ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2024, Volume 15, Number 3

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russia

YESMUKHANBET SAIDAKHMETOVICH SMAILOV

Doctor of physical and mathematical sciences, Professor Smailov Esmuhanbet Saidakhmetovich passed away on May 24, 2024, at the age of 78 years.

Esmuhanbet Saidakhmetovich was well known to the scientific community as a high qualified specialist in science and education, and an outstanding organizer. Fundamental scientific articles and textbooks written in various fields of the theory of functions of several variables and functional analysis, the theory of approximation of functions, embedding theorems, and harmonic analysis are a significant contribution to the development of mathematics.

E.S. Smailov was born on October 18, 1946, in the village of Kyzyl Kesik, Aksuat district, Semipalatinsk region. In 1963, he graduated from high school with a silver medal, and in the same year he entered the Faculty of Mechanics and Mathematics of the Kazakh State University (Almaty) named after Kirov (now named after Al-Farabi). In 1971 he graduated from graduate school at the Institute of Mathematics and Mechanics.

He defended his PhD thesis in 1973 (supervisor was K.Zh. Nauryzbaev) and defended his doctoral thesis “Fourier multipliers, embedding theorems and related topics” in 1997. In 1993 he was awarded the academic title of professor.

E.S. Smailov since 1972 worked at the Karaganda State University named after E.A. Buketov as an associate professor (1972-1978), the head of the department of mathematical analysis (1978-1986, 1990-2000), the dean of the Faculty of Mathematics (1983-1987) and was the director of the Institute of Applied Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan in Karaganda (2004 -2018).

Professor Smailov was one of the leading experts in the theory of functions and functional analysis and a major organizer of science in the Republic of Kazakhstan. He had a great influence on the formation of the Mathematical Faculty of the Karaganda State University named after E.A. Buketov and he made a significant contribution to the development of mathematics in Central Kazakhstan. Due to the efforts of Y.S. Smailov, in Karaganda an actively operating Mathematical School on the function theory was established, which is well known in Kazakhstan and abroad.

He published more than 150 scientific papers and 2 monographs. Under his scientific advice, 4 doctoral and 10 candidate theses were defended.

In 1999 the American Biographical Institute declared professor Smailov “Man of the Year” and published his biography in the “Biographical encyclopedia of professional leaders of the Millennium”.

For his contribution to science and education, he was awarded the Order of “Kurmet” (=“Honour”).

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Esmuhanbet Saidakhmetovich Smailov.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 15, Number 3 (2024), 38 – 45

WEAK VERSION OF SYMMETRIC SPACE

T.N. Bekjan

Communicated by K.N. Ospanov

Key words: symmetric space, fundamental function of a symmetric space, noncommutative symmetric space, von Neumann algebra.

AMS Mathematics Subject Classification: 46L52, 47L05.

Abstract. In this paper, we defined weak versions of symmetric spaces and established Hölder and Chebyshev type inequalities for noncommutative spaces associated with these spaces.

DOI: https://doi.org/10.32523/2077-9879-2024-15-3-38-45

1 Introduction

In the realm of the classical analysis, the utilization of weak L_p spaces in both harmonic analysis and martingale theory has received significant scholarly attention. These spaces have proven instrumental in various areas such as interpolation theory, rearrangement-invariant function spaces, weighted inequalities, singular integral operators, and beyond, playing pivotal roles in advancing theoretical frameworks and facilitating analytical investigations. For example, using the weak L_p norm, Ledoux and Talagrand [16] conducted an investigation into the integrability properties and tail probability behavior of p-stable random variables. Soria [19] delved into the discussion of weak-type Lorentz space $\Lambda_{p,\infty}(\omega)$ for $0 < p < \infty$. Fefferman and Soria [11] also addressed various properties of the weak Hardy space H_1. Weisz [23, 24] dedicated his studies to the weak atom decompositions of martingales and martingale inequalities within weak Hardy spaces. Furthermore, Cwikel and other scholars extensively examined the dual of weak L_p spaces (cf. [6, 7]).

Liu/Hou/Wang [17] introduced the weak version of Orlicz spaces and proved the Burkholder-Gundy inequalities for martingales in these weak Orlicz spaces. The noncommutative version of the weak Orlicz spaces was investigated in [1] and was utilized in the theory of noncommutative martingales. In [3], Raikhan and the author considered the weak noncommutative Orlicz space cases associated with arbitrary faithful normal locally finite weights on a semi-finite von Neumann algebra $\mathcal{M}$, and characterized the dual spaces of the noncommutative weak Orlicz-Hardy spaces.

