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YESMUKHANBET SAIDAKHMETOVICH SMAILOV

Doctor of physical and mathematical sciences, Professor Smailov Esmuhanbet Saidakhmetovich passed away on May 24, 2024, at the age of 78 years.

Esmuhanbet Saidakhmetovich was well known to the scientific community as a high qualified specialist in science and education, and an outstanding organizer. Fundamental scientific articles and textbooks written in various fields of the theory of functions of several variables and functional analysis, the theory of approximation of functions, embedding theorems, and harmonic analysis are a significant contribution to the development of mathematics.

E.S. Smailov was born on October 18, 1946, in the village of Kyzyl Kesik, Aksuat district, Semipalatinsk region. In 1963, he graduated from high school with a silver medal, and in the same year he entered the Faculty of Mechanics and Mathematics of the Kazakh State University (Almaty) named after Kirov (now named after Al-Farabi). In 1971 he graduated from graduate school at the Institute of Mathematics and Mechanics.

He defended his PhD thesis in 1973 (supervisor was K.Zh. Nauryzbaev) and defended his doctoral thesis “Fourier multipliers, embedding theorems and related topics” in 1997. In 1993 he was awarded the academic title of professor.

E.S. Smailov since 1972 worked at the Karaganda State University named after E.A. Buketov as an associate professor (1972-1978), the head of the department of mathematical analysis (1978-1986, 1990-2000), the dean of the Faculty of Mathematics (1983-1987) and was the director of the Institute of Applied Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan in Karaganda (2004 -2018).

Professor Smailov was one of the leading experts in the theory of functions and functional analysis and a major organizer of science in the Republic of Kazakhstan. He had a great influence on the formation of the Mathematical Faculty of the Karaganda State University named after E.A. Buketov and he made a significant contribution to the development of mathematics in Central Kazakhstan. Due to the efforts of Y.S. Smailov, in Karaganda an actively operating Mathematical School on the function theory was established, which is well known in Kazakhstan and abroad.

He published more than 150 scientific papers and 2 monographs. Under his scientific advice, 4 doctoral and 10 candidate theses were defended.

In 1999 the American Biographical Institute declared professor Smailov “Man of the Year” and published his biography in the “Biographical encyclopedia of professional leaders of the Millennium”.

For his contribution to science and education, he was awarded the Order of “Kurmet” (=“Honour”).

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Esmuhanbet Saidakhmetovich Smailov.

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BARRIER COMPOSED OF PERFORATED RESONATORS AND BOUNDARY CONDITIONS

I.Y. Popov, E.S. Trifanova, A.S. Bagmutov, I.V. Blinova

Communicated by B.E. Kanguzhin

Key words: spectrum, Helmholtz resonator, boundary condition.

AMS Mathematics Subject Classification: 47B25, 35P15.

Abstract. We consider the Laplace operator with the Neumann boundary condition in a two-dimensional domain divided by a barrier composed of many small Helmholtz resonators coupled with the both parts of the domain through small windows of diameter $2a$. The main terms of the asymptotic expansions in a of the eigenvalues and eigenfunctions are considered in the case in which the number of the Helmholtz resonators tends to infinity. It is shown that such a homogenization procedure leads to some energy-dependent boundary condition in the limit. We use the method of matching the asymptotic expansions of boundary value problem solutions.

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1 Introduction

Construction of unusual boundary conditions, particularly, energy-dependent (see, e.g., [2]), is important for many physical applications. A possible way for doing this is as follows. Let us take a resonator with the boundary composed of many small resonators coupling to the main cavity. After performing the limiting procedure when the number of these small resonators tends to infinity one obtains the system with a boundary condition as a result of homogenization. The problem of such type was considered in [7]. Later it was investigated in the framework of the model of point-like windows [16]. A number of works were devoted to the problem of homogenization [3, 4, 5, 6, 11, 20, 23]. As for physical applications, the additional interest was recently excited by the problem of metamaterials creation [13, 20, 12, 15].

In the present paper, we consider the two-dimensional system of two resonators Ω^+ and Ω^- separated by a barrier composed of N identical small resonators Ω^N coupled to each big resonator through small windows of width $2a$. The geometry of the domain is shown in Fig. 1a.

