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YESMUKHANBET SAIDAKHMETOVICH SMAILOV

Doctor of physical and mathematical sciences, Professor Smailov Esmuhanbet Saidakhmetovich passed away on May 24, 2024, at the age of 78 years.

Esmuhanbet Saidakhmetovich was well known to the scientific community as a high qualified specialist in science and education, and an outstanding organizer. Fundamental scientific articles and textbooks written in various fields of the theory of functions of several variables and functional analysis, the theory of approximation of functions, embedding theorems, and harmonic analysis are a significant contribution to the development of mathematics.

E.S. Smailov was born on October 18, 1946, in the village of Kyzyl Kesik, Aksuat district, Semipalatinsk region. In 1963, he graduated from high school with a silver medal, and in the same year he entered the Faculty of Mechanics and Mathematics of the Kazakh State University (Almaty) named after Kirov (now named after Al-Farabi). In 1971 he graduated from graduate school at the Institute of Mathematics and Mechanics.

He defended his PhD thesis in 1973 (supervisor was K.Zh. Nauryzbaev) and defended his doctoral thesis "Fourier multipliers, embedding theorems and related topics" in 1997. In 1993 he was awarded the academic title of professor.

E.S. Smailov since 1972 worked at the Karaganda State University named after E.A. Buketov as an associate professor (1972-1978), the head of the department of mathematical analysis (1978-1986, 1990-2000), the dean of the Faculty of Mathematics (1983-1987) and was the director of the Institute of Applied Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan in Karaganda (2004 -2018).

Professor Smailov was one of the leading experts in the theory of functions and functional analysis and a major organizer of science in the Republic of Kazakhstan. He had a great influence on the formation of the Mathematical Faculty of the Karaganda State University named after E.A. Buketov and he made a significant contribution to the development of mathematics in Central Kazakhstan. Due to the efforts of Y.S. Smailov, in Karaganda an actively operating Mathematical School on the function theory was established, which is well known in Kazakhstan and abroad.

He published more than 150 scientific papers and 2 monographs. Under his scientific advice, 4 doctoral and 10 candidate theses were defended.

In 1999 the American Biographical Institute declared professor Smailov "Man of the Year" and published his biography in the "Biographical encyclopedia of professional leaders of the Millennium".

For his contribution to science and education, he was awarded the Order of "Kurmet" (="Honour").

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Esmuhanbet Saidakhmetovich Smailov.

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THE SPECTRUM AND PRINCIPAL FUNCTIONS OF A NONSELF-ADJOINT STURM–LIOUVILLE OPERATOR WITH DISCONTINUITY CONDITIONS

N.P. Kosar, O. Akcay

Communicated by B.E. Kanguzhin

Key words: nonself-adjoint Sturm–Liouville operator, discontinuity conditions, eigenvalues and spectral singularities, principal functions.

AMS Mathematics Subject Classification: 34B24, 34L05, 47A10.

Abstract. This paper deals with the nonself-adjoint Sturm–Liouville operator (or one-dimensional time-independent Schrödinger operator) with discontinuity conditions on the positive half line. In this study, the spectral singularities and the eigenvalues are investigated and it is proved that this problem has a finite number of spectral singularities and eigenvalues with finite multiplicities under two additional conditions. Moreover, we determine the principal functions with respect to the eigenvalues and the spectral singularities of this operator.

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1 Introduction

In mathematical physics, when we investigate solutions of partial differential equations under given initial and boundary conditions using the Fourier method, we encounter the following types of problems: to determine the eigenvalues and eigenfunctions of differential operators and to expand an arbitrary function as a series of eigenfunctions. Therefore, since it is interesting to study these types of problems, many works have been done on such problems and continue to be done. An important and interesting problem is that of the examination of the spectrum and expanding a given function via eigenfunctions of a differential operator which is not self-adjoint.

In the present paper, we examine the spectrum and the principal functions of a nonself-adjoint Sturm–Liouville operator with discontinuity conditions on the positive half plane. That is, we deal with in the following nonself-adjoint problem for the Sturm–Liouville equation

$$-\omega'' + q(x)\omega = \mu^2\omega, \quad x \in (0,a) \cup (a,\infty), \tag{1.1}$$

with the discontinuity conditions

$$\omega(a-0) = \alpha\omega(a+0), \qquad \omega'(a-0) = \alpha^{-1}\omega'(a+0) \tag{1.2}$$

and the boundary condition

$$\omega(0) = 0, \tag{1.3}$$

where $a > 0$, $0 < \alpha \neq 1$, μ is a complex parameter, $q(x)$ is a complex-valued function which satisfies the condition

$$\int_0^\infty x|q(x)|dx < \infty. \tag{1.4}$$

The spectral theory of nonself-adjoint operator in the classical case (i.e., $\alpha = 1$) was studied by Naimark [17, 18], he showed that some poles of the resolvent kernel are not the eigenvalues of the operator and belong to the continuous spectrum, moreover, these poles are called spectral singularities and were first introduced by Schwartz [22]. In the self-adjoint case (i.e. $\text{Im}q(x) \equiv 0$), the number of the eigenvalues of the operator is finite under condition (1.4) (see [15]).

In the nonself-adjoint case, Naimark demonstrated that the number of eigenvalues is finite under the condition (see [17, 18])

$$\int_0^\infty \exp(\epsilon x)|q(x)|dx < \infty, \quad \epsilon > 0.$$

This condition is too strict and Pavlov weakened this condition as follows (see [19]):

$$\sup_{0\leq x<\infty} \left\{|q(x)| \exp(\epsilon\sqrt{x})\right\} < \infty, \quad \epsilon > 0$$

and he proved that if $q(x)$ satisfies the above condition, then there is a finite number of eigenvalues of the operator.

