ function* will be denoted by f. It is a prescribed function belonging to $\mathcal{L}^p(I)$. Next we shall define a family of fractal functions which are self-referential functions associated to f.

For $T \in \mathbb{N}$ denote by $\mathbb{N}_T$ the subset of $\mathbb{N}$ consisting of the first T natural numbers. For $t \in \mathbb{N}_{T-1}$, let $I_t = [x_{t-1}, x_t)$ and $I_T = [x_{T-1}, x_T]$. Note that $I = \cup_{t\in\mathbb{N}_t} I_t$ and each point in the partition is exactly in one of the subintervals I_t. For $t \in \mathbb{N}_T$, suppose that $L_t : [x_0, x_T] \to [x_{t-1}, x_t]$ are affine maps of the form $L_t(x) = a_t x + b_t$, where a_t and b_t are determined such that $L_t(x_0) = x_{t-1}$ and $L_t(x_T) = x_t$.

For a fixed base function $\beta \in \mathcal{L}^p(I)$, we pick a *scale factor* and *scale vector* (or a *scaling vector*) respectively, defined as $\alpha_t \in (-1, 1)$ for $t \in \mathbb{N}_T$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_T) \in (-1, 1)^T$.

To each $\rho \in \mathcal{L}^p(I)$, we associate a Read-Bajraktarević type operator $B^{\alpha,\beta}_{f,\Delta} : \mathcal{L}^p(I) \to \mathcal{L}^p(I)$ in contrast with the germ function f and the parameters Δ, β and α, and is as follows,

$$B^{\alpha,\beta}_{f,\Delta}\rho(x) = f(x) + \alpha_t(\rho - \beta) \circ L_t^{-1}(x)$$

for $x \in I_t, t \in \mathbb{N}_T$.

The steps to show that $B^{\alpha,\beta}_{f,\Delta}$ is a contraction map are fairly direct and, hence, by the Banach fixed point theorem, $B^{\alpha,\beta}_{f,\Delta}$ admits a unique fixed point $f^{\alpha,\beta}_{\Delta}$. Correspondingly, we have: $f^{\alpha,\beta}_{\Delta}$ satisfies the self-referential equation:

$$f^{\alpha,\beta}_{\Delta} = f(x) + \alpha_t(f^{\alpha,\beta}_{\Delta} - \beta) \circ L_t^{-1}(x). \tag{2.1}$$

Let us define

$$\Lambda = \max\{|\alpha_t| : t \in \mathbb{N}_T < 1\}$$

The following inequality can be easily obtained using equation (2.1):

$$\|f^{\alpha,\beta}_{\Delta} - f\| \leq \frac{\Lambda}{1-\Lambda}\|f - \beta\|. \tag{2.2}$$

Remark 3. Note that, in particular, $f^{\alpha,\beta}_{\Delta} = f$, at $\alpha = 0$. It is clear that $f^{\alpha,\beta}_{\Delta}$ may agree with f in specified subintervals, taking the corresponding zero scale factors.

We call, $f^{\alpha,\beta}_{\Delta}$ an α-fractal function or "fractalization" of f. In fact, with different choices of the parameters as specified above, we obtain a family of fractal functions $\{f^{\alpha,\beta}_{\Delta}\}$ corresponding to f.

In the remaining part of the work, for any $\rho \in \mathcal{L}^p(I)$ and a sequence $(a_m) \in L^p(\mathrm{I})$, the existence of $k := \sum_{m=0}^{\infty}\|\rho - a_m\| < +\infty$, will be termed as that the sequence (ρ, a_m) satisfies the *k-condition.*

2.3 Fractal convolutions

Navascues and Massopust have defined and studied the fractal convolution operator in detail for L^p spaces in [18, 22]. More recently, an extensive study of perturbation of bases and frames have been presented via fractal convolution in [24]. The underlying rule considers "fractalization" of f as convolution. The fractal convolution associated with an Read-Bajraktarević operator B acting on the Lebesgue spaces $L^p(I)$. With respect to a given partition Δ, and a fixed scaling vector α, it is simply a binary operation on $\mathcal{L}^p(I) \times \mathcal{L}^p(I)$ given by

Definition 7 (fractal convolution)**.** For a germ f and a base β in $L^p(\mathrm{I})$

$$\digamma_{\Delta,\alpha} : L^p(\mathrm{I}) \longrightarrow L^p(\mathrm{I})$$
$$\digamma_{\Delta,\alpha}(f, \beta) = f *_{\Delta,\alpha} \beta$$

where $f *_{\Delta,\alpha} \beta$ is the α-fractal function $f^{\alpha,\beta}_{\Delta}$.