We consider the asymptotics of an eigenvalue and an eigenfunction close to some eigenvalue of Ω^-. We use the method of matching the asymptotic expansions of boundary value problems [10, 9, 14, 22, 18, 8]. Briefly speaking, the scheme is as follows. We construct to circles of radii $\sqrt{a}$, $\sqrt{2a}$ centered at the center of each window (see Fig. 1b). One constructs the external asymptotic expansion in the exterior of each small circle and the internal asymptotic expansion in the interior of each larger circle. In the domain between the two circles we make matching of the asymptotic expansions. We obtain the main terms of the asymptotic expansion of the eigenvalue and the corresponding eigenfunction. We construct formal asymptotic expansion. The justification of the matching method is, e.g., in [9].

The next step is performing the limiting procedure as $N \to \infty$. We show that the limit of the eigenfunction satisfies the integral equation at the boundary which corresponds to some energy-dependent boundary condition analogous to delta-potential supported by line in $\mathbb{R}$.

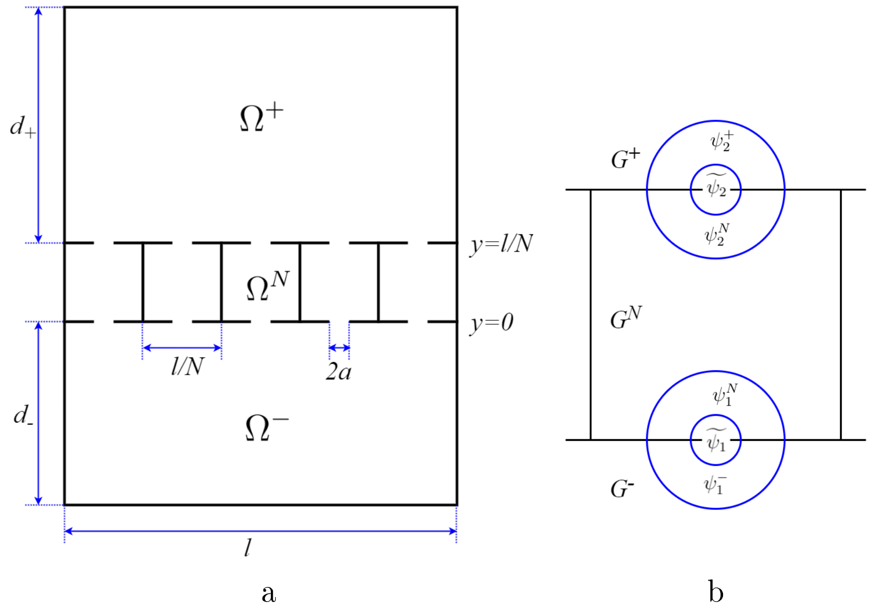

Figure 1: The geometry of the system: a – the whole system; b – construction of the domains for matching the asymptotic expansions. The Green function (G^+, G^N or G^-) which is used in the corresponding domain is indicated.

2 Matching of asymptotic expansions

First, let us consider the spectral problem for the Neumann Laplacian in $\Omega = \Omega^+ \cup \Omega^N \cup \Omega^-$, i.e. we deal with the following boundary problem

$$\Delta u + k^2 u = 0, \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \tag{2.1}$$

where $\partial\Omega$ is the domain boundary.

The Green functions for the upper (G^+), intermediate (G^N) and the lower (G^-) resonators with the Neumann boundary condition can be expressed using the corresponding eigenfunctions:

$$G^+(X, X', k) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{4}{ld_+} \frac{\cos(\frac{\pi n x}{l})\cos(\frac{\pi n x'}{l})\cos(\frac{\pi m y}{d_+})\cos(\frac{\pi m y'}{d_+})}{\left(k^2 - \frac{\pi^2 n^2}{l^2} - \frac{\pi^2 m^2}{d_+^2}\right)(\delta_{nm}+1)}, \tag{2.2}$$

$$G^N(X, X', k) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{4N^2}{l^2} \frac{\cos(\frac{\pi n N x}{l})\cos(\frac{\pi n N x'}{l})\cos(\frac{\pi m N y}{l})\cos(\frac{\pi m N y'}{l})}{\left(k^2 - \frac{\pi^2 N^2}{l^2}(n^2+m^2)\right)(\delta_{nm}+1)}, \tag{2.3}$$

