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The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russia

YESMUKHANBET SAIDAKHMETOVICH SMAILOV

Doctor of physical and mathematical sciences, Professor Smailov Esmuhanbet Saidakhmetovich passed away on May 24, 2024, at the age of 78 years.

Esmuhanbet Saidakhmetovich was well known to the scientific community as a high qualified specialist in science and education, and an outstanding organizer. Fundamental scientific articles and textbooks written in various fields of the theory of functions of several variables and functional analysis, the theory of approximation of functions, embedding theorems, and harmonic analysis are a significant contribution to the development of mathematics.

E.S. Smailov was born on October 18, 1946, in the village of Kyzyl Kesik, Aksuat district, Semipalatinsk region. In 1963, he graduated from high school with a silver medal, and in the same year he entered the Faculty of Mechanics and Mathematics of the Kazakh State University (Almaty) named after Kirov (now named after Al-Farabi). In 1971 he graduated from graduate school at the Institute of Mathematics and Mechanics.

He defended his PhD thesis in 1973 (supervisor was K.Zh. Nauryzbaev) and defended his doctoral thesis “Fourier multipliers, embedding theorems and related topics” in 1997. In 1993 he was awarded the academic title of professor.

E.S. Smailov since 1972 worked at the Karaganda State University named after E.A. Buketov as an associate professor (1972-1978), the head of the department of mathematical analysis (1978-1986, 1990-2000), the dean of the Faculty of Mathematics (1983-1987) and was the director of the Institute of Applied Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan in Karaganda (2004 -2018).

Professor Smailov was one of the leading experts in the theory of functions and functional analysis and a major organizer of science in the Republic of Kazakhstan. He had a great influence on the formation of the Mathematical Faculty of the Karaganda State University named after E.A. Buketov and he made a significant contribution to the development of mathematics in Central Kazakhstan. Due to the efforts of Y.S. Smailov, in Karaganda an actively operating Mathematical School on the function theory was established, which is well known in Kazakhstan and abroad.

He published more than 150 scientific papers and 2 monographs. Under his scientific advice, 4 doctoral and 10 candidate theses were defended.

In 1999 the American Biographical Institute declared professor Smailov “Man of the Year” and published his biography in the “Biographical encyclopedia of professional leaders of the Millennium”.

For his contribution to science and education, he was awarded the Order of “Kurmet” (=“Honour”).

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Esmuhanbet Saidakhmetovich Smailov.

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SYMMETRY GROUPS OF PFAFFIANS OF SYMMETRIC MATRICES

A.S. Dzhumadil'daev

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AMS Mathematics Subject Classification: 15A15.

Abstract. We prove that the symmetry group of the pfaffian polynomial of a symmetric matrix is a dihedral group.

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1 Introduction

At a first glance the title of the paper is not correct. Pfaffians usually are connected with determinants of skew-symmetric matrices. If $a_{i,j} = -a_{j,i}$, for any $1 \le i, j \le 2n$, then the determinant of a skew-symmetric matrix $A = (a_{i,j})$ is a complete square and the square root of the determinant is a pfaffian, so

$$det\, A = (pf_{2n}A)^2.$$

In fact, the pfaffian polynomial is defined by using not the whole matrix A. To construct pfaffians it suffices to know the upper triangular part of A.

The connection between determinants of skew-symmetric matrices and pfaffians was first noted in [2]. For details of pfaffian constructions see also [1] and [3].