Since the weak versions of L_p spaces and Orlicz spaces have opened new research avenues in (noncommutative) harmonic analysis and (noncommutative) martingale theory, we are investigating a weak version of symmetric spaces. We will apply them in the study of (noncommutative) harmonic analysis and (noncommutative) martingale theory. Notice that for a symmetric (quasi-) Banach space E, we define the weak version of E as the usual Marcinkiewicz space M_{φ_E} associated with the fundamental function φ_E of E. In the rearrangement-invariant Banach space case, it is the space $M(E)$ ([4, Definition 2.5.2]).

The purpose of this paper is to investigate a weak version of symmetric spaces and to study some properties of noncommutative spaces associated with the weak version of symmetric spaces.

2 Preliminaries

Let $L_0(0,1)$ be the set of all Lebesgue measurable almost everywhere finite real-valued functions on $(0,1)$. For $f \in L_0(0,1)$ we define the distribution function $\lambda(f)$ of f by

$$\lambda_s(f) = m(\{\omega \in (0,1) : |f(\omega)| > s\}), \quad s > 0$$

and its decreasing rearrangement $\mu(f)$ by

$$\mu_t(f) = \inf\{s > 0 \, : \, \lambda_s(f) \le t\}, \quad t > 0.$$

If $f, g \in L_0(0,1)$ and

$$\int_0^t \mu_s(f)ds \le \int_0^t \mu_s(g)ds, \qquad \text{for all} \quad t > 0,$$

we say f is *majorized* by g, and write $f \preccurlyeq g$.

If E is a (quasi-)Banach lattice of measurable functions on $(0,1)$ (with the Lebesgue measure) and satisfies the following properties:

if $f \in E$, $g \in L_0(0,1)$ and $\mu(g) \le \mu(f)$ implies that $g \in E$ and $\|g\|_E \le \|f\|_E$,

then E is called a symmetric (quasi-)Banach space on $(0,1)$. E is called fully symmetric if, in addition,

for $x \in L_0(I)$ and $y \in E$ with $x \preceq y$ it follows that $x \in E$ and $\|x\|_E \le \|y\|_E$.

For $0 < p < \infty$, $E^{(p)}$ will denote the quasi-Banach lattice defined by

$$E^{(p)} = \{f : \ |f|^p \in E\},$$

equipped with the quasi-norm

$$\|f\|_{E^{(p)}} = \||f|^p\|_E^{\frac{1}{p}}.$$

Observe that, if $0 < p,\ q < \infty$, then $(E^{(p)})^{(q)} = E^{(pq)}$. It is to be noted that, if E is a Banach space and $p > 1$, then the space $E^{(p)}$ is a Banach space and is usually called the p-convexification of E.

Let $0 < \alpha, \beta < \infty$. If there a constant $C > 0$ such that for all finite sequences $(f_n)_{n\ge1}$ in E

$$\|(\sum |f_n|^\alpha)^{\frac{1}{\alpha}}\|_E \le C(\sum \|f_n\|_E^\alpha)^{\frac{1}{\alpha}}$$
$$(\text{respectively, } \|(\sum |f_n|^\beta)^{\frac{1}{\beta}}\|_E \ge C^{-1}(\sum \|f_n\|_E^\beta)^{\frac{1}{\beta}}),$$

then E is called α-convex (respectively, β-concave). The least such constant C is called the α-convexity (respectively, β-concavity) constant of E and is denoted by $M^{(\alpha)}(E)$ (respectively, $M_{(\beta)}(E)$). If E is α-convex and β-concave, then $E^{(p)}$ is $p\alpha$-convex and $p\beta$-concave with $M^{(p\alpha)}(E^{(p)}) = M^{(\alpha)}(E)^{\frac{1}{p}}$ and $M_{(p\beta)}(E^{(p)}) = M_{(\beta)}(E)^{\frac{1}{p}}$ (see [9, Proposition 3.1]). Therefore, if E is α-convex then $E^{(\frac{1}{\alpha})}$ is 1-convex, so it can be renormed as a Banach lattice (see [15, Proposition 1.d.8] and [22, p. 544]).