$$G^-(X, X', k) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{4}{ld_-} \frac{\cos(\frac{\pi n x}{l})\cos(\frac{\pi n x'}{l})\cos(\frac{\pi m y}{d_-})\cos(\frac{\pi m y'}{d_-})}{\left(k^2 - \frac{\pi^2 n^2}{l^2} - \frac{\pi^2 m^2}{d_-^2}\right)(\delta_{nm}+1)}, \tag{2.4}$$

where $X = (x, y)$, $X' = (x', y')$, δ_{nm} is the Kronecker symbol, ($\delta_{nm} = 1$ for $n^2 + m^2 = 0$, otherwise $\delta_{nm} = 0$).

We will investigate the asymptotics in the small parameter a of the eigenvalues of the corresponding operator in the case in which there are small coupling windows (of width $2a$) connecting Ω^N with Ω^- and Ω^+. Naturally, we pose the Meixner condition at the windows edges. Let the considered eigenvalue k^2 be close to the following eigenvalue (λ_{11}) of the lower resonator: $k^2 \approx \lambda_{11} = \frac{\pi^2}{l^2} + \frac{\pi^2}{d_-^2}$.

We assume the following ansatz for the asymptotics:

$$\gamma_a := k_a^2 - \frac{\pi^2}{l^2} - \frac{\pi^2}{d_-^2} = \tau_1 \ln^{-1} a + \tau_2 \ln^{-2} a + o(\ln^{-2} a), \quad a \to 0+. \tag{2.5}$$

In such a case, the term $n = m = 1$ in (2.4) has a singularity.

To find the coefficients of asymptotic expansion (2.5), we use the conventional scheme of matching the asymptotic expansions of boundary value problems (see, e.g., [10, 9, 14, 22, 18, 8, 17]. Briefly speaking, it is as follows. Let the coupling windows be centered at points $(\frac{2jl+1}{2N}, 0), (\frac{2jl+1}{2N}, \frac{l}{N}), j = 0, \ldots N-1$. For each point, let us form two circles of radii $\sqrt{a}$ and $\sqrt{2a}$ centered at these points (see Fig. 1b). One constructs the internal asymptotic expansion of the solution inside the larger circle and the external asymptotic expansion of the solution outside the smaller circle. Correspondingly, in the ring between the two circles one has two asymptotic expansions. The proper expansion is obtained by matching of these two expansions in each ring.

We search a solution to (2.1) near the i-th coupling window in the following form:

Near the line $y = l/N$ we have $\psi_2(x)$:

$$\begin{cases} \psi_2^+(x, \frac{l}{N}) = -\gamma_a \sum_{j=1}^N \beta_j G^+((x, \frac{l}{N}), (x_j, \frac{l}{N}), k); & x \in \Omega^+, \\ \tilde{\psi}_2(x), & \text{matching domain}, \\ \psi_2^N = \gamma_a \Big[\alpha_i G^N((x, \frac{l}{N}), (x_i, 0), k) + \beta_i G^N((x, \frac{l}{N}), (x_i, \frac{l}{N}), k)\Big], & x \in \Omega^N; \end{cases} \tag{2.6}$$

near the line $y = 0$, $\psi_1(x)$:

$$\begin{cases} \psi_1^N = \gamma_a \Big[\alpha_i G^N((x, 0), (x_i, 0), k) + \beta_i G^N((x, 0), (x_i, \frac{l}{N}), k)\Big], & x \in \Omega^N, \\ \tilde{\psi}_1(x) & \text{matching domain}, \\ \psi_1^+(x, 0) = -\gamma_a \sum_{j=1}^N \alpha_j G^-((x, 0), (x_j, 0), k); & x \in \Omega^-. \end{cases} \tag{2.7}$$

The asymptotics of the Green functions near the coupling windows are as follows (here $\xi = \frac{x - x_i}{a}$). We have two small parameters: the width of window a and the distance between the neighbor windows centers ε. Correspondingly, one has the following asymptotics.