Let S_{2n} be set the of all permutations of the set $[2n] = \{1, 2, \dots, 2n\}$ and $S_{2n,pf}$ its subset of all permutations called *Pfaff permutations*,

$$S_{2n,pf} = \{\sigma = (i_1, j_1, \dots, i_n, j_n) \in S_{2n} | i_1 < i_2 < \dots < i_n, i_s < j_s, 1 \le s \le n\}.$$

For any $\sigma \in S_{2n,pf}$ we define *Pfaff aggregates* a_σ by

$$a_\sigma = a_{i_1,j_1} \cdots a_{i_n,j_n}.$$

We see that the Pfaff aggregates are defined for any triangular array $\bar{A} = (a_{i,j})_{1 \le i < j \le 2n}$. Then the pfaffian of order $2n$ is the polynomial defined as the alternating sum of Pfaff aggregates

$$pf_n = \sum_{\sigma \in S_{2n,pf}} sign\ \sigma\ a_\sigma.$$

Here $sign\, \sigma$ is the signature of the permutation σ,

$$sign\, \sigma = (-1)^{k(\sigma)},$$

where $k(\sigma)$ is the number of inversions

$$k(\sigma) = |\{\ (i, j) \mid \sigma(i) > \sigma(j),\ 1 \le i < j \le n\ \}|.$$

Suppose now that $\{a_{i,j}, 1 \leq i,j \leq 2n\}$ are n^2 generators and endow the space of polynomials $K[a_{i,j}|1 \leq i,j \leq n]$ with the structure of S_{2n}-module by the following action on generators

$$\sigma a_{i,j} = a_{\sigma^{-1}(i),\sigma^{-1}(j)}.$$

In particular, if $A = (a_{i,j})$ has a skew-symmetric set of generators, $a_{i,j} = -a_{j,i}$ then this action induces the structure of S_{2n}-module on the space of polynomials with $\binom{n}{2}$ generators $K[a_{i,j}|1 \leq i < j \leq n]$. Similarly, we obtain one more structure of S_{2n}-module on this space if generators are symmetric, $a_{i,j} = a_{j,i}$. In both cases natural questions appear about invariants under these actions of permutation groups. In particular, we can ask about symmetry and skew-symmetry groups of a given polynomial $f \in K[a_{i,j}]$,

$$Sym\ f = \{\sigma \in S_n | \sigma\ f = f\},$$

$$SSym\ f = \{\sigma \in S_n | \sigma\ f = sign\,\sigma\ f\}.$$

For example, the determinant polynomial $det\,A$ for $A = (a_{ij})_{1 \leq i,j \leq n}$ is a polynomial of degree n and its symmetry group is isomorphic to S_n.

Another example: if a matrix A is skew-symmetric, then the pfaffian polynomial $pf_{2n} = pf_{2n}A$ is a polynomial of degree n and

$$SSym\ pf_{2n} \cong S_{2n}.$$

Let the characteristic of the main field be $p \neq 2$ and

$$g_{2n}(x_1, \ldots, x_{2n}) = (x_1 - x_2)(x_2 - x_3) \cdots (x_{2n-1} - x_{2n})(x_{2n} - x_1).$$

Theorem 1.1. *Let* $\bar{A} = (a_{i,j})_{1 \leq i < j \leq 2n}$ *be the triangular array with components* $a_{i,j} = (x_i - x_j)^2$ *for* $1 \leq i < j \leq 2n$. *Then*

$$pf_{2n}\ \bar{A} = -(-2)^{n-1}\ g_{2n}.$$

Theorem 1.2. *The symmetry group of the polynomial* g_{2n} *is isomorphic to the dihedral group* D_{2n}.

Based on these two results our main result is as follows.

Theorem 1.3. *If generators* $a_{i,j}$ *are symmetric,* $a_{i,j} = a_{j,i}$, *then the symmetry group of the pfaffian polynomial* $pf_{2n} = pf_{2n}\bar{A}$ *is isomorphic to the dihedral group*

$$Sym\ pf_{2n} \cong D_{2n}.$$

Recall that the dihedral group D_n is the symmetry group of a regular n-gon. It can be generated by n rotations and n reflections,

$$D_n = \langle a, b \mid a^n = e, b^2 = e,\ bab = a^{n-1} \rangle.$$

In our paper we use the following notation for permutations. The standard notation for a permutation is a two row notation