In the spectral analysis of nonself-adjoint operators, the spectral singularities have an essential role and the influence of the spectral singularities in the spectral expansion with respect to the principal functions of the operator is investigated by Lyantse [12, 13]. The investigations on the spectrum, principal functions and the spectral expansion by the principal functions of the nonself-adjoint operator are very attractive and there are many works on the nonself-adjoint operator under different boundary conditions (see [2, 4, 5, 6, 8, 10, 11, 14, 16, 20, 23, 24, 25] and the references therein). Moreover, the nonself-adjoint operator with discontinuous coefficient is studied in [1], some spectral properties of the Sturm–Liouville operator with impulsive condition is worked in [3].

The distinction between this work and other studies is that the nonself-adjoint boundary value problem (1.1)-(1.3) has discontinuity conditions at $x = a \in (0, \infty)$. The presence of discontinuity condition (1.2) in problem (1.1)-(1.3) seriously affects the structure of a Jost solution to (1.1), i.e., a Jost solution is not expressed as a transformation operator, it has the integral representation which is obtained by Huseynov and Osmanova [9] and in this work. It is seen from this representation that the triangular property of a Jost solution is lost and the kernel function has a discontinuity along the line $s = 2a - x$ for $x \in (0, a)$. In this paper, we will obtain our results using this integral representation.

The conclusions drawn from this paper are as follows: in Section 2, we give an estimate of the kernel $k(x, s)$ of a Jost solution to equation (1.1) with discontinuity conditions (1.2) and examine the spectrum and the resolvent of problem (1.1)-(1.3). Moreover, it is demonstrated that under additional conditions, the number of the eigenvalues and the spectral singularities of this problem is finite. In Section 3, the principal functions are determined and their convergence properties are examined.

2 The spectrum and resolvent of L

Assume that a function $e(x, \mu)$ satisfies equation (1.1), discontinuity conditions (1.2) and the following condition at infinity

$$\lim_{x\to\infty} e^{-i\mu x} e(x, \mu) = 1.$$

Then, the function $e(x, \mu)$ is called a Jost solution to equation (1.1). When $q(x) \equiv 0$ in (1.1), the Jost solution has the form:

$$e_0(x, \mu) = \begin{cases} e^{i\mu x}, & x > a \\ \alpha^+ e^{i\mu x} + \alpha^- e^{i\mu(2a-x)}, & 0 < x < a \end{cases}$$

where $\alpha^\pm = \frac{1}{2}\left(\alpha \pm \frac{1}{\alpha}\right)$.

Theorem 2.1. [9] *Let a complex-valued function $q(x)$ satisfy* (1.4)*. Then for any μ from the closed upper half-plane, there exists a Jost solution $e(x,\mu)$ to equation* (1.1) *with discontinuity conditions* (1.2)*, it is unique and representable in the form*

$$e(x,\mu) = e_0(x,\mu) + \int_x^\infty k(x,s)e^{i\mu s}ds, \tag{2.1}$$

where for every fixed $x \in (0,a)\cup(a,\infty)$, the kernel $k(x,.) \in L_1(x,\infty)$ and satisfies the inequality

$$\int_x^\infty |k(x,s)|ds \le e^{c\sigma_1(x)} - 1, \qquad \sigma_1(x) = \int_x^\infty t|q(t)|dt, \qquad c = \alpha^+ + |\alpha^-|.$$

Moreover, the function $k(x,s)$ is continuous for $s \neq 2a - x$.

Remark 1. The following estimate holds:

$$|k(x,s)| \le \frac{c}{2}\sigma\left(\frac{x+s}{2}\right)e^{(c+1)\sigma_1(x)} \tag{2.2}$$

with $\sigma(x) = \int_x^\infty |q(u)|du$ and $c = \alpha^+ + |\alpha^-|$. This estimate is obtained as follows.

The function $k(x,s)$ is of the form for $0 < x < a$ (see [9]):

$$\begin{aligned} k(x,s) &= k_0(x,s) + \frac{1}{2}\int_x^a q(\zeta)\int_{s-\zeta+x}^{s+\zeta-x} k(\zeta,u)dud\zeta \\ &\quad +\frac{\alpha^+}{2}\int_a^\infty q(\zeta)\int_{s-\zeta+x}^{s+\zeta-x} k(\zeta,u)dud\zeta \\ &\quad -\frac{\alpha^-}{2}\int_a^{2a-x} q(\zeta)\int_{s+\zeta-2a+x}^{s-\zeta+2a-x} k(\zeta,u)dud\zeta \\ &\quad +\frac{\alpha^-}{2}\int_{2a-x}^\infty q(\zeta)\int_{s-\zeta+2a-x}^{s+\zeta-2a+x} k(\zeta,u)dud\zeta, \end{aligned}$$

where

$$\begin{aligned} k_0(x,s) &= \frac{\alpha^+}{2}\int_{\frac{x+s}{2}}^\infty q(\zeta)d\zeta + \frac{\alpha^-}{2}\int_{\frac{2a+x-s}{2}}^a q(\zeta)d\zeta \\ &\quad -\frac{\alpha^-}{2}\int_a^{\frac{s+2a-x}{2}} q(\zeta)d\zeta, \qquad x < s < 2a - x, \end{aligned} \tag{2.3}$$

$$k_0(x,s) = \frac{\alpha^+}{2}\int_{\frac{x+s}{2}}^\infty q(\zeta)d\zeta + \frac{\alpha^-}{2}\int_{\frac{s+2a-x}{2}}^\infty q(\zeta)d\zeta, \qquad s > 2a - x \tag{2.4}$$

and for $x > a$

$$k(x,s) = k_0(x,s) + \frac{1}{2}\int_x^\infty q(\zeta)\int_{s-\zeta+x}^{s+\zeta-x} k(\zeta,u)dud\zeta,$$