A symmetric (quasi-)Banach space E on $(0,1)$ is said to have the Fatou property if for every net $(x_i)_{i\in I}$ in E satisfying $0 \le x_i \uparrow$ and $\sup_{i\in I}\|x_i\|_E < \infty$ the supremum $x = \sup_{i\in I} x_i$ exists in E and $\|x_i\|_E \uparrow \|x\|_E$; We say that E has order continuous norm, if for every net $(f_i)_{i\in I}$ in E such that $f_i \downarrow 0$, $\|f_i\|_E \downarrow 0$ holds; E is called a rearrangement invariant space if it has order continuous (quasi-)norm or the Fatou property.

Let E_i be a symmetric (quasi-)Banach space on $(0,1)$, $i = 1, 2$. We define the pointwise product space $E_1 \odot E_2$ as

$$E_1 \odot E_2 = \{f : \ f = f_1 f_2, \ f_i \in E_i, \ i = 1, \ 2\} \tag{2.1}$$

with a functional $\|f\|_{E_1 \odot E_2}$ defined by

$$\|f\|_{E_1 \odot E_2} = \inf\{\|f_1\|_{E_1}\|f_2\|_{E_2} : \ f = f_1 f_2, \ f_i \in E_i, \ i = 1, \ 2\}.$$

If E_i is a symmetric quasi-Banach space on $(0,1)$, $i = 1, 2$, then by [3, Corollary 1], there is an equivalent quasi-norm $\|\cdot\|$ such that $(E_1 \odot E_2, \ \|\cdot\|)$ is a symmetric quasi-Banach space on $(0,1)$.

It is clear that if E is a symmetric (quasi-)Banach space on $(0,1)$, then for different Lebesgue measurable subsets A of $(0,1)$ with the same measure $m(A) = t$, the value of $\|\chi_A\|$ remains constant, where χ_A is the characteristic function of A.

Definition 1. Let E be a symmetric (quasi-)Banach space on $(0,1)$. The fundamental function φ_E is defined by $\varphi_E(t) = \|\chi_A\|$, where $t \in [0,1)$ and A is a Lebesgue measurable subset of $(0,1)$ with $m(A) = t$.

Note that $\varphi_{L_1(0,1)} = t$ (see [4, p. 65]). Let $0 < p < \infty$. If $A \subset (0,1)$ with $m(A) = t$ $(0 \le t < 1)$, then

$$\varphi_{L_p(0,1)}(t) = \|\chi_A\|_p = \|\chi_A\|_1^{\frac{1}{p}} = t^{\frac{1}{p}}.$$

Let $M_{\varphi_E}(0,1)$ be the usual Marcinkiewicz space:

$$M_{\varphi_E}(0,1) = \{f \in L_0(0,1) : \ \|f\|_{M_{\varphi_E}} = \sup_{t>0} \frac{\varphi_E(t)}{t} \int_0^t \mu_s(f) ds < \infty\}.$$

Definition 2. Let E be a symmetric (quasi-)Banach space on $(0,1)$. We call $M_{\varphi_E}(0,1)$ is a weak version of E and denote it by E_∞.

The classical weak L_p space $L_{p,\infty}(0,1)$ $(1 \le p < \infty)$ is defined as the set of all measurable functions f on $(0,1)$ such that

$$\|f\|_{L_{p,\infty}} = \sup_{t>0} t^{\frac{1}{p}} \mu_t(f) < \infty.$$

For $p > 1$, $L_{p,\infty}(0,1)$ can be renormed into a Banach space. More precisely,

$$f \mapsto \sup_{t>0} t^{-1+\frac{1}{p}} \int_0^t \mu_s(f) ds$$

gives an equivalent norm on $L_{p,\infty}(0,1)$. We refer to [12] for more information about weak L_p spaces.

If $E = L_p(0,1)$ $(1 < p < \infty)$, then $E_\infty = L_{p,\infty}(0,1)$. But for $0 < p \le 1$, if $f \in (L_p(0,1))_\infty$, then

$$\|f\|_{(L_p(0,1))_\infty} = \sup_{t>0} t^{\frac{1}{p}-1} \int_0^t \mu_s(f) ds = \int_0^1 \mu_s(f) ds = \|f\|_1.$$

Hence, $(L_p(0,1))_\infty = L_1(0,1)$ and it is different from the classical weak L_p space.