Near the line $y = l/N$:

$$G^+\Big((x, \frac{l}{N}), (x_i, \frac{l}{N}), k_a\Big) = -\frac{1}{\pi} \ln a + g_1^+(x) - \frac{1}{\pi} \ln|\xi|, \quad a \to 0; \tag{2.8}$$

$$G^+\Big((x, \frac{l}{N}), (x_i, \frac{l}{N}), k_a\Big) = -\frac{1}{\pi} \ln \varepsilon + h^+(x) - \frac{1}{\pi} \ln|\xi|, \quad \varepsilon \to 0; \tag{2.9}$$

$$G^N\Big((x, \frac{l}{N}), (x_i, \frac{l}{N}), k_a\Big) = -\frac{1}{\pi} \ln a + g_2^N(x) - \frac{1}{\pi} \ln|\xi|, \quad a \to 0; \tag{2.10}$$

$$G^N\Big((x, \frac{l}{N}), (x_i, 0), k_a\Big) = -\frac{1}{\pi} \ln \varepsilon + g_3^N(x) - \frac{1}{\pi} \ln|\xi|, \quad \varepsilon \to 0; \tag{2.11}$$

near the line $y = 0$:

$$G^-\Big((x, 0), (x_i, 0), k_a\Big) = -\frac{1}{\pi} \ln a + \frac{4}{ld_-} \frac{\cos(\frac{\pi x}{l}) \cos(\frac{\pi x_i}{l})}{k^2 - \frac{\pi^2}{l^2} - \frac{\pi^2}{d_-^2}} + g_4^-(x) - \frac{1}{\pi} \ln|\xi|, \quad a \to 0; \tag{2.12}$$

$$G^-\Big((x, 0), (x_i, 0), k_a\Big) = -\frac{1}{\pi} \ln \varepsilon + \frac{4}{ld_-} \frac{\cos(\frac{\pi x}{l}) \cos(\frac{\pi x_i}{l})}{k^2 - \frac{\pi^2}{l^2} - \frac{\pi^2}{d_-^2}} + h^-(x) - \frac{1}{\pi} \ln|\xi|, \quad \varepsilon \to 0; \tag{2.13}$$

$$G^N\Big((x,0),(x_i,0),k_a\Big) = -\frac{1}{\pi}\ln a + g_5^N(x) - \frac{1}{\pi}\ln|\xi|, \quad a\to 0; \tag{2.14}$$

$$G^N\Big((x,0),(x_i,\frac{l}{N}),k_a\Big) = -\frac{1}{\pi}\ln\varepsilon + g_6^N(x) - \frac{1}{\pi}\ln|\xi|, \quad \varepsilon\to 0. \tag{2.15}$$

Here g_1^+, h^+, $g_{2,3}^N$, g_4^-, h^-, $g_{5,6}^N$ are regular functions in the corresponding domains.

Let us assume that there is a relation between the small parameters:

$$\varepsilon = ma^\delta, \quad \ln\varepsilon = \delta\ln a + \text{const}, \quad \delta\in(0,1). \tag{2.16}$$

Taking equal terms of zero order of $\psi_2(x)$ on the line $y = l/N$ near the i-th window, one obtains the following relation

$$-\Big[\beta_i\big(-\frac{\tau_1}{\pi}\big) + \sum_{j\neq i}\beta_j\big(-\frac{\delta\tau_1}{\pi}\big)\Big] = \alpha_i\big(-\frac{\delta\tau_1}{\pi}\big) + \beta_i\big(-\frac{\tau_1}{\pi}\big), \tag{2.17}$$

or

$$\delta\alpha_i + 2\beta_i + \delta\sum_{j\neq i}\beta_j = 0. \tag{2.18}$$

The analogous operation with zero order terms of $\psi_1(x)$ on the line $y=0$ gives one the relation:

$$\begin{gathered}\alpha_i\big(-\tfrac{\tau_1}{\pi}\big) + \beta_i\big(-\tfrac{\delta\tau_1}{\pi}\big) = -\alpha_i\Big[-\tfrac{\tau_1}{\pi} + \tfrac{4}{ld_-}\cos^2\big(\tfrac{\pi x_i}{l}\big)\Big] - \\ \textstyle\sum_{j\neq i}\alpha_j\Big[-\tfrac{\delta\tau_1}{\pi} + \tfrac{4}{ld_-}\cos\big(\tfrac{\pi x_j}{l}\big)\cos\big(\tfrac{\pi x_i}{l}\big)\Big]\end{gathered} \tag{2.19}$$