For the rest of the paper, for brevity, let us denote

$$\mathcal{F}_{\Delta,\alpha} = \mathcal{F}$$
$$f *_{\Delta,\alpha} \beta = f * \beta$$

It is easy to check that for any $f, \beta \in L^p(\mathrm{I})$, the fractal convolution operator $\mathcal{F}$ is linear and bounded.

The goal of the paper is to link weaving frames and fractal convolutions. The conceptual approach focuses on the core of convolutions but in a one-sided way. We shall now look at the partial convolution or one-sided convolutions via left or via right as defined with the help of operator $\mathcal{F}$.

Definition 8 (left partial fractal convolution)**.** For a fixed f in $L^p(\mathrm{I})$,

$$\mathcal{F}^l_f(\beta) = f * \beta,\ \ \beta \in L^p(\mathrm{I})$$

This is called f-left partial fractal convolution.

Definition 9 (right partial fractal convolution)**.** For a fixed β,

$$\mathcal{F}^r_\beta(f) = f * \beta,\ \ f \in L^p(\mathrm{I})$$

This is called β-*right* partial fractal convolution.

The authors in [24] listed several straight forward properties of fractal convolution operator $\mathcal{F}$ and the partial fractal convolutions $\mathcal{F}^l_f$, $\mathcal{F}^r_\beta$ respectively. Let us now review these properties for our later use in woven.

1. The fractal convolution $f * \beta$ satisfies a fixed point equation analogous to equation (2.1)
$$(f * b)(x) = f(x) + \alpha_n((f * b) - b) \circ L_n^{-1}(x), \forall\ x \in I_n, \tag{2.3}$$
where $f^{\alpha,\beta}_\Delta$ has been replaced by $f * \beta$.

2. The following inequalities are direct corollaries of above equation (2.3), algebraically observed in [22]
$$\|f * \beta - f * \beta'\| = \|\mathcal{F}^l_f(\beta) - \mathcal{F}^l_f(\beta')\| \leq \frac{\Lambda}{1-\Lambda}\|\beta - \beta'\|, \tag{2.4}$$
$$\|f * \beta - f' * \beta\| = \|\mathcal{F}^r_\beta(f) - \mathcal{F}^r_\beta(f')\| \leq \frac{1}{1-\Lambda}\|f - f'\|. \tag{2.5}$$

3. The f-left partial convolution $\mathcal{F}^l_f$ is nonlinear and Lipschitz continuous. Furthermore, if $\Lambda < \frac{1}{2}$, then it is a contraction whose unique fixed point is f. Similarly, the operator $\mathcal{F}^r_b$ is nonlinear and Lipschitz continuous

4. The closeness of $f * \beta$ to f and β are transferred in the form of inequalities as a result of the above first two items in this list in conjunction with (a): the uniqueness of the fixed point of the operator $B^{\alpha,\beta}_{f,\Delta}$ and (b): establishing that the operator $\mathcal{F}$ is idempotent, that is, $b * b = b$, for any b. Therefore,
$$\|f * \beta - f\| = \|\mathcal{F}^l_f(\beta) - \mathcal{F}^l_f(f)\| \leq \frac{\Lambda}{1-\Lambda}\|f - \beta\|, \tag{2.6}$$
$$\|f * \beta - \beta\| = \|\mathcal{F}^r_\beta(f) - \mathcal{F}^r_\beta(\beta)\| \leq \frac{1}{1-\Lambda}\|f - \beta\|. \tag{2.7}$$

In the current work, we start by linking weaving frames with left partial fractal convolutions.

3 Background on frames and fractals

3 Main results

So far, we have not seen a mechanism to construct weaving frames linked to fractal convolutions. The results presented in the mentioned literature in Section 2.3 are a stepping stone to derive links between fractal convolutions and perturbation theory of bases and frames for Banach spaces and Hilbert spaces. The primary objective in this section is to provide interesting connections between fractal convolutions and cases in which weaving frames can be obtained. In the presence of left-partial fractal convolutions, our approach is well aligned to prove the following theorems on obtaining sufficient conditions for existence of weaving frames. Note that these characterisations of weaving of left convolved frames is studied in a Hilbert space setting.