Let Φ be an N-function, we define

$$a_\Phi = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)} \quad \text{and} \quad b_\Phi = \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

If $b_\Phi < \infty$, then the fundamental function of Orlicz space $L_\Phi(0,1)$ on $(0,1)$ equipped with the Luxemburg norm, is the following

$$\varphi_{L_\Phi(\Omega)}(t) = 1/\Phi^{-1}(\frac{1}{t}), \quad t > 0,$$

where the Luxemburg norm is defined by

$$\|x\|_\Phi = \inf\{\lambda > 0 : \int_0^1 \Phi(\frac{|x|}{\lambda})dx \leq 1\}.$$

Hence, if $E = L_\Phi(0,1)$ and $1 < a_\Phi \leq b_\Phi < \infty$, then $E_\infty = L_{\Phi,\infty}(0,1)$.

For more details on symmetric (quasi-)Banach space and Orlicz spaces we refer to [4, 5, 9, 14, 15, 18, 21, 25].

Let $\mathcal{M}$ be a finite von Neumann algebra with a normal finite faithful trace τ ($\tau(1) = 1$) and $L_0(\mathcal{M})$ be the topological $*$-algebra of measurable operators with respect to $(\mathcal{M}, \tau)$. For $x \in L_0(\mathcal{M})$, we define the distribution function $\lambda(x)$ of x as follows:

$$\lambda_t(x) = \tau(e_{(t,\infty)}(|x|)) \quad \text{for} \quad t > 0,$$

where $e_{(t,\infty)}(|x|)$ is the spectral projection of $|x|$ in the interval (t, ∞). We also define the generalized singular numbers $\mu(x)$ of x as

$$\mu_t(x) = \inf\{s > 0 : \ \lambda_s(x) \leq t\} \quad \text{for} \quad t > 0.$$

Recall that both functions $\lambda(x)$ and $\mu(x)$ are decreasing and continuous from the right on $(0, \infty)$ (for further information, see [10]).

For a symmetric quasi-Banach function space E on $(0, 1)$, set

$$E(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \ \mu(x) \in E\};$$

$$\|x\|_E = \|\mu(x)\|_E, \qquad x \in E(\mathcal{M}).$$

Recall that $(E(\mathcal{M}), \|.\|_E)$ is a Banach space and we call $(E(\mathcal{M}), \|.\|_E)$ a noncommutative symmetric Banach space (see for reference [8, 20]).

3 Properties

If E_1 and E_2 are symmetric Banach spaces on $(0, 1)$, then by [13, Theorem 2], we know that

$$\varphi_{E_1 \odot E_2}(t) = \varphi_{E_1}(t)\varphi_{E_2}(t), \quad t \geq 0. \tag{3.1}$$

We claim that if E is a symmetric (quasi-)Banach space on $(0, 1)$ and $0 < p < \infty$, then

$$\varphi_{E^{(p)}}(t) = \varphi_E(t)^{\frac{1}{p}}, \quad t \geq 0. \tag{3.2}$$

Indeed, if $A \subset (0, 1)$ with $m(A) = t$ $(0 \leq t < 1)$, then

$$\varphi_{E^{(p)}}(t) = \||\chi_A|^p\|_E^{\frac{1}{p}} = \|\chi_A\|_E^{\frac{1}{p}} = \varphi_E(t)^{\frac{1}{p}}.$$

Proposition 3.1. *Let E_i be a symmetric (quasi-)Banach space on $(0, 1)$ which is α_i-convex for some $0 < \alpha_i < \infty$ $(i = 1, 2)$. Then E_1 and E_2 can be equipped with equivalent quasi norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, so that $\varphi_{E_1 \odot E_2}(t) = \varphi_{E_1}(t)\varphi_{E_2}(t)$, for any $t \geq 0$.*

Proof. Let $n \in \mathbb{N}$ such that $n\alpha_i \geq 1$ $(i = 1, 2)$. Then $E_i^{(n)} = (E_i^{\alpha_1})^{(n\alpha_i)}$ can be renormed as a symmetric Banach space $(i = 1, 2)$. In the following, we consider $E_j^{(n)}$ with this new symmetric norm $(j = 1, 2)$. Using [13, Theorem 1 (iii)], we get that $(E_1 \odot E_2)^{(n)} = E_1^{(n)} \odot E_2^{(n)}$. Applying (3.1), we get $\varphi_{(E_1 \odot E_2)^{(n)}}(t) = \varphi_{E_1^{(n)}}(t)\varphi_{E_2^{(n)}}(t)$, for each $t \geq 0$. Hence, by (3.2), we have that

$$\varphi^{\frac{1}{n}}_{E_1 \odot E_2}(t) = \varphi^{\frac{1}{n}}_{E_1}(t)\varphi^{\frac{1}{n}}_{E_2}(t), \quad t \geq 0.$$