or

$$\begin{gathered}\alpha_i\Big[\cos^2\big(\tfrac{\pi x_i}{l}\big) - \tfrac{\tau_1 ld_-}{2\pi}\Big] + \\ \textstyle\sum_{j\neq i}\alpha_j\Big[\cos\big(\tfrac{\pi x_i}{l}\big)\cos\big(\tfrac{\pi x_j}{l}\big) - \tfrac{\delta\tau_1 ld_-}{4\pi}\Big] - \beta_i\cdot\tfrac{\tau_1 ld_-}{4\pi} = 0.\end{gathered} \tag{2.20}$$

Let us denote for brevity

$$b = \frac{\tau_1 ld_-}{4\pi}, \quad \tilde{x}_i = \frac{\pi x_i}{l}. \tag{2.21}$$

Then

$$\alpha_i\Big[\cos^2(\tilde{x}_i) - 2b] + \sum_{j\neq i}\alpha_j\Big[\cos(\tilde{x}_i)\cos(\tilde{x}_j) - \delta b\Big] - \delta b\beta_i = 0. \tag{2.22}$$

Incorporating (2.18) and (2.22), we come to the following theorem.

Theorem 2.1. *If $\varepsilon = ma^\delta$, $\delta\in(0,1)$, then matching of terms of zero order in asymptotic expansion of a solution to* (2.1) *leads to the following system of equations for the coefficients α_i,β_i of representations* (2.7), (2.6)*:*

$$\begin{pmatrix}
\cos^2\tilde{x}_1 - 2b & \cos\tilde{x}_1\cos\tilde{x}_2 - \delta b & \dots & \cos\tilde{x}_1\cos\tilde{x}_N - \delta b & -\delta b & 0 & \dots & 0\\
\cos\tilde{x}_2\cos\tilde{x}_1 - \delta b & \cos^2\tilde{x}_2 - 2b & \dots & \cos\tilde{x}_2\cos\tilde{x}_N - \delta b & 0 & -\delta b & \dots & 0\\
\dots & \dots & \dots & \dots & 0 & \dots & \dots & 0\\
\cos\tilde{x}_N\cos\tilde{x}_1 - \delta b & \cos\tilde{x}_N\cos\tilde{x}_2 - \delta b & \dots & \cos^2\tilde{x}_N - 2b & 0 & 0 & \dots & -\delta b\\
\delta & 0 & \dots & 0 & 2 & \delta & \dots & \delta\\
0 & \delta & \dots & 0 & \delta & 2 & \dots & \delta\\
\dots & \dots & \dots & \dots & \delta & \delta & \dots & \delta\\
0 & 0 & \dots & \delta & \delta & \delta & \dots & 2
\end{pmatrix}
\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_N\\ \beta_1\\ \beta_2\\ \vdots\\ \beta_N\end{pmatrix} = 0. \tag{2.23}$$

A necessary and sufficient condition for the existence of non-trivial solutions to (2.23) is vanishing of the system determinant. The determinant can be exactly calculated [19]. This gives one the expression for b:

$$b = \frac{N}{8} \cdot \frac{2-\delta}{1-\delta}. \tag{2.24}$$

Taking into account (2.21), one obtains the expression for τ_1. This is resulted in the following theorem.

Theorem 2.2. *If* $\varepsilon = ma^{\delta}$, $\delta \in (0,1)$, *then the coefficient* τ_1 *of the main term of asymptotic expansion* (2.5) *of the eigenvalue close to* $\frac{\pi^2}{l^2} + \frac{\pi^2}{d_-^2}$ *is as follows:*

$$\tau_1 = \frac{\pi N}{2ld_-} \cdot \frac{2-\delta}{1-\delta}. \tag{2.25}$$

3 Integral equation

Let us consider the limiting case $N \to \infty$. Our goal is obtaining the integral equation for the limit of the eigenfunction corresponding to eigenvalue k_a^2.