Theorem 3.1. *In a Hilbert space $\mathcal{H} = L^2(I)$, where (δ_m) is an orthonormal basis, and a given convolved Riesz basis $(\rho * \{\delta_m\}_{m\in I})$ with bounds M_l and M_u respectively, where,*

$$M_l = \frac{1-\lambda-k}{1-\lambda} \quad \& \quad M_u = \frac{1-\lambda+k}{1-\lambda}$$

- *λ is as defined in Section 2.2.*
- *(ρ, δ_m) satisfies the k-*condition *which is bounded by $(1-\lambda)$.*

then for every $\sigma \subset I$, the family $(\rho * \{\delta_m\}_{m\in\sigma}) \bigcup (\rho * \{\delta_{\pi(m)}\}_{m\in\sigma^c})$ is a frame sequence with bounds M_l and $2M_u$.

Moreover, $(\rho * \{\delta_m\}_{m\in I})$ and $(\rho * \{\delta_{\pi(m)}\}_{m\in I})$ are woven if and only if $\pi = I_d$, π is a permutation of I.

We shall present the proof of Theorem 3.1 a little later after we have discussed the existence of the frame $(\rho * \phi_m)$ as a frame for $\mathcal{H} = L^2(\mathrm{I})$ (left convolved fractal frame), for any given frame (ϕ_m), see Proposition 2. But even before that, let us observe some sufficient conditions for the existence of a Schauder basis $(\rho * \beta_m)$ for $\mathcal{L}^p(I)$ (left convolved fractal basis) for any given Schauder basis (β_m) for $\mathcal{L}^p(I)$.

For a given Banach space $\mathcal{L}^p(I)$, we begin by considering a Schauder basis (β_m) of $\mathcal{L}^p(I), (1 \le p < \infty, m = 0, 1, \ldots)$, and a fixed $\rho \in \mathcal{L}^p(I)$. It is obvious that, for any f, we have $f = \sum_{m=0}^{\infty} c_m(f)\beta_m$, where for M^{th} partial sum operator R_M, the coefficients $c_m(f)$ satisfy

$$R_M(f) = \sum_{m=0}^{M} c_m(f)\beta_m,$$

Let us write, $\Re = \sup_M \| R_M \|$

Proposition 3.1. *For an arbitrary $\rho \in \mathcal{L}^p(I)$ and a normalised Scahuder basis (β_m) of $\mathcal{L}^p(I), (1 \le p < \infty, m = 0, 1, \ldots)$, if (ρ, δ_m) satisfies the k-*condition*, then the operator*

$$\Theta_\rho{}^l(f) = \sum_{m=0}^{\infty} c_m(f)(\rho * \beta_m)$$

is well defined, linear and bounded, where $c_m(f)$ are the coefficients of the expansion of f along the basis (β_m) as before.

A related phenomenon is that of obtaining left convolved Schauder basis of type $(\rho * \beta_m)$ for $\mathcal{L}^p(I)$, where it is mathematically observed that the process demands to drop the condition of normality in the given Schauder basis (β_m).

It is also worth noticing that in a Hilbert space $L^2(\mathrm{I})$, for any given orthonormal basis (δ_m), $(\rho * \delta_m), \rho \in L^2(\mathrm{I})$ is a Bessel sequence, provided (ρ, δ_m) satisfies the *k-condition.*

We now provide a proof of Theorem 3.1 in which the approach encompasses the use of Proposition 1 and the boundedness property of the Schauder basis (β_m) plus the norm of c_m, given in [24].

Proof of Theorem 3.1. Since any subsequence of a Reisz basis is a Reisz sequence with the same bounds, for every $\sigma \subset I$, we have that for any $f \in \overline{\text{span}}(\rho * \{\delta_m\}_{m\in\sigma}) \bigcup (\rho * \{\delta_{\pi(m)}\}_{m\in\sigma^c})$,

$$\sum_{m\in\sigma} |\langle f, \rho * \{\delta_m\}\rangle|^2 + \sum_{m\in\sigma^c} |\langle f, \rho * \{\delta_{\pi(m)}\}\rangle|^2 \\ \geq \sum_{\sigma\cup(\sigma^c\cap\pi(\sigma^c))} |\langle f, \rho * \{\delta_m\}\rangle|^2 \geq M_l \| f \|^2.$$

Clearly, we note that the upper bound is $2 * M_u$. Let us prove the woven part. If we assume that $\pi \neq I_d$, π is a permutation of I, which implies that $\pi(m_0) = r_0 \neq s_0$ for some $r_0, s_0 \in I$ $(\rho * \{\delta_m\}_{m\in I})$ and $(\rho * \{\delta_{\pi(m)}\}_{m\in I})$. Excluding s_0 from I, we can obtain a set in which δ_{r_0} appears twice and δ_{s_0} is absent. This is a contradiction to the fact that the closure of the span is the entire space. □