Thus, we obtain the desired result. □

In rest of this paper, $\mathcal{M}$ will always denote a finite von Neumann algebra with a normal finite faithful trace τ ($\tau(1)=1$).

Theorem 3.1. *Let E_i be a symmetric (quasi-)Banach space on $(0,1)$ which is α_i-convex for some $0<\alpha_i<\infty$ $(i=1,\ 2)$ and $0<a<1$. If $x\in((E_1^{(a)})_\infty)^{(\frac{1}{a})}(\mathcal{M})$ and $y\in((E_2^{(1-a)})_\infty)^{(\frac{1}{1-a})}(\mathcal{M})$, then $xy\in(E_1\odot E_2)_\infty(\mathcal{M})$ and the following Hölder type inequality holds*

$$\|xy\|_{(E_1\odot E_2)_\infty}\le\|x\|_{((E_1^{(a)})_\infty)^{(\frac{1}{a})}}\|y\|_{((E_2^{(1-a)})_\infty)^{(\frac{1}{1-a})}}.$$

Proof. Let $x\in((E_1^{(a)})_\infty)^{(\frac{1}{a})}(\mathcal{M})$ and $y\in((E_2^{(1-a)})_\infty)^{(\frac{1}{1-a})}(\mathcal{M})$. By Proposition 3.1, [10, Theorem 4.2, Lemma 2.3(iv)], classical Hölder inequality and (3.2), we have that

$$\begin{aligned}
\|xy\|_{(E_1\odot E_2)_\infty} &= \sup_{t>0}\tfrac{\varphi_{E_1\odot E_2}(t)}{t}\int_0^t\mu_s(xy)ds\\
&= \sup_{t>0}\tfrac{\varphi_{E_1}(t)\varphi_{E_2}(t)}{t}\int_0^t\mu_s(xy)ds\\
&\le \sup_{t>0}\tfrac{\varphi_{E_1}(t)\varphi_{E_2}(t)}{t}\int_0^t\mu_s(x)\mu_s(y)ds\\
&\le \sup_{t>0}\tfrac{\varphi_{E_1}(t)\varphi_{E_2}(t)}{t}\Big(\int_0^t\mu_s(x)^{\frac{1}{a}}ds\Big)^a\Big(\int_0^t\mu_s(y)^{\frac{1}{1-a}}ds\Big)^{1-a}\\
&\le \sup_{t>0}\tfrac{\varphi_{E_1}(t)\varphi_{E_2}(t)}{t}\Big(\int_0^t\mu_s(|x|^{\frac{1}{a}})ds\Big)^a\Big(\int_0^t\mu_s(|y|^{\frac{1}{1-a}})ds\Big)^{1-a}\\
&= \sup_{t>0}\Big(\tfrac{\varphi_{E_1}(t)^{\frac{1}{a}}}{t}\int_0^t\mu_s(|x|^{\frac{1}{a}})ds\Big)^a\Big(\tfrac{\varphi_{E_2}(t)^{\frac{1}{1-a}}}{t}\int_0^t\mu_s(|y|^{\frac{1}{1-a}})ds\Big)^{1-a}\\
&= \sup_{t>0}\Big(\tfrac{\varphi_{E_1^{(a)}}(t)}{t}\int_0^t\mu_s(|x|^{\frac{1}{a}})ds\Big)^a\Big(\tfrac{\varphi_{E_2^{(1-a)}}(t)}{t}\int_0^t\mu_s(|y|^{\frac{1}{1-a}})ds\Big)^{1-a}\\
&\le \sup_{t>0}\Big(\tfrac{\varphi_{E_1^{(a)}}(t)}{t}\int_0^t\mu_s(|x|^{\frac{1}{a}})ds\Big)^a\sup_{t>0}\Big(\tfrac{\varphi_{E_2^{(1-a)}}(t)}{t}\int_0^t\mu_s(|y|^{\frac{1}{1-a}})ds\Big)^{1-a}\\
&= \||x|^{\frac{1}{a}}\|^a_{(E_1^{(a)})_\infty}\||y|^{\frac{1}{1-a}}\|^{(1-a)}_{(E_2^{(1-a)})_\infty}=\|x\|_{((E_1^{(a)})_\infty)^{(\frac{1}{a})}}\|y\|_{((E_2^{(1-a)})_\infty)^{(\frac{1}{1-a})}}.
\end{aligned}$$