Using asymptotics of the Green functions (2.8), (2.9), (2.12), (2.13) and expansion (2.5), one can obtain the asymptotic expansion for the eigenfunction at the upper and lower sides of the barrier:

$$\psi_1^-(x_i,0) = \alpha_i\Big(\frac{\tau_1}{\pi} - \frac{4}{ld_-}\cos^2 x_i\Big) + \sum_{j\neq i}\alpha_j\Big(\frac{\tau_1}{\pi}\delta - \frac{4}{ld_-}\cos x_i \cos x_j\Big) + o(1), \tag{3.1}$$

$$\psi_2^+(x_i,\frac{l}{N}) = \frac{\tau_1}{\pi}\beta_i + \frac{\tau_1}{\pi}\delta\sum_{j\neq i}\beta_j + o(1). \tag{3.2}$$

It is convenient to return from τ_1 to b: $\tau_1 = \frac{4\pi}{ld_-}b$. Then, one rewrites the equations:

$$-\frac{ld_-}{4}\psi_1^-(x_i,0) = \alpha_i(\cos^2 x_i - b) + \sum_{j\neq i}\alpha_j(\cos x_i\cos x_j - \delta b) + o(1), \tag{3.3}$$

$$\frac{ld_-}{4b}\psi_2^+(x_i,\frac{l}{N}) = \beta_i + \delta\sum_{j\neq i}\beta_j + o(1). \tag{3.4}$$

Let us join (2.18), (2.22), (3.3) and (3.4) into the system:

$$\begin{cases} \delta\alpha_i + 2\beta_i + \delta\sum_{j\neq i}\beta_j = 0, \\ \alpha_i(\cos^2(x_i) - 2b) - \delta b\beta_i + \sum_{j\neq i}\alpha_j(\cos x_i\cos x_j - \delta b) = 0, \\ \alpha_i(\cos^2 x_i - b) + \sum_{j\neq i}\alpha_j(\cos x_i\cos x_j - \delta b) = -\frac{ld_-}{4}\psi_1^-(x_i,0), \\ \beta_i + \delta\sum_{j\neq i}\beta_j = \frac{ld_-}{4b}\psi_2^+(x_i,\frac{l}{N}). \end{cases} \tag{3.5}$$

Solving (3.5) with respect to α_i and β_i, one obtains:

$$\alpha_i = \frac{ld_-}{4(1-\delta^2)b}\Big[\delta\psi_2^+(x_i,\frac{l}{N}) - \psi_1^-(x_i,0)\Big], \tag{3.6}$$

$$\beta_i = \frac{ld_-}{4(1-\delta^2)b}\Big[\delta\psi_1^-(x_i,0) - \psi_2^+(x_i,\frac{l}{N})\Big]. \tag{3.7}$$

Note that if one poses $\psi_1^-(x_i,0) = \psi_2^+(x_i,\frac{l}{N})$ then $\alpha_i = \beta_i$.

Let us substitute the obtained expressions in (2.6) and (2.7), summarize them and use expression (2.24) for b. Then one obtains

$$\psi_1^-(x,0)+\psi_2^+(x,\frac{l}{N}) = \frac{2d_-}{(1+\delta)(2-\delta)}\gamma_a \times \sum_{i=1}^{N}\frac{l}{N}\Big[G^-(x,x_i)\psi_1^-(x_i,0)+G^+(x,x_i)\psi_2^+(x_i,\frac{l}{N})\Big]+ \frac{l}{N}\delta\Big[G^-(x,x_i)\psi_2^+(x_i,\frac{l}{N})+G^+(x,x_i)\psi_1^-(x_i,0)\Big]. \quad (3.8)$$

We assume that $\psi_1^-(x,0)=\psi_2^+(x,\frac{l}{N})=\psi(x)$. It leads to simplifying the expression:

$$2\psi(x) = \frac{2d_-(1-\delta)}{(1+\delta)(2-\delta)}\gamma_a\sum_{i=1}^{N}\frac{l}{N}\Big(G^-(x,x_i)+G^+(x,x_i)\Big)\psi(x_i). \quad (3.9)$$

One can see that the sum in the right hand side is the integral sum which turns in the integral over Γ, i.e. over segment $[0,l]$ in our case. Correspondingly, we come to the integral equation. As a result, we obtain the following theorem.