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} \in S_n.$$

The one row notation of σ is $i_1 i_2 \cdots i_n$. If σ is a cycle on the set $i_1, i_2, \ldots, i_k$, i.e., $\sigma(i_1) = i_2, \sigma(i_2) = i_3, \ldots, \sigma(i_{k-1}) = i_k, \sigma(i_k) = i_1$, then we will write $\sigma = (i_1, i_2, \ldots, i_k)$. For example,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 4 & 1 & 6 & 2 & 8 & 7 \end{pmatrix} \in S_8 \Rightarrow \sigma = 35416287 = (1,3,4)(2,5,6)(7,8).$$

2 Pfaffian of $(x_i - x_j)^2$

If $A = (a_{i,j})$ is skew-symmetric, then

$$det\,A = (pf_{2n}\bar{A})^2.$$

If a matrix A is not skew-symmetric, say if A is symmetric, then the determinant polynomial $det\,A$ and the pfaffian polynomial $pf_{2n}(\bar{A})$ have no such connection. For example, if $A_n = ((x_i - x_j)^2)_{1\le i,j\le n}$, then

$$det\,A_n = \begin{cases} -(x_1 - x_2)^4, & \text{if } n = 2,\\ 2((x_2 - x_2)(x_2 - x_3)(x_3 - x_1))^2, & \text{if } n = 3,\\ 0, & \text{otherwise,} \end{cases}$$

while, by Theorem 1.1, pfaffians are non-trivial for any even n.

Proof of Theorem 1.1. Let $\psi(x, y) = (x - y)^2$ and $a_{i,j} = \psi(x_i, x_j) = (x_i - x_j)^2$. Note that the pfaffian $pf_{2n}A$ is a polynomial in the variables $x_1, \ldots, x_{2n}$. We have to prove that

$$pf_{2n} = -(-2)^{n-1}g_{2n}.$$

Since $\psi(x, x) = 0$, the polynomial $pf_{2n}(x_1, \ldots, x_s, x_{s+1}, \ldots, x_{2n})$ is divisable by $x_s - x_{s+1}$ for any $1 \le s \le 2n$. Here we set $x_{2n+1} = x_1$. Note that the degree of the polynomial $g_{2n}(x_1, \ldots, x_{2n})$ is $2n$ and the degree of $pf_{2n}((x_i - x_j)^2)$ is also $2n$. Therefore,

$$pf_{2n}(x_1, x_2, \ldots, x_{2n}) = c\; g_{2n}(x_1, x_2, \ldots, x_{2n}),$$

for some constant c. Take $x_i = i$. It is easy to see that

$$g_{2n}(1, 2, \ldots, 2n) = (1 - 2)(2 - 3)\cdots(2n - 1 - 2n)(2n - 1) = -(2n - 1).$$

It remains to prove that

$$pf_{2n}(1, 2, \ldots, 2n) = (-2)^{n-1}(2n - 1) \tag{2.1}$$

to obtain that $c = -(-2)^{n-1}$.

By induction on n we will prove that

$$pf_{2n}(x_1, x_2, \ldots, x_{2n}) = -(-2)^{n-1}g_{2n}(x_1, x_2, \ldots, x_{2n}).$$

For $n = 1$ our statement is evident:

$$pf_2\bar{A} = a_{1,2} = -(x_1 - x_2)(x_2 - x_1).$$

Suppose that our statement is true for $n - 1$,

$$pf_{2n-2}\bar{A} = -(-2)^{n-2}(x_1 - x_2)(x_2 - x_3)\cdots(x_{2n-3} - x_{2n-2})(x_{2n-2} - x_1).$$

Let us prove it for n.