where

$$k_0(x,s) = \frac{1}{2}\int_{\frac{x+s}{2}}^\infty q(\zeta)d\zeta.$$

When $x > a$, we face the classical case (see [18]). In this case, we have

$$|k(x,s)| \le \frac{1}{2}e^{\sigma_1(x)}\sigma\left(\frac{x+s}{2}\right).$$

Now, let us examine the case $0 < x < a$. Set, for $n \in \mathbb{N}$

$$\begin{aligned} k_n(x,s) &= \frac{1}{2}\int_x^a q(\zeta)\int_{s-\zeta+x}^{s+\zeta-x} k_{n-1}(\zeta,u)dud\zeta \\ &+\frac{\alpha^+}{2}\int_a^\infty q(\zeta)\int_{s-\zeta+x}^{s+\zeta-x} k_{n-1}(\zeta,u)dud\zeta \\ &-\frac{\alpha^-}{2}\int_a^{2a-x} q(\zeta)\int_{s+\zeta-2a+x}^{s-\zeta+2a-x} k_{n-1}(\zeta,u)dud\zeta \\ &+\frac{\alpha^-}{2}\int_{2a-x}^\infty q(\zeta)\int_{s-\zeta+2a-x}^{s+\zeta-2a+x} k_{n-1}(\zeta,u)dud\zeta \end{aligned}$$

and $k_0(x,s)$ is specified by relations (2.3) and (2.4). Then, we obtain

$$|k_0(x,s)| \le \frac{c}{2}\sigma\left(\frac{x+s}{2}\right), \qquad |k_n(x,s)| \le \frac{c}{2}\sigma\left(\frac{x+s}{2}\right)\frac{(c+1)^n(\sigma_1(x))^n}{n!}.$$

This implies that the series $\sum_{n=0}^{\infty} k_n(x,s)$ converges and its sum $k(x,s)$ satisfies inequality (2.2). Consequently, for $x \in (0,a)\cup(a,\infty)$ inequality (2.2) is valid.

Now, we define $\hat{e}(x,\mu)$ as a solution to equation (1.1) with discontinuity conditions (1.2) and the following condition at infinity

$$\lim_{x\to\infty} e^{i\mu x}\hat{e}(x,\mu) = 1.$$

When $q(x) \equiv 0$ in equation (1.1), the solution has the form:

$$\hat{e}_0(x,\mu) = \begin{cases} e^{-i\mu x}, & x > a, \\ \alpha^+ e^{-i\mu x} + \alpha^- e^{-i\mu(2a-x)}, & 0 < x < a \end{cases}$$

for $\text{Im}\mu \ge 0$. The Wronskian of the solutions $e(x,\mu)$ and $\hat{e}(x,\mu)$ is

$$W[e(x,\mu),\hat{e}(x,\mu)] = -2i\mu, \quad \text{Im}\mu \ge 0.$$

Now, we consider problem (1.1)-(1.3) as an operator L operating on the Hilbert space $L_2(0,\infty)$. The values $\lambda = \mu^2$ for which L has a non-zero solution are called eigenvalues and the corresponding solutions are called eigenfunctions.

Consider $\tilde{e}(x,\mu) = e(x,-\mu)$ with $\text{Im}\mu \le 0$ and the expression of the Wronskian of $e(x,\mu)$ and $\tilde{e}(x,\mu)$ is

$$W[e(x,\mu),\tilde{e}(x,\mu)] = -2i\mu, \quad \text{Im}\mu = 0. \tag{2.5}$$

Lemma 2.1. *The nonself-adjoint operator L does not have positive eigenvalues.*

Proof. It follows from (2.5) that for $\lambda > 0$, the general solution to (1.1) is of the form $\omega = c_1 e(x,\mu) + c_2\tilde{e}(x,\mu)$ and as $x \to \infty$, $\omega = c_1 e^{i\mu x} + c_2 e^{-i\mu x} + o(1)$. This function does not belong to $L_2(0,\infty)$ if both c_1 and c_2 are not equal to zero. □

Let us define $s(x,\mu)$ as a solution to (1.1) under discontinuity conditions (1.2) and the initial conditions

$$s(0,\mu) = 0, \qquad s'(0,\mu) = 1.$$

Now, consider non-positive or complex λ. Since the general solution to (1.1) satisfying the initial condition $\omega(0) = 0$ has the form $\omega(x) = cs(x,\mu)$, it follows that $\lambda = \mu^2$ is an eigenvalue of the operator L if and only if $s(.,\mu) \in L_2(0,\infty)$. Moreover,

$$s(x,\mu) = \frac{\hat{e}(0,\mu)e(x,\mu) - e(0,\mu)\hat{e}(x,\mu)}{2i\mu}, \quad \text{Im}\mu > 0. \tag{2.6}$$

Lemma 2.2. *The necessary and sufficient conditions for $\lambda \neq 0$ to be an eigenvalue of L are*

$$e(0,\mu) = 0, \quad \lambda = \mu^2, \quad \mathrm{Im}\mu > 0.$$

Proof. It follows from the representations of $e(x,\mu)$ and $\hat{e}_0(x,\mu)$ that $e(.,\mu) \in L_2(0,\infty)$ and $\hat{e}(x,\mu) \notin L_2(0,\infty.)$ Then, from (2.6), $s(.,\mu) \in L_2(0,\infty)$ if and only if $e(0,\mu) = 0$. □

Lemma 2.3. *The set of eigenvalues of L is bounded, is no more than countable and its limit points can lie only on the half-axis $\lambda \geq 0$.*