□

Proposition 3.2. *Let E be a symmetric (quasi-)Banach space on $(0,1)$.*

(i) *If $1\le p<\infty$, then $(E_\infty)^{(p)}(\mathcal{M})\hookrightarrow(E^{(p)})_\infty(\mathcal{M})$.*

(ii) *If $0<p\le1$, then $(E^{(p)})_\infty(\mathcal{M})\hookrightarrow(E_\infty)^{(p)}(\mathcal{M})$.*

Proof. (i) Let $x\in(E_\infty)^{(p)}(\mathcal{M})$. Using Jensen's inequality and [10, Lemma 2.3(iv)], we obtain that

$$\begin{aligned}
\|x\|_{(E^{(p)})_\infty} &= \sup_{t>0}\tfrac{\varphi_{E^{(p)}}(t)}{t}\int_0^t\mu_s(x)ds\\
&= \sup_{t>0}\tfrac{\varphi_E(t)^{\frac{1}{p}}}{t}\int_0^t\mu_s(x)ds\\
&= \Big(\sup_{t>0}\varphi_E(t)(\tfrac{1}{t}\int_0^t\mu_s(x)ds)^p\Big)^{\frac{1}{p}}\\
&\le \Big(\sup_{t>0}\tfrac{\varphi_E(t)}{t}\int_0^t\mu_s(x)^pds\Big)^{\frac{1}{p}}\\
&= \Big(\sup_{t>0}\tfrac{\varphi_E(t)}{t}\int_0^t\mu_s(|x|^p)ds\Big)^{\frac{1}{p}}\\
&= \||x|^p\|_{E_\infty}^{\frac{1}{p}}=\|x\|_{(E_\infty)^{(p)}}.
\end{aligned}$$

The proof of (ii) is similar to the proof of (i). □

In general, $(E_\infty)^{(p)}(\mathcal{M})\ne(E^{(p)})_\infty(\mathcal{M})$. For example, let $E=L_1(\mathcal{M})$. If $1<p<\infty$, then $(E_\infty)^{(p)}(\mathcal{M})=L_p(\mathcal{M})$ and $(E^{(p)})_\infty(\mathcal{M})=L_{p,\infty}(\mathcal{M})$. If $0<p<1$, then $(E_\infty)^{(p)}(\mathcal{M})=L_p(\mathcal{M})$ and $(E^{(p)})_\infty(\mathcal{M})=L_1(\mathcal{M})$.

Theorem 3.2. *Let E be a symmetric (quasi-)Banach space on $(0,1)$. Then we have the following Chebyshev type inequality*

$$t\varphi_E(\tau(e_{(t,\infty)}(|x|))) \leq \|x\|_{E\infty}, \quad \forall x \in E_\infty(\mathcal{M}).$$

Proof. It is clear that for $s \geq 0$,

$$\mu_s(e_{(t,\infty)}(|x|)) = \chi_{[0,\tau(e_{(t,\infty)}(|x|))}.$$

Since $|x|e_{(t,\infty)}(|x|) \geq te_{(t,\infty)}(|x|)$,

$$\begin{aligned}\varphi_E(\tau(e_{(t,\infty)}(|x|))) &\leq \sup_{s>0} \tfrac{\varphi_E(s)}{s} \int_0^s \mu_\nu(e_{(t,\infty)}(|x|))d\nu \\ &= \|e_{(t,\infty)}(|x|)\|_{E_\infty} \leq \|\tfrac{1}{t}|x|e_{(t,\infty)}(|x|)\|_{E_\infty} \\ &= \tfrac{1}{t}\||x|e_{(t,\infty)}(|x|)\|_{E_\infty} \leq \tfrac{1}{t}\||x|\|_{E_\infty} = \tfrac{1}{t}\|x\|_{E_\infty}.\end{aligned}$$

□

Acknowledgments

The author thanks the unknown referee for useful comments, which improved the paper.