Let us decompose the pfaffian along the first row

$$pf_{2n}\bar{A} = \sum_{i=2}^{2n}(-1)^i a_{1,i}\; pf_{2n-2}A_{\hat{1},\hat{i}}.$$

We see that

$$pf_{2n}\bar{A} = R_1 + R_2 + R_3,$$

where

$$R_1 = a_{1,2}\ pf_{2n}\bar{A}_{\hat{1}\hat{2}},$$

$$R_2 = \sum_{i=3}^{2n-1}(-1)^i a_{1,i}\ pf_{2n-2}\bar{A}_{\hat{1},\hat{i}},$$

$$R_3 = a_{1,2n}\ pf_{2n-2}\bar{A}_{\hat{1},\hat{2n}}.$$

By the inductive suggestion

$$R_1 = -(-2)^{n-2}(x_1 - x_2)^2(x_3 - x_4)\cdots(x_{2n-1} - x_{2n})(x_{2n} - x_3).$$

Hence,

$$R_1|_{x_i \to i} = -(-2)^{n-2}(1-2)^2(3-4)\cdots(2n-1-2n)(2n-3) = (-2)^{n-2}(2n-3),$$

$$R_3|_{x_i \to i} = -(-2)^{n-2}(1-2n)^2(2-3)(3-4)\cdots(2n-2-2n+1)(2n-1-2) =$$

$$-(-2)^{n-2}(2n-1)^2(-1)^{2n-3}(2n-3) = (-2)^{n-2}(2n-3)(2n-1)^2.$$

Further, if $2 < i < 2n$, then

$$(-1)^i a_{1,i}\ pf_{2n}\bar{A}_{\hat{1},\hat{i}}|_{x_j \to j} =$$

$$(-1)^i(-(-2)^{n-2})(x_1 - x_i)^2(x_2 - x_3)(x_3 - x_4)\cdots(x_{i-1} - x_{i+1})(x_{i+1} - x_{i+2}) \times \cdots$$

$$\times(x_{2n-1} - x_{2n})(x_{2n} - x_2)|_{x_j \to j} =$$

$$(-1)^i(-(-2)^{n-2})(i-1)^2(-2)(2n-2) = (-1)^i(i-1)^2(-2)^{n-2}4(n-1).$$

Hence,

$$R_2|_{x_i \to i} = -(-2)^{n-2}\sum_{i=3}^{2n-1} -(-1)^i(i-1)^2 4(n-1) = (-2)^{n-2}4(n-1)(2n^2 - 3n + 2).$$

So, we see that (2.1) is true for n,

$$f_{2n}(1, 2, \ldots, 2n) = R_1 + R_2 + R_3 =$$

$$(-2)^{n-2}[(2n-3) - 4(n-1)(2n^2 - 3n + 2) + (2n-3)(2n-1)^2] =$$

$$-(-2)^{n-2}2(2n-1) = (-2)^{n-1}(2n-1).$$

□

3 Symmetry group of the polynomial g_{2n}

Proof of Theorem 1.2. First, we check that any dihedral permutation $\sigma \in D_{2n}$ is a symmetry of the polynomial g_{2n}.

Let us take the realization of a dihedral group as the symmetry group of the regular n-gon whose vertices are clockwise labelled by $1, 2, \ldots, 2n$. Elements of a dihedral group might have:

I. one up-run: $\sigma = \begin{pmatrix} 1 & 2 & \cdots & 2n \\ 1 & 2 & \cdots & 2n \end{pmatrix}$,

II. one down-run: $\sigma = \begin{pmatrix} 1 & 2 & \cdots & 2n \\ 2n & 2n-1 & \cdots & 1 \end{pmatrix}$,

III. two up-run

$$\sigma(1) = s < \sigma(2) = s+1 < \cdots < \sigma(2n-s+1) = 2n, \quad \sigma(2n-s+2) = 1 < \cdots < \sigma(2n) = s-1,$$

for some $1 < s \le 2n$,

IV. or two down-run

$$\sigma(1) = s > \sigma(2) = s-1 > \cdots > \sigma(s) = 1, \quad \sigma(s+1) = 2n > \cdots > \sigma(2n) = s+1.$$

for some $1 \le s < 2n$.

In cases **I** and **II** our statement is evident.