Proof. Using the representation of solution $e(x,\mu)$ given by (2.1), as $|\mu| \to \infty$, we have $e(0,\mu) \to \alpha^+$ for $\mathrm{Im}\mu > 0$. Therefore, the set of the zeros of $e(0,\mu)$ is bounded in the half plane $\mathrm{Im}\mu > 0$. Since $e(0,\mu)$ is holomorphic in the half plane $\mathrm{Im}\mu > 0$, the set of its zeros is no more than countable and can have limit points only on the real axis. □

All numbers λ of the form $\lambda = \mu^2$, $\mathrm{Im}\mu > 0$, $e(0,\mu) \neq 0$ belong to the resolvent set of L. The resolvent operator $R_{\mu^2} = \left(L - \mu^2 I\right)^{-1}$ exists and has the following form:

$$\omega(x,\mu) =: R_{\mu^2}(L)f(x) = \int_0^\infty g(x,s;\mu^2)f(s)ds,$$

where

$$g(x,s;\mu^2) = \begin{cases} \frac{\hat{e}(0,\mu)e(x,\mu)e(s,\mu)}{2i\mu e(0,\mu)} - \frac{\hat{e}(x,\mu)e(s,\mu)}{2i\mu}, & x < s < \infty, \\ \\ \frac{\hat{e}(0,\mu)e(x,\mu)e(s,\mu)}{2i\mu e(0,\mu)} - \frac{e(x,\mu)\hat{e}(s,\mu)}{2i\mu}, & 0 < s < x \end{cases}$$

and $\omega(x,\mu)$ is a solution to the following nonhomogeneous problem:

$$\begin{gathered} -\omega'' + q(x)\omega = \mu^2\omega + f(x), \\ \omega(a-0) = \alpha\omega(a+0), \quad \omega'(a-0) = \alpha^{-1}\omega'(a+0), \\ \omega(0) = 0. \end{gathered}$$

Note that all numbers $\lambda \geq 0$ belong to the continuous spectrum of L (see [18]). Moreover, the spectral singularities defined as the poles of the kernel function of the resolvent operator belong to the continuous spectrum. The set of spectral singularities of L is closed and its Lebesgue measure is zero which can be seen from the boundary uniqueness theorem for analytic functions [21] (also, see [1]).

Now, let us use the notation $\sigma_d(L)$ and $\sigma_{ss}(L)$ for the eigenvalues and spectral singularities of L, respectively.

$$\sigma_d(L) = \left\{\lambda : \lambda = \mu^2,\ \mathrm{Im}\mu > 0,\ e(0,\mu) = 0\right\},$$

$$\sigma_{ss}(L) = \left\{\lambda : \lambda = \mu^2,\ \mathrm{Im}\mu = 0,\ \mu \neq 0,\ e(0,\mu) = 0\right\}.$$

Moreover, the multiplicity m_k of a root μ_k of the equation $e(0,\mu)$ is called the multiplicity of μ_k.

Now, we will show that the nonself-adjoint operator L has a finite number of eigenvalues and spectral singularities under the following additional restrictions

$$\int_0^\infty e^{\epsilon x}|q(x)|dx < \infty, \quad \epsilon > 0, \tag{2.7}$$

$$\sup_{0 \leq x < \infty} \left\{\exp(\epsilon\sqrt{x})|q(x)|\right\} < \infty, \quad \epsilon > 0. \tag{2.8}$$

First, assume that condition (2.7) introduced by M.A. Naimark holds. This condition implies that

$$\sigma(x) = \int_x^\infty |q(t)|dt \leq C_\epsilon e^{-\epsilon x},$$

$$\sigma_1(x) = \int_x^\infty t|q(t)|dt \leq C_{\epsilon'} e^{-\epsilon' x},$$

where $C_\epsilon > 0$, $C_{\epsilon'} > 0$ and $0 < \epsilon' < \epsilon$ (see [18]). Using these relations and estimate (2.2), we have

$$|k(x,s)| \leq C \exp\left\{-\epsilon\left(\frac{x+s}{2}\right)\right\}, \tag{2.9}$$

where $C = cc_\epsilon e^{(c+1)d_\epsilon}$, $c = \alpha^+ + |\alpha^-|$, $c_\epsilon > 0$ and $d_\epsilon > 0$.

Theorem 2.2. *Suppose that condition* (2.7) *is valid. Then, the operator* L *has finite number of eigenvalues and spectral singularities with finite multiplicity.*

Proof. It is obtained from (2.9) that the function $e(0,\mu)$ has an analytic continuation from the real axis to the half plane $\text{Im}\mu > -\frac{\epsilon}{2}$. Then, there are no limit points of the sets of eigenvalues $\sigma_d(L)$ and spectral singularities $\sigma_{ss}(L)$ on the positive real line. Since $\sigma_d(L)$ and $\sigma_{ss}(L)$ are bounded and $e(0,\mu)$ is holomorphic in the half plane $\text{Im}\mu > -\frac{\epsilon}{2}$, L has finite number of eigenvalues and spectral singularities with finite multiplicity. □

Now, let condition (2.8) be satisfied. We need to show that the numbers of the spectral singularities and the eigenvalues under condition (2.8) are finite. First, we define the set of zeros of $e(0,\mu)$ in the closed upper half plane $\text{Im}\mu \geq 0$:

$$S_1 := \{\mu : \ \mu \in \mathbb{C}_+, \ e(0,\mu) = 0\}, \quad S_2 := \{\mu : \ \mu \in \mathbb{R}, \ \mu \neq 0, \ e(0,\mu) = 0\}.$$

Moreover, let us take into account that the sets S_3 and S_4 contain all limit points of S_1 and S_2 respectively, and the set S_5 has all infinite multiple zeros of $e(0,\mu)$. We can write

$$S_1 \cap S_5 = \emptyset, \quad S_3 \subset S_2, \quad S_4 \subset S_2, \quad S_5 \subset S_2$$

from the uniqueness theorem of analytic functions (see [7]) and

$$S_3 \subset S_5, \quad S_4 \subset S_5 \tag{2.10}$$

from the continuity of all derivatives of $e(0,\mu)$ up to the real axis.