Theorem 3.1. *If* $\varepsilon = ma^\delta$, $\delta\in(0,1)$, *then the eigenfunction corresponding to the eigenvalue close to* $\frac{\pi^2}{l^2}+\frac{\pi^2}{d_-^2}$ *tends as* $N\to\infty$ *to the function satisfying the following integral equation:*

$$\psi(x) = \frac{d_-(1-\delta)}{(1+\delta)(2-\delta)}\gamma_a\int_0^l \Big(G^-(x,x',k_a)+G^+(x,x',k_a)\Big)\psi(x')dx'. \quad (3.10)$$

4 Boundary condition and integral equation

After performing the limiting procedure $N\to\infty$, one comes to a rectangle $\Omega^+\cup\Omega^-$ divided by line $\Gamma=\{(x,y),y=0\}$. Let us consider the following boundary problem for $u=u(x,y)$:

$$\begin{gathered}\Delta u + k^2 u = 0, (x,y)\in\Omega^+\cup\Omega^-,\\ \frac{\partial u}{\partial n}\Big|_+ = \frac{\alpha}{2}u\Big|_\Gamma,\quad \frac{\partial u}{\partial n}\Big|_- = \frac{\alpha}{2}u\Big|_\Gamma,\quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega^\pm\setminus\Gamma}=0,\\ u(x,0+)=u(x,0-).\end{gathered} \quad (4.1)$$

Here n is the inward normal for the corresponding domain.
Remark. Note that boundary conditions (4.1) on Γ correspond to the jump of the derivative at Γ:

$$\frac{\partial u}{\partial y}\Big|_+ - \frac{\partial u}{\partial y}\Big|_- = \alpha u\Big|_\Gamma,$$

i.e. to 1D delta-potential at $y=0$ in the transverse direction.

Theorem 4.1. *A solution* u *to boundary problem* (4.1) *satisfies the following integral equation:*

$$u(x) = \frac{\alpha}{2}\int_\Gamma \Big(G^-(x,x',k_a)+G^+(x,x',k_a)\Big)u(x')dx',\quad x'\in\Gamma. \quad (4.2)$$

Proof. Let us multiply the normal derivative of u by the Green function for the Neumann Laplacian $G^{\pm}((x,y),(x',y'),k)$ in $\Omega^{\pm}$ and integrate over the boundary of $\partial\Omega^{-} \cup \partial\Omega^{+}$. Due to the properties of the Green function, one obtains the value of u in $\Omega^{\pm}$

$$u(x) = \int\limits_{\partial\Omega^{\pm}} G^{\pm}(x,x',k_a)\frac{\partial u}{\partial n}(x')dx', \quad x \in \Omega^{\pm}. \tag{4.3}$$

Note that $\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega^{\pm}\setminus\Gamma} = 0$. Correspondingly, only the integral over Γ is non-zero in the right hand side of (4.3). Considering the limiting value of u at $\partial\Omega^{\pm}$ and taking into account boundary conditions (4.1), one comes to the following integral equation:

$$u(x) = \frac{\alpha}{2}\int\limits_{\Gamma}\Big(G^{-}(x,x',k_a) + G^{+}(x,x',k_a)\Big)u(x')dx', \quad x' \in \Gamma. \tag{4.4}$$

One can see that equation (4.4) coincides with equation (3.10) if

$$\alpha = 2\frac{d_{-}(1-\delta)}{(1+\delta)(2-\delta)} \quad \gamma_a.$$

□

Remark. Numerical results for resonator with a boundary composed of many Helmholtz resonators [1] show that the ratio of the eigenfunction normal derivative and the eigenfunction value stabilizes near a value depending on energy but independent of the point position at the boundary. It is in agreement with the results obtained above.

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References

[1] A.S. Bagmutov, E.S. Trifanova, I.Y. Popov, *Resonator with corrugated boundary: Numerical results.* Phys. Part. Nucl. Letters. 20 (2023), no. 2, 110–115.

[2] J. Behrndt, *Elliptic boundary value problems with k-dependent boundary conditions.* J. Differential Equations. 249 (2010), 2663–2687.

[3] M.Sh. Birman, T.A. Suslina, *Homogenization with corrector term for periodic elliptic differential operators.* St. Petersburg Math. J. 17 (2006), 897–973.

[4] D. Borisov, R. Bunoiu, G. Cardone, *Homogenization and asymptotics for a waveguide with an infinite number of closely located small windows.* J. Math. Sci. 176 (2011), 774–785.