Theorem 3.2. *If a family $\{\phi_m\}$ is a frame for $\mathcal{H}$ with bounds $T, D > 0$, and $\rho \in L^2(I)$ is such that*

$$\sum_{m=0}^{\infty} \| (\rho - \phi_m) \|^2 \leq T(1-\lambda)^2, \tag{3.1}$$

$0 < \lambda < 1$, then $\gamma = \sum_{m=0}^{\infty} \| (\rho * \phi_m - \phi_m) \|^2 < T$ and $(\rho * \phi_m)$ constitutes a frame with the following frame bounds,

$$T\left(1 - \sqrt{\frac{\gamma}{T}}\right)^2 \quad \& \quad D\left(1 + \sqrt{\frac{\gamma}{D}}\right)^2$$

Proof. Using equation (2.7) and equation (3.1), it is an easy algebra to drive that,

$$\gamma = \sum_{m=0}^{\infty} \| (\rho * \phi_m - \phi_m) \|^2 < \frac{1}{(1-\lambda)^2} \sum_{m=0}^{\infty} \| (\rho - \phi_m) \|^2 < T.$$

The remaining algebra is given by the authors in Theorem 1 of [8]. □

Corollary 3.1. *Let families $\{\phi_m\}$ and $\{\psi_m\}$ be frames with bounds $T, D > 0$, and $T^*, D^* > 0$ respectively. Let $\rho \in L^2(I)$, be such that*

$$\sum_{m=0}^{\infty} \| (\rho - \phi_m) \|^2 \leq T_1(1-\lambda)^2 \quad \& \quad \sum_{m=0}^{\infty} \| (\rho - \psi_m) \|^2 \leq T_2(1-\lambda)^2.$$

Then for $\gamma_1 = \sum_{m=0}^{\infty} \parallel (\rho * \phi_m - \phi_m) \parallel^2$, $(\rho * \phi_m)$ *and* $(\rho * \psi_m)$ *constitute frames for* $\mathcal{H}$ *with the bounds*

$$P_1 = T\left(1 - \sqrt{\frac{\gamma_1}{T}}\right)^2 \quad \& \quad Q_1 = D\left(1 + \sqrt{\frac{\gamma_1}{D}}\right)^2.$$

Analogously for $\gamma_2 = \sum_{m=0}^{\infty} \parallel (\rho * \psi_m - \psi_m) \parallel^2$

$$P_2 = T^*\left(1 - \sqrt{\frac{\gamma_2}{T^*}}\right)^2 \quad \& \quad Q_2 = D^*\left(1 + \sqrt{\frac{\gamma_2}{D^*}}\right)^2.$$

Corollary 1 enables the interest in studying the perturbation of woven family of frames in a Hilbert space.

4 Perturbation of woven frames linked with fractal convolutions

We are interested to explore the connections of weaving frames linked with fractal convolutions with the classical perturbation result by Paley and Wiener [7], stating that a sequence that is sufficiently close to an (orthonormal) basis in a Hilbert space automatically forms a basis. Some basic results in perturbation of frames are provided in [5, 15, 16, 17].

The following theorem tells us that perturbations can be linked with woven convoluted frames by considering the closeness property of a family of frames in a special manner as you will see.

Theorem 4.1. *Let* $(\rho * \{\phi_m\}_{m\in\Xi})$ *and* $(\rho * \{\psi_m\}_{m\in\Xi})$ *be frames for* $\mathcal{H}$ *with the bounds* P_1, Q_1, *and* P_2, Q_2 *respectively. Suppose that there exists* $\mu \in (0,1)$ *such that*

$$\mu\left(\sqrt{Q_1} + \sqrt{Q_2}\right) \leq \frac{P_1}{2}. \tag{4.1}$$

Then, for all sequences $\{\xi_m\}_{m\in\Xi}$, *we have*

$$\| \sum_{m\in\Xi} \xi_m(\rho * \phi_m - \rho * \psi_m) \| \leq \mu \| \{\xi_m\}_{m\in\Xi} \|. \tag{4.2}$$

Moreover, for every $\sigma \subset \Xi$, *the family* $(\rho * \{\phi_m\}_{m\in\sigma^c}) \bigcup (\rho * \{\psi_m\}_{m\in\sigma})$ *is a frame for* $\mathcal{H}$ *with frame bounds* $\frac{P_1}{2}, Q_1 + Q_2$.