Lemma 2.4. *Assume that condition* (2.8) *is satisfied, then* $S_5 = \emptyset$.

Proof. To prove this lemma, we use the following theorem (see [19], also [1, 2]): Suppose that the function φ is analytic in $\mathbb{C}_+$, all of its derivatives are continuous up to the real axis, and there exists $M > 0$ such that

$$|\varphi^{(v)}(z)| \leq K_v, \quad v = 0,1,... \ z \in \mathbb{C}_+, \ |z| < 2M, \tag{2.11}$$

and

$$\left|\int_{-\infty}^{-M} \frac{\ln|\varphi(x)|}{1+x^2}dx\right| < \infty, \quad \left|\int_M^\infty \frac{\ln|\varphi(x)|}{1+x^2}dx\right| < \infty. \tag{2.12}$$

If the set Q with the one-dimensional Lebesgue measure zero is the set of all zeros of the function φ with infinite multiplicity and the relation

$$\int_0^u \ln H(s) d\mu(Q_s) = -\infty \tag{2.13}$$

holds, then $\varphi(z) \equiv 0$, where u is an arbitrary positive constant, $H(s) = \inf_{\upsilon} \frac{K_\upsilon s^\upsilon}{\upsilon!}$, $\upsilon = 0, 1, \ldots$ and $\mu(Q_s)$ is the Lebesgue measure of s-neighborhood of Q.

Now, it follows from relation (2.2) and condition (2.8) that

$$|k(x,s)| \leq \widetilde{C} \exp\left\{-\epsilon \left(\frac{x+s}{2}\right)^\delta\right\}, \quad \widetilde{C} = cc_\epsilon e^{(c+1)c_\epsilon}, \quad c = \alpha^+ + |\alpha^-| > 0.$$

Then, the function $e(0,\mu)$ is analytic in $\mathbb{C}_+$, all of its derivatives are continuous up to the real axis and we have

$$\left|\frac{d^\upsilon e(0,\mu)}{d\mu^\upsilon}\right| \leq K_\upsilon, \ \ \mu \in \overline{\mathbb{C}}_+, \ \ \upsilon = 1, 2, \ldots, \tag{2.14}$$

where

$$K_\upsilon = \widetilde{C}(2a)^\upsilon \left(1 + \int_0^\infty s^\upsilon \exp\left\{-\epsilon\left(\frac{s}{2}\right)^\delta\right\} ds\right), \quad \upsilon = 1, 2, \ldots$$

Moreover, since the set of zeros of $e(0,\mu)$ is bounded, for sufficiently large M the function $e(0,\mu)$ satisfies condition (2.12). Thus, it follows from this fact and relation (2.14) that $e(0,\mu)$ provides conditions (2.11) and (2.12). Since the function $e(0,\mu) \neq 0$, we have from (2.13)

$$\int_0^u \ln H(s) d\mu(S_{5,s}) > -\infty, \tag{2.15}$$

where $H(s) = \inf_\upsilon \frac{K_\upsilon s^\upsilon}{\upsilon!}$ and $\mu(S_{5,s})$ is the Lebesgue measure of the s-neighborhood of S_5. The following estimate holds

$$K_\upsilon \leq \left(\widetilde{C}(2a)^\upsilon + Dd^\upsilon\right) \upsilon^\upsilon \upsilon!, \tag{2.16}$$

where $D = 4\frac{\widetilde{C}e}{\delta}\epsilon^{-\frac{1}{\delta}}(\upsilon+1)$ and $d = 8a\epsilon^{-\frac{1}{\delta}}$. In fact, we can write

$$\begin{aligned} K_\upsilon &= \widetilde{C}(2a)^\upsilon \left(1 + \int_0^\infty s^\upsilon \exp\left\{-\epsilon\left(\frac{s}{2}\right)^\delta\right\} ds\right) \\ &\leq \widetilde{C}(2a)^\upsilon \left(1 + \frac{2^{(\upsilon+1)}}{\delta}\epsilon^{-\frac{(\upsilon+1)}{\delta}}(2\upsilon+2)^{\upsilon+1}\upsilon!\right) \\ &\leq \widetilde{C}(2a)^\upsilon \left(1 + \frac{2^{2(\upsilon+1)}}{\delta}\epsilon^{-\frac{(\upsilon+1)}{\delta}}\left(1+\frac{1}{\upsilon}\right)^\upsilon (\upsilon+1)\upsilon^\upsilon\upsilon!\right) \\ &\leq \left(\widetilde{C}(2a)^\upsilon + Dd^\upsilon\right)\upsilon^\upsilon\upsilon!. \end{aligned}$$

Putting estimate (2.16) into $H(s)$, we get

$$\begin{aligned} H(s) &\leq \widetilde{C}\inf_\upsilon\{(2a)^\upsilon\upsilon^\upsilon\upsilon!\} + D\inf_\upsilon\{d^\upsilon\upsilon^\upsilon s^\upsilon\} \\ &\leq \widetilde{C}\exp\left\{-(2a)^{-1}s^{-1}e^{-1}\right\} + D\exp\left\{-d^{-1}s^{-1}e^{-1}\right\}. \end{aligned} \tag{2.17}$$

Then, taking into account (2.15) and (2.17), we have

$$\int_0^u \frac{1}{s} d\mu(S_{5,s}) < \infty.$$

This inequality is valid for an arbitrary s if and only if $d\mu(S_{5,s}) = 0$ or $S_5 = \emptyset$. □

Theorem 2.3. *If condition (2.8) is satisfied, then L has finite number of eigenvalues and spectral singularities with finite multiplicity.*

Proof. It follows from (2.10) and Lemma 2.4 that $S_3 = \emptyset$ and $S_4 = \emptyset$. For this reason, the bounded sets S_1 and S_2 do not have limit points. Thus, the finiteness of the sets of $\sigma_d(L)$ and $\sigma_{ss}(L)$ are established. Moreover, due to $S_5 = \emptyset$, the eigenvalues and spectral singularities have finite multiplicities. □

3 Principal functions

Now, we examine the principal functions of L. Assume that condition (2.8) is satisfied.