In case **III** we have

$$g_{2n}(x_{\sigma(1)}, \ldots, x_{\sigma(2n)}) =$$

$$(x_s - x_{s+1})(x_{s+1} - x_{s+2}) \cdots (x_{2n-1} - x_{2n})(x_{2n} - x_1)\ (x_1 - x_2) \cdots (x_{s-2} - x_{s-1})(x_{s-1} - x_s) =$$

$$(x_1 - x_2) \cdots (x_{2n-1} - x_{2n})(x_{2n} - x_1) = g_{2n}(x_1, \ldots, x_{2n}).$$

In case **IV**

$$g_{2n}(x_{\sigma(1)}, \ldots, x_{\sigma(2n)}) =$$

$$(x_s - x_{s-1})(x_{s-1} - x_{s-2}) \cdots (x_2 - x_1)(x_1 - x_{2n})\ (x_{2n} - x_{2n-1}) \cdots (x_{s+1} - x_s) =$$

$$(-1)^s (x_{s-1} - x_s)(x_{s-2} - x_{s-1}) \cdots (x_1 - x_2)(x_{2n} - x_1)\ (-1)^{2n-s} (x_{2n-1} - x_{2n}) \cdots (x_s - x_{s+1}) =$$

$$(x_1 - x_2) \cdots (x_{2n-1} - x_{2n}) = g_{2n}(x_1, \ldots, x_{2n}).$$

So,

$$D_{2n} \subseteq Sym(g_{2n}).$$

Now we will prove that any $\sigma \in Sym(g_{2n})$ is a dihedral permutation.

Let $M_{2n} = \{1, 2, \ldots, 2n\}$. For $i, j \in M_{2n}$ we say that they are connected, if $|i - j| = 1$ or $|i - j| = 2n - 1$. So, if $i < j < 2n$, then i, j are connected iff $j = i + 1$. If $j = 2n$, and i, j are connected, then $i = 2n - 1$ or $i = 1$. It is clear that this relation is symmetric: i, j are connected iff j, i are connected. So, $i, j \in M_{2n}$ are connected, if $|i - j| = 1$ or $(i, j) = (1, 2n)$ or $(i, j) = (2n, 1)$.

Note that the polynomial $g_{2n}(x_1, \ldots, x_{2n})$ is a product of polynomials $x_i - x_j$, $i < j$, where i and j are connected. Therefore, any symmetry $\sigma \in Sym(g_{2n})$ has the following property: if i and j are connected, then $\sigma(i)$ and $\sigma(j)$ are also connected.

Let $\sigma \in Sym(g_{2n})$ and $\sigma(1) = i_1$. The following possibilities may arise.

Case A. Suppose that $\sigma(1) = i_1 < \sigma(2)$. Take $k > 1$, such that $\sigma(k-1) < \sigma(k)$ and $\sigma(k+1) < \sigma(k)$. Since $\sigma(1)$ and $\sigma(2)$ are connected and $\sigma(2) > \sigma(1)$, then $\sigma(2) = i_1 + 1$. By similar arguments,

$$\sigma(3) = i_1 + 2, \ldots, \sigma(k) = i_1 + k - 1,$$

but $\sigma(k+1) \neq i_1 + k$. Such situation is possible only in one case: $i_1 = 2n - k + 1$ and $\sigma(k+1) = 1$. So,

$$\sigma(k+1) = 1, \sigma(k+2) = 2, \ldots, \sigma(2n) = i_1 - 1.$$

In other words,

$$\sigma = i_1 \ (i_1 + 1) \ \ldots \ (2n) \ 1 \ 2 \ \ldots (i_1 - 1).$$

We obtained a permutation σ that has exactly one up-run if $i_1 = 1$, or two up-runs if $i_1 > 1$. So, we obtain permutations of type **I** or **III**. Therefore, $\sigma \in D_{2n}$.

Case B. Now consider the case $\sigma(1) = i_1 > \sigma(2)$. Take $k > 1$, such that $\sigma(k-1) > \sigma(k)$ and $\sigma(k+1) > \sigma(k)$.