This work was supported by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan, project no. AP14870431.

References

[1] T.N. Bekjan, Z. Chen, P. Liu, Y. Jiao, *Noncommutative weak Orlicz spaces and martingale inequalities.* Studia Math. 204 (2011), 195–212.

[2] T.N. Bekjan, M.N. Ospanov, *On products of noncommutative symmetric quasi Banach spaces and applications.* Positivity 25 (2021), 121–148.

[3] T.N. Bekjan, M. Raikhan, *On noncommutative weak Orlicz-Hardy spaces.* Ann. Funct. Anal. 13 (2022), article number 7.

[4] C. Bennett, R. Sharpley, *Interpolation of operators.* Academic Press Inc., Boston, MA, 1988.

[5] N.A. Bokayev, A. Gogatishvili, A.N. Abek, *On estimates of non-increasing rearrangement of generalized fractional maximal function*, Eurasian Math. J., 14 (2023), no. 2, 13–23.

[6] M. Cwikel, *The dual of weak Lp.* Ann Inst Fourier 25 (1975), 85–126.

[7] M. Cwikel, C. Fefferman, *The canonical seminorm on weak* L^1. Studia Math. 78 (1984), 275–278.

[8] P.G. Dodds, T.K. Dodds, B. de Pagter, *Fully symmetric operator spaces.* Integ. Equ. Oper. Theory 15 (1992), 942–972.

[9] P.G. Dodds, T.K. Dodds, F.A. Sukochev, *On* p*-convexity and* q*- concavity in non-commutative symmetric spaces.* Integ. Equ. Oper. Theory 78 (2014), no. 1, 91–114.

[10] T. Fack, H. Kosaki, *Generalized* s*-numbers of* τ*-measurable operators.* Pac. J. Math. 123 (1986), 269–300.

[11] R. Fefferman, F. Soria, *The space weak* H^1. Studia Math 85 (1987), 1–16.

[12] L. Grafakos, *Classical and modern Fourier analysis.* Pearson Education, London, 2004.

[13] P. Kolwicz, K. Leśnik, L. Maligranda, *Pointwise products of some Banach function spaces and factorization.* J. Funct. Anal. 266 (2014), 616–659.

[14] S.G. Krein, J.I. Petunin, E.M. Semenov, *Interpolation of linear operators*, Translations of Mathematical Monographs, vol.54, Amer. Math. Soc., 1982.

[15] J. Lindenstraus, L. Tzafriri, *Classical Banach space* II. Springer-Verlag, Berlin, 1979.

[16] M. Ledoux, M. Talagrand, *Probability in Banach spaces (Isoperimetry and processes).* Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, 1991.

[17] P. Liu, Y. Hou, M. Wang, *Weak Orlicz space and its applications to martingale theory.* Sci. China Math. 53 (2010), no. 4, 905–916.

[18] M. Rao, Z. Ren, *Application of Orlicz Spaces.* Marcel Dekker, New York, 2002.

[19] J. Soria, *Lorentz spaces of weak-type.* Quart. J. Math. Oxford 49 (1998), 93–103.

[20] F. Sukochev, *Completeness of quasi-normed symmetric operator spaces.* Indag. Math. (N.S.) 25 (2014), no. 2, 376–388.

[21] K.S. Tulenov, *Optimal rearrangement-invariant Banach function range for the Hilbert transform.* Eurasian Mathematical Journal 12 (2021), no. 2, 90–103.

[22] Q. Xu, *Analytic functions with values in lattices and symmetric spaces of measurable operators.* Math. Proc. Camb. Phil. Soc. 109 (1991), 541–563.

[23] F. Weisz, *Weak martingale Hardy spaces.* Prob. Math. Stat. 18 (1998), 133–148.

[24] F. Weisz, *Bounded operators on weak Hardy spaces and applications.* Acta Math Hungarica 80 (1998), 249–264.

[25] N. Zhangabergenova, A. Temirkhanova, *Iterated discrete Hardy-type inequalities*, Eurasian Math. J., 14 (2023), no.1, 81–95.

Turdebek Nurlybekuly Bekjan
Department of Computational and Data Science
Astana IT University
55/11 Mangilik El Avenue,
010000 Astana, Kazakhstan
E-mails: bekjant@yahoo.com

Received: 26.08.2023