Denote $\mu_1, \mu_2, ..., \mu_\ell$ by the zeros of $e(0,\mu)$ in $\mathbb{C}_+$ with multiplicities $m_1, m_2, ..., m_\ell$ respectively (note that $\mu_1^2, \mu_2^2, ..., \mu_\ell^2$ are the eigenvalues of L). We can write

$$\left\{\frac{d^\nu}{d\mu^\nu}W[e(x,\mu), s(x,\mu)]\right\}_{\mu=\mu_\eta} = \left\{\frac{d^\nu}{d\mu^\nu}e(0,\mu)\right\}_{\mu=\mu_\eta} = 0 \tag{3.1}$$

for $\nu = \overline{0, m_\eta - 1}$, $\eta = \overline{1,\ell}$. In case of $\nu = 0$, we have

$$e(x,\mu_\eta) = \kappa_0(\mu_\eta)s(x,\mu_\eta), \quad \kappa_0(\mu_\eta) \neq 0, \quad \eta = \overline{1,\ell}. \tag{3.2}$$

Lemma 3.1. *The following relation*

$$\left\{\frac{\partial^\nu}{\partial\mu^\nu}e(x,\mu)\right\}_{\mu=\mu_\eta} = \sum_{i=0}^{\nu}\binom{\nu}{i}\kappa_{\nu-i}\left\{\frac{\partial^i}{\partial\mu^i}s(x,\mu)\right\}_{\mu=\mu_\eta} \tag{3.3}$$

is valid for $\nu = \overline{0, m_\eta - 1}$, $\eta = \overline{1,\ell}$ *and here* $\kappa_0, \kappa_1 ..., \kappa_\nu$ *depend on* μ_η.

Proof. To prove of this lemma, we use the mathematical induction. Consider $\nu = 0$. It follows from relation (3.2) that the proof is trivial. Now, suppose that formula (3.3) is valid for ν_0 such that $0 < \nu_0 \leq m_\eta - 2$. That is,

$$\left\{\frac{\partial^{\nu_0}}{\partial\mu^{\nu_0}}e(x,\mu)\right\}_{\mu=\mu_\eta} = \sum_{i=0}^{\nu_0}\binom{\nu_0}{i}\kappa_{\nu_0-i}\left\{\frac{\partial^i}{\partial\mu^i}s(x,\mu)\right\}_{\mu=\mu_\eta}. \tag{3.4}$$

Then, we will show that formula (3.3) is satisfied for $\nu_0 + 1$. If $\omega(x,\mu)$ is a solution to (1.1), then we find

$$\left\{-\frac{d^2}{dx^2} + q(x) - \mu^2\right\}\frac{\partial^\nu}{\partial\mu^\nu}\omega(x,\mu) = 2\mu\nu\frac{\partial^{\nu-1}}{\partial\mu^{\nu-1}}\omega(x,\mu) + \nu(\nu-1)\frac{\partial^{\nu-2}}{\partial\mu^{\nu-2}}\omega(x,\mu). \tag{3.5}$$

Since $e(x,\mu)$ and $s(x,\mu)$ are solutions to equation (1.1), using (3.4) and (3.5) we calculate

$$\left\{-\frac{d^2}{dx^2} + q(x) - \mu_\eta^2\right\}h_{\nu_0+1}(x,\mu_\eta) = 0,$$

where

$$h_{\nu_0+1}(x,\mu_\eta) = \left\{\frac{\partial^{\nu_0+1}}{\partial\mu^{\nu_0+1}}e(x,\mu)\right\}_{\mu=\mu_\eta} - \sum_{i=0}^{\nu_0+1}\binom{\nu_0+1}{i}\kappa_{\nu_0+1-i}\left\{\frac{\partial^i}{\partial\mu^i}s(x,\mu)\right\}_{\mu=\mu_\eta}.$$

It follows from (3.1) that

$$W[h_{\nu_0+1}(x,\mu_\eta), s(x,\mu_\eta)] = \left\{\frac{d^{\nu_0+1}}{d\mu^{\nu_0+1}}W[e(x,\mu), s(x,\mu)]\right\}_{\mu=\mu_\eta} = 0.$$

Then, this shows that

$$h_{\nu_0+1}(x,\mu_\eta) = \kappa_{\nu_0+1}(\mu_\eta)s(x,\mu_\eta), \quad \eta = \overline{1,\ell}.$$

Consequently, we obtain that formula (3.3) is satisfied for $\nu = \nu_0 + 1$. □

Define the functions

$$\psi_\nu(x,\lambda_\eta)=\left\{\frac{\partial^\nu}{\partial\mu^\nu}e(x,\mu)\right\}_{\mu=\mu_\eta}=\sum_{i=0}^{\nu}\left(\begin{array}{c}\nu\\ i\end{array}\right)\kappa_{\nu-i}\left\{\frac{\partial^i}{\partial\mu^i}s(x,\mu)\right\}_{\mu=\mu_\eta} \tag{3.6}$$

for $\nu=\overline{0,m_\eta-1}$, $\eta=\overline{1,\ell}$ and $\lambda_\eta=\mu_\eta^2$.