[5] R. Brizzi, J.-P. Chalot, *Boundary homogenization and Neumann boundary value problem.* Ricerche Mat. 46 (1997), 341–387.

[6] G. Cardone, T. Durante, *Asymptotic analysis of a planar waveguide perturbed by a non periodic perforation.* Nanosyst. Phys. Chem. Math. 13 (2022), 5–13.

[7] G. Cardone, A. Khrabustovskyi, *Neumann spectral problem in a domain with very corrugated boundary.* J. Differential Equations. 259 (2015) no. 6, 2333–2367 . https://doi.org/10.1016/j.jde.2015.03.031

[8] S.V. Frolov, I.Yu. Popov, *Resonances for laterally coupled quantum waveguides.* J. Math. Phys. 41 (2000), no. 7, 4391–4405.

[9] R.R. Gadilshin, *Surface potentials and method of matching of asymptotic expansions in the problem of Helmholtz resonator.* Alg. Anal. (Leningrad Math. J.) 4 (1992), 88-115.

[10] A.M. Il'in, *Matching of asymptotic expansions of solutions of boundary value problems.* Nauka, Moscow, 1989 (in Russian); English transl. (Transl. Math. Monographs, 102), Amer. Math. Soc., Providence (1992).

[11] A. Khrabustovskyi, *Homogenization of eigenvalue problem for Laplace-Beltrami operator on Riemannian manifold with complicated "bubble-like" microstructure.* Math. Methods Appl. Sci. 32 (2009), 2123–2137.

[12] A. Maurel, J.-J. Marigo, J.-F. Mercier, K. Pham, *Modelling resonant arrays of the Helmholtz type in the time domain.* Proc. R. Soc. A. 474 (2018), 20170894. http://dx.doi.org/10.1098/rspa.2017.0894

[13] X.Ni, K. Chen, M. Weiner, D.J. Apigo, C. Prodan, A. Alu, E. Prodan, A.B. Khanikaev, *Observation of Hofstadter butterfly and topological edge states in reconfigurable quasi-periodic acoustic crystals.* Commun. Phys. 2 (2019), 55. https://doi.org/10.1038/s42005-019-0151-7

[14] I.Yu. Popov, *Asymptotics of bound states for laterally coupled waveguides.* Rep. Math. Phys. 43 (1999), 427–437.

[15] I.Y. Popov, *A model of charged particle on the flat Möbius strip in a magnetic field.* Nanosystems: Phys. Chem. Math., 14 (2023), no. 4, 418-420.

[16] I.Yu. Popov, I.V. Blinova, A.I. Popov, *A model of a boundary composed of the Helmholtz resonators.* Complex Variables and Elliptic Equations, 66 (2021), no. 8, 1256-1263. https://doi.org/10.1080/17476933.2020.1751138

[17] I. Popov, T.S. Yurova, *Resonances for Laplacian perturbed on surface and cell membrane model.* Boletin de la Sociedad Matematica Mexicana, 29 (2023), no. 3, 85.

[18] I.Yu. Popov, E.S. Tesovskaya, *Electron in a multilayered magnetic structure: resonance asymptotics.* Theoretical and Mathematical Physics, 146 (2006), no. 3, 361–372.

[19] I.V. Proskuryakov, *Problems in linear algebra.* English translation, Mir Publishers, Moscow, 1978.

[20] L. Schwan, O. Umnova, C. Boutin, J.-P. Groby, *Nonlocal boundary conditions for corrugated acoustic metasurface with strong near-field interactions.* Journal of Applied Physics. 123 (2018), 091712. https://doi.org/10.1063/1.5011385

[21] T.A. Suslina, *Homogenization of elliptic operators with periodic coefficients depending on the spectral parameter.* St. Petersburg Math. J. 27 (2016), no. 4, 651–708.

[22] A.M. Vorobiev, A.S. Bagmutov, A.I. Popov, *On formal asymptotic expansion of resonance for quantum waveguide with perforated semitransparent barrier.* Nanosyst. Phys. Chem. Math. 10 (2019), no. 4, 415–419.

[23] V.V. Zhikov, *Spectral method in homogenization theory.* Proc. Steklov Inst. Math. 250 (2005), 85–94.

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