Since $\sigma(1)$ and $\sigma(2)$ are connected and $\sigma(2) < \sigma(1)$, then $\sigma(2) = i_1 - 1$. By similar arguments,

$$\sigma(3) = i_1 - 2, \ldots, \sigma(k) = i_1 - k + 1.$$

but $\sigma(k+1) \neq i_1 - k$. Such situation is possible only in one case: $i_1 = k, \sigma(k+1) = 2n$. So,

$$\sigma(k+1) = 2n, \sigma(k+2) = 2n - 1, \ldots, \sigma(2n) = i_1 + 1.$$

In other words,

$$\sigma = i_1 \ (i_1 - 1) \ \ldots \ 1 \ 2n \ (2n - 1) \ \ldots (i_1 + 1).$$

We obtained a permutation σ that has exactly one down-run if $i_1 = 2n$ or two down-runs if $i_1 < 2n$. In other words we obtained a permutations of type **II** or **IV**. Thus, $\sigma \in D_{2n}$. □

4 Proof of Theorem 1.3

First we prove that $D_{2n} \subseteq Sym\, pf_{2n}$.

Lemma 4.1. *If $A = (a_{i,j})$ is symmetric, then the pfaffian is invariant under action of the dihedral group D_{2n},*

$$\mu(pf_{2n}) = pf_{2n}$$

for any $\mu \in D_{2n}$.

Proof. The dihedral group D_{2n} has order $4n$ and is generated by the cyclic permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & 2n-1 & 2n \\ 2 & 3 & 4 & \cdots & 2n & 1 \end{pmatrix}$$

and reflection

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & n & n+1 & n+2 & \cdots & 2n-1 & 2n \\ 1 & 2n & 2n-1 & \cdots & n+2 & n+1 & n & \cdots & 3 & 2 \end{pmatrix}.$$

To prove our lemma it suffices to establish that

$$\sigma(pf_{2n}) = pf_{2n},$$

$$\tau(pf_{2n}) = pf_{2n},$$

if $a_{i,j} = a_{j,i}$, for any $1 \le i < j \le 2n$.

Recall that $\alpha = (i_1, i_2, \dots, i_{2n-1}, i_{2n})$ is a Pfaff permutation, if

$$i_1 < i_3 < i_5 < \cdots < i_{2n-1},$$

$$i_1 < i_2, i_3 < i_4, \dots, i_{2n-1} < i_{2n}.$$

Let $S_{2n,pf}$ be set of all Pfaff permutations. Below we use the one-line notation for permutations. We write $\alpha = (i_1, i_2, \dots, i_{2n-1}, i_{2n})$ instead of

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & i_2 & \dots & i_{2n-1} & i_{2n} \end{pmatrix}.$$

Note that

$$\tau(i) + i = \begin{cases} 2n+2, & \text{if } 1 < i \le 2n, \\ 2 & \text{if } i = 1. \end{cases}$$

Set

$$\bar{i} = 2n + 2 - i,$$

if $i > 1$.

Now we study the action of the generator σ on pfaffian polynomials, when generators are symmetric, $a_{i,j} = a_{j,i}$, for any $1 \le i, j \le 2n$. Let $\alpha = (1, i_2, i_3, \dots, i_{2n}) \in S_{2n,pf}$, and $l = \alpha^{-1}(2n)$. Then l is even, $l = 2k$, and

$$\sigma(a_\alpha) = \sigma(a_{i_1,i_2} \cdots a_{i_{2n-1} i_{2n}}) =$$

$$a_{i_1+1,i_2+1} \cdots a_{i_{2k-3}+1,i_{2k-2}+1} a_{i_{2k-1}+1,1} a_{i_{2k+1}+1,i_{2k+2}+1} \cdots a_{i_{2n-1}+1,i_{2n}+1} = a_{\tilde{\alpha}},$$

where

$$\tilde{\alpha} = (1, i_{2k-1}+1, i_1+1, i_2+1, \dots, i_{2k-3}+1, i_{2k-2}+1, i_{2k+1}+1, i_{2k+2}+1, \\ \dots, i_{2n-1}+1, i_{2n}+1).$$