Theorem 3.1. $\psi_\nu(x,\lambda_\eta)\in L_2(0,\infty)$ *for* $\nu=\overline{0,m_\eta-1}$, $\eta=\overline{1,\ell}$.

Proof. Since

$$|k(x,s)|\le\widetilde{C}\exp\left\{-\epsilon\left(\frac{x+s}{2}\right)^\delta\right\},\quad \widetilde{C}=cc_\epsilon e^{(c+1)c_\epsilon},\quad c=\alpha^++|\alpha^-|>0,$$

using integral representation (2.1) we have for $0<x<a$

$$\begin{aligned}\left|\left\{\frac{\partial^\nu}{\partial\mu^\nu}e(x,\mu)\right\}_{\mu=\mu_\eta}\right| &\le x^\nu\alpha^+e^{-\mathrm{Im}\mu_\eta x}+(2a-x)^\nu|\alpha^-|e^{-\mathrm{Im}\mu_\eta(2a-x)}\\ &\quad+\widetilde{C}\int_x^\infty s^\nu\exp\left\{-\epsilon\left(\frac{x+s}{2}\right)^\delta\right\}e^{-\mathrm{Im}\mu_\eta}ds\end{aligned} \tag{3.7}$$

and for $a<x<\infty$

$$\left|\left\{\frac{\partial^\nu}{\partial\mu^\nu}e(x,\mu)\right\}_{\mu=\mu_\eta}\right|\le x^\nu e^{-\mathrm{Im}\mu_\eta x}+\widetilde{C}\int_x^\infty s^\nu\exp\left\{-\epsilon\left(\frac{x+s}{2}\right)^\delta\right\}e^{-\mathrm{Im}\mu_\eta}ds. \tag{3.8}$$

Since $\lambda_\eta=\mu_\eta^2$, $\eta=\overline{1,\ell}$ are eigenvalues of operator L, it follows from (3.7) and (3.8) for $\mathrm{Im}\mu_\eta>0$ that

$$\left\{\frac{\partial^\nu}{\partial\mu^\nu}e(x,\mu)\right\}_{\mu=\mu_\eta}\in L_2(0,\infty),\quad \nu=\overline{0,m_\eta-1},\quad \eta=\overline{1,\ell}.$$

Consequently, from (3.6) we have $\psi_\nu(x,\lambda_\eta)\in L_2(0,\infty)$, $\nu=\overline{0,m_\eta-1}$, $\eta=\overline{1,\ell}$. □

Definition 1. Functions $\psi_0(x,\lambda_\eta)$, $\psi_1(x,\lambda_\eta)$,...,$\psi_{m_\eta-1}(x,\lambda_\eta)$ are called principle functions associated with the eigenvalues $\lambda_\eta=\mu_\eta^2$, $\eta=\overline{1,\ell}$ of L. The function $\psi_0(x,\lambda_\eta)$ is the eigenfunction, $\psi_1(x,\lambda_\eta)$, $\psi_2(x,\lambda_\eta)$,..., $\psi_{m_\eta-1}(x,\lambda_\eta)$ are the associated functions of $\psi_0(x,\lambda_\eta)$ corresponding to the eigenvalue λ_η.

Now, we define the spectral singularities of L: $\mu_{\ell+1},\mu_{\ell+2},...,\mu_\beta$ are the zeros of $e(0,\mu)$ in $\mathbb{R}-\{0\}$ with multiplicities $m_{\ell+1},m_{\ell+2},...,m_\beta$, respectively. Then, similarly to the proof of Lemma 3.1, we obtain

$$\left\{\frac{\partial^v}{\partial\mu^v}e(x,\mu)\right\}_{\mu=\mu_\gamma}=\sum_{j=0}^{v}\left(\begin{array}{c}v\\ j\end{array}\right)\tau_{v-j}(\mu_\gamma)\left\{\frac{\partial^j}{\partial\mu^j}s(x,\mu)\right\}_{\mu=\mu_\gamma}$$

for $v=\overline{0,m_\gamma-1}$, $\gamma=\ell+1,\ell+2,...,\beta$. Consider the functions

$$\psi_v(x,\lambda_\gamma)=\left\{\frac{\partial^v}{\partial\mu^v}e(x,\mu)\right\}_{\mu=\mu_\gamma}=\sum_{j=0}^{v}\left(\begin{array}{c}v\\ j\end{array}\right)\tau_{v-j}(\mu_\gamma)\left\{\frac{\partial^j}{\partial\mu^j}s(x,\mu)\right\}_{\mu=\mu_\gamma} \tag{3.9}$$

for $v=\overline{0,m_\gamma-1}$, $\gamma=\ell+1,\ell+2,...,\beta$ and $\lambda_j=\mu_j^2$.

Theorem 3.2. *The functions $\psi_v(x,\lambda_\gamma)$ do not belong to $L_2(0,\infty)$ for $v=\overline{0,m_\gamma-1}$, $\gamma=\ell+1,\ell+2,...,\beta$.*

Proof. Take into account relations (3.7) and (3.8) for $\mu=\mu_\gamma$, $\gamma=\ell+1,\ell+2,...,\beta$ and since $\operatorname{Im}\mu_\gamma=0$ for the spectral singularities, we have

$$\left\{\frac{\partial^v}{\partial\mu^v}e(x,\mu)\right\}_{\mu=\mu_\gamma}\notin L_2(0,\infty),\ v=\overline{0,m_\gamma-1},\ \gamma=\overline{\ell+1,\beta}.$$

As a result, from the definition of functions (3.9), we find $\psi_v(x,\lambda_\gamma)\notin L_2(0,\infty)$ for $v=\overline{0,m_\gamma-1}$, $\gamma=\ell+1,\ell+2,...,\beta$. □

Now, we introduce the Hilbert spaces

$$H_\rho=\left\{f:\|f\|_\rho<\infty\right\},\quad H_{-\rho}=\left\{f:\|f\|_{-\rho}<\infty\right\},\quad \rho=1,2,...$$

with the norms

$$\|f\|_\rho^2=\int_0^\infty(1+s)^{2\rho}|f(s)|^2ds,\quad \|f\|_{-\rho}^2=\int_0^\infty(1+s)^{-2\rho}|f(s)|^2ds$$

respectively and evidently, $H_0=L_2(0,\infty)$.