Proof. The goal is to obtain the lower and upper frame bounds of the weaving frame which are worked out separately below. The upper frame bound of the weaving can be obtained by the claim that for Bessel sequence $(\rho * \{\phi_m\}_{m\in\Xi})$ and $(\rho * \{\psi_m\}_{m\in\Xi})$, every weaving is bounded above by sum of their respective upper bounds, i.e $Q_1 + Q_2$ in this proof, see [2].

The algebra for calculating lower frame bound is more complex and we shall begin by introducing synthesis operator given by χ and ω respectively for the two frames linked with fractal convolutions $(\rho * \{\phi_m\}_{m\in\Xi})$ and $(\rho * \{\psi_m\}_{m\in\Xi})$. For any $\sigma \subset \Xi$ and any given orthogonal projection $\acute{P}$ onto span of standard orthonormal basis of $l^2(\Xi)$, let

$$\chi_\sigma(\{\xi_m\}_{m\in\Xi}) = \chi\acute{P}_\sigma(\{\xi_m\}_{m\in\Xi}) = \sum_{m\in\sigma} \xi_m \rho * \phi_m \tag{4.3}$$

Combining equation (4.3) with its analogous equation using ω and $\acute{P}$, we obtain a restatement of inequality (4.2) to be written as $\parallel \chi - \omega \parallel < \mu$.
Now for any $f \in \mathcal{H}$, consider

$$\| \sum_{m\in\sigma} \langle f, \rho * \psi_m \rangle \rho * \psi_m + \sum_{m\in\sigma^c} \langle f, \rho * \phi_m \rangle \rho * \phi_m \| . \tag{4.4}$$

Taking into account that $\sigma^c = \Xi - \sigma$, we can obtain

$$\| \sum_{m\in\sigma} \langle f, \rho * \psi_m \rangle \rho * \psi_m + \sum_{m\in\sigma^c} \langle f, \rho * \phi_m \rangle \rho * \phi_m \|$$

$$\begin{aligned}\geq & \| \sum_{m\in\Xi} \langle f, \rho * \phi_m \rangle \rho * \phi_m + \left(\sum_{m\in\sigma} \langle f, \rho * \psi_m \rangle \rho * \psi_m - \sum_{m\in\sigma} \langle f, \rho * \phi_m \rangle \rho * \phi_m \right) \| \\ & \geq P_1 \| f \| - \| \sum_{m\in\sigma} \langle f, \rho * \psi_m \rangle \rho * \psi_m - \sum_{m\in\sigma} \langle f, \rho * \phi_m \rangle \rho * \phi_m \| .\end{aligned}$$

Applying the fact that synthesis operator for the two frames linked with fractal convolutions $(\rho * \{\phi_m\}_{m\in\Xi})$ and $(\rho * \{\psi_m\}_{m\in\Xi})$, are given by χ and ω respectively, combining equation (4.1) and equation (4.2), with the help of basic algebra over properties of norms, we can compute,

$$\begin{aligned}\| \sum_{m\in\sigma} \langle f, \rho * \psi_m \rangle \rho * \psi_m - \sum_{m\in\sigma} \langle f, \rho * \phi_m \rangle \rho * \phi_m \| & \\ = \| \chi_\sigma {\chi_\sigma}^* f - \omega_\sigma {\omega_\sigma}^* f \| & \\ \leq \frac{P_1}{2} \| f \| .\end{aligned}$$

This leads us to prove that expression (4.4) $\geq \frac{P_1}{2} \| f \|$. □

In Remark 4, we observe the perturbation of weaving frames linked with fractal convolutions as image of bounded invertible operator for a given frame.

Remark 4. Let $(\rho * \{\phi_m\}_{m\in\Xi})$ be a frame for $\mathcal{H}$ with bounds P and Q respectively. For any bounded operator τ, such that

$$\|I_d - \tau\|^2 < \frac{P}{Q},$$

we have $(\rho * \{\phi_m\}_{m\in\Xi})$ and $(\rho * \{\tau\phi_m\}_{m\in\Xi})$ are woven.

The famous Minkowski's inequality implies that τ is invertible, and the fact that $(\rho * \{\tau\phi_m\}, m \in \Xi)$ is automatically a frame underpin the remark.

5 Conclusion and future plan

To conclude, we remind that in this article, we developed weaving properties of frames linked with fractal convolutions. One of the many problems concerned with diagnosis of woven frames, which are actually Reisz bases for Hilbert spaces have been studied. The finding in this paper have created broader impact in planning the future work in the direction of obtaining links of fractal convolutions and a variety of other frames. Not only this, there is an ample scope of solving problems that connect fractals with frames and bases.

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