Here we replace $a_{i_{2k-1}+1,1}$ by $a_{1,i_{2k-1}+1}$. We see that the map

$$S_{2n,pf} \to S_{2n,pf}, \quad \alpha \mapsto \tilde{\alpha}$$

is a bijection and

$$sign\, \tilde{\alpha} = sign\, \alpha.$$

Hence,

$$\sigma(pf_{2n}) = \sum_{\alpha \in S_{2n,pf}} sign\, \alpha\, \sigma(a_\alpha) = \sum_{\alpha \in S_{2n,pf}} sign\, \tilde{\alpha}\, a_{\tilde{\alpha}} = pf_{2n}.$$

So, we have estabilshed that the pfaffian is invariant under action $\sigma \in D_{2n}$. □

Let us study the action of the generator τ on pfaffian polynomials.

We have

$$\tau : a_\alpha \mapsto a_{1,\, \overline{i_2}} a_{\overline{i_3},\, \overline{i_4}} \cdots a_{\overline{i_{2n-1}},\, \overline{i_{2n}}}.$$

Since

$$\overline{i_{2k-1}} > \overline{i_{2k}}, \quad 1 < k \le n,$$

we have to replace $a_{\overline{i_{2k-1}},\, \overline{i_{2k}}}$ by $a_{\overline{i_{2k}},\, \overline{i_{2k-1}}}$. Further,

$$\overline{i_3} > \overline{i_5} > \cdots > \overline{i_{2n-1}} > 1.$$

Therefore,

$$\tau : a_\alpha \mapsto a_{\bar{\alpha}},$$

where

$$a_{\bar{\alpha}} = a_{1,\,\overline{i_2}} a_{\overline{i_{2n}},\,\overline{i_{2n-1}}} a_{\overline{i_{2n-2}},\,\overline{i_{2n-3}}} \cdots a_{\overline{i_4},\,\overline{i_3}}.$$

We see that

$$sign\ \alpha = sign\ \bar{\alpha}.$$

Note that the map

$$S_{2n,pf} \to S_{2n,pf}, \quad \alpha \mapsto \bar{\alpha},$$

is a bijection. Therefore,

$$\tau(pf_{2n}) = \sum_{\alpha \in S_{2n,pf}} sign\,\alpha\ \tau(a_\alpha) = \sum_{\alpha \in S_{2n,pf}} sign\,\bar{\alpha}\ a_{\bar{\alpha}} = pf_{2n}.$$

So, we have proved that the pfafian pf_{2n} is invariant under the action of the dihedral group D_{2n} of order $4n$, if the matrix $(a_{i,j})_{1 \le i,j \le 2n}$ is symmetric.
Example. Let

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}, \mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

Then

$$\tau(pf_4) = \tau(a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}) =$$

$$a_{1,4}a_{3,2} - a_{1,3}a_{4,2} + a_{1,2}a_{4,3} =$$

$$a_{1,4}a_{2,3} - a_{1,3}a_{2,4} + a_{1,2}a_{3,4} = pf_4,$$

$$\mu(pf_4) = \mu(a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}) =$$

$$a_{4,3}a_{2,1} - a_{4,2}a_{3,1} + a_{4,1}a_{3,2} =$$

$$a_{3,4}a_{1,2} - a_{2,4}a_{1,3} + a_{1,4}a_{2,3} = pf_4.$$

Proof of Theorem 1.3. . Let $\sigma \in Sym\, pf_{2n}$ i.e.,

$$\sigma\,(pf_{2n}) = pf_{2n}$$

for any $a_{i,j}$, such that $a_{i,j} = a_{j,i}$. In particular, σ is a symmetry of the pfaffian polynomial $pf_{2n}((x_i - x_j)^2)_{1 \le i < j \le 2n}$. By Theorems 1.1 and 1.2 and Lemma 4.1 our theorem is valid. □

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Askar Serkulovich Dzhumadil'daev
Kazakh-British Technical University
59 Tole bi,
050000 Almaty, Kazakhstan
E-mail: dzhuma@hotmail.com

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