Let m_0 denote the greatest of the multiplicities of the spectral singularities of L:

$$m_0=\max\left\{m_{\ell+1},m_{\ell+2},...,m_\beta\right\}.$$

We put

$$H_+=H_{m_0+1},\quad H_-=H_{-(m_0+1)}$$

Then, we have

$$H_+\subset L_2(0,\infty)\subset H_-$$

and for all $f\in H_+$, $\|f\|_+\geq\|f\|\geq\|f\|_-$, where $\|.\|_\pm=\|.\|_{\pm(m_0+1)}$, $\|.\|=\|.\|_0$ (see [18]). We are particularly interested in the space H_- because the space H_- contains the principal functions for the spectral singularities. Now, we will prove the above claim by using the following lemma.

Lemma 3.2. *The following relations hold:*

$$\sup_{0\leq x<\infty}\frac{\left|e^{(m)}(x,\mu)\right|}{(1+x)^m}<\infty,\quad e^{(m)}=\left(\frac{d}{d\mu}\right)^m e,\quad \operatorname{Im}\mu=0,\quad m=0,1,2,... \tag{3.10}$$

Proof. Using integral representation (2.1), we obtain for $\operatorname{Im}\mu=0$

$$\begin{aligned}\left|e^{(m)}(x,\mu)\right| \leq\ & x^m\alpha^+ +(2a-x)^m|\alpha^-|\\ &+\widetilde{C}\int_x^\infty s^m\exp\left\{-\epsilon\left(\frac{x+s}{2}\right)^\delta\right\}ds,\quad 0<x<a\end{aligned} \tag{3.11}$$

and

$$\left|e^{(m)}(x,\mu)\right|\leq x^m+\widetilde{C}\int_x^\infty s^m\exp\left\{-\epsilon\left(\frac{x+s}{2}\right)^\delta\right\}ds,\quad a<x<\infty. \tag{3.12}$$

Then, taking into account (3.11) and (3.12), we find $\sup_{0\leq x<\infty}\frac{\left|e^{(m)}(x,\mu)\right|}{(1+x)^m}<\infty$. □

Theorem 3.3. *$\psi_v(x,\lambda_\gamma) \in H_{-(v+1)}$ for $v = 0,1,...,m_\gamma - 1$, $\gamma = \ell+1, \ell+2, ..., \beta$.*

Proof. Using relation (3.10), we have

$$\left\| e^{(v)}(x,\mu) \right\|^2_{-(v+1)} = \int_0^\infty \left| \frac{e^{(v)}(x,\mu)}{(1+x)^{v+1}} \right|^2 dx < \infty.$$

That is, the functions $e^{(v)}(x,\mu) = \frac{\partial^v}{\partial \mu^v} e(x,\mu) \in H_{-(v+1)}$ for $\text{Im}\mu = 0$ and $v = 0,1,2,....$ Then, we get

$$\left\{ \frac{\partial^v}{\partial \mu^v} e(x,\mu) \right\}_{\mu=\mu_\gamma} \in H_{-(v+1)}$$

for $\text{Im}\mu_\gamma = 0$, $v = 0,1,...,m_\gamma - 1$ and $\gamma = \ell+1, \ell+2, ..., \beta$. Consequently, it follows from formula (3.9) that $\psi_v(x,\lambda_\gamma) \in H_{-(v+1)}$ for $v = 0,1,...,m_\gamma - 1$, $\gamma = \ell+1, \ell+2, ..., \beta$. □

Definition 2. The functions $\psi_0(x,\lambda_\gamma), \psi_1(x,\lambda_\gamma), ..., \psi_{m_\gamma - 1}(x,\lambda_\gamma)$ are called the principal functions associated with the spectral singularities $\lambda_\gamma = \mu_\gamma^2$, $\gamma = \ell+1, \ell+2, ..., \beta$ of operator L. The function $\psi_0(x,\lambda_\gamma)$ is the generalized eigenfunction, $\psi_1(x,\lambda_\gamma), ..., \psi_{m_\gamma - 1}(x,\lambda_\gamma)$ are the generalized associated functions of $\psi_0(x,\lambda_\gamma)$ corresponding to the spectral singularity λ_γ.

4 Conclusion

In this paper, we examine the spectrum and principal functions of a nonself-adjoint Sturm-Liouville operator with discontinuity conditions at the point $x = a \in (0,\infty)$. When examining the spectrum of problem (1.1)-(1.3), we use the Jost solution to equation (1.1) with discontinuity condition (1.2) which is obtained by Huseynov and Osmanova [9] and in this work. The triangular property of the Jost solution is lost and the kernel function has a discontinuity along the line $s = 2a - x$ for $x \in (0,a)$. Under two different additional conditions, it is proved that problem (1.1)-(1.3) has finite number of eigenvalues and spectral singularities with finite multiplicity. Finally, since restriction (2.8) is weaker than restriction (2.7), we determine the principal functions corresponding to the eigenvalues and spectral singularities of problem (1.1)-(1.3) under additional restriction (2.8).

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