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YESMUKHANBET SAIDAKHMETOVICH SMAILOV

Doctor of physical and mathematical sciences, Professor Smailov Esmuhanbet Saidakhmetovich passed away on May 24, 2024, at the age of 78 years.

Esmuhanbet Saidakhmetovich was well known to the scientific community as a high qualified specialist in science and education, and an outstanding organizer. Fundamental scientific articles and textbooks written in various fields of the theory of functions of several variables and functional analysis, the theory of approximation of functions, embedding theorems, and harmonic analysis are a significant contribution to the development of mathematics.

E.S. Smailov was born on October 18, 1946, in the village of Kyzyl Kesik, Aksuat district, Semipalatinsk region. In 1963, he graduated from high school with a silver medal, and in the same year he entered the Faculty of Mechanics and Mathematics of the Kazakh State University (Almaty) named after Kirov (now named after Al-Farabi). In 1971 he graduated from graduate school at the Institute of Mathematics and Mechanics.

He defended his PhD thesis in 1973 (supervisor was K.Zh. Nauryzbaev) and defended his doctoral thesis “Fourier multipliers, embedding theorems and related topics” in 1997. In 1993 he was awarded the academic title of professor.

E.S. Smailov since 1972 worked at the Karaganda State University named after E.A. Buketov as an associate professor (1972-1978), the head of the department of mathematical analysis (1978-1986, 1990-2000), the dean of the Faculty of Mathematics (1983-1987) and was the director of the Institute of Applied Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan in Karaganda (2004 -2018).

Professor Smailov was one of the leading experts in the theory of functions and functional analysis and a major organizer of science in the Republic of Kazakhstan. He had a great influence on the formation of the Mathematical Faculty of the Karaganda State University named after E.A. Buketov and he made a significant contribution to the development of mathematics in Central Kazakhstan. Due to the efforts of Y.S. Smailov, in Karaganda an actively operating Mathematical School on the function theory was established, which is well known in Kazakhstan and abroad.

He published more than 150 scientific papers and 2 monographs. Under his scientific advice, 4 doctoral and 10 candidate theses were defended.

In 1999 the American Biographical Institute declared professor Smailov “Man of the Year” and published his biography in the “Biographical encyclopedia of professional leaders of the Millennium”.

For his contribution to science and education, he was awarded the Order of “Kurmet” (=“Honour”).

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Esmuhanbet Saidakhmetovich Smailov.

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OPTIMAL CUBATURE FORMULAS FOR MORREY TYPE FUNCTION CLASSES ON MULTIDIMENSIONAL TORUS

Sh.A. Balgimbayeva, D.B. Bazarkhanov

Communicated by E.D. Nursultanov

Key words: Nikol'skii-Besov/Lizorkin-Triebel smoothness spaces related to Morrey space, multidimensional torus, optimal cubature formula.

AMS Mathematics Subject Classification: 42B35, 41A63, 41A55.

Abstract. In the paper, we establish estimates, sharp in order, for the error of optimal cubature formulas for the smoothness spaces $B^{s\,\tau}_{p\,q}(\mathbb{T}^m)$ of Nikol'skii – Besov type and $F^{s\,\tau}_{p\,q}(\mathbb{T}^m)$ of Lizorkin – Triebel type, both related to Morrey spaces, on multidimensional torus, for some range of the parameters s, p, q, τ ($0 < s < \infty$, $1 \le p$, $q \le \infty$, $0 \le \tau \le 1/p$). In particular, we obtain those estimates for the isotropic Lizorkin – Triebel function spaces $F^{s}_{\infty\,q}(\mathbb{T}^m)$.

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1 Introduction

Let Ω be a compact set in $\mathbb{R}^m$ ($m \ge 2$) (with nonempty interior), F a set (class) of complex–valued continuous functions with domain Ω. In numerical integration, for the approximation of the integral

$$\int_\Omega f(x)dx, \quad f \in \mathrm{F},$$

expressions of the form (cubature formulas)

$$\mathscr{Q}(f, C_N, \Lambda_N) := \sum_{k=1}^{N} c(k) f(\lambda(k)), \tag{1.1}$$

are used; here $C_N := (c(1), \ldots, c(N)) \in \mathbb{C}^N$ is the collection of weights and $\Lambda_N := (\lambda(1), \ldots, \lambda(N)) \subset \Omega^N$ is the grid of nodes of the cubature formula, and

$$\mathscr{R}(f, \Omega, C_N, \Lambda_N) := \int_\Omega f(x)dx - \mathscr{Q}(f, C_N, \Lambda_N)$$

is its error on a function f. Denote

$$\mathscr{R}(\mathrm{F}, \Omega, C_N, \Lambda_N) := \sup\{|\mathscr{R}(f, \Omega, C_N, \Lambda_N)| \ \mid \ f \in \mathrm{F}\}.$$

The problem of optimal numerical integration under consideration here consists in determining the exact order (in N) of the quantity

$$\mathscr{R}_N(\mathrm{F}, \Omega) := \inf\{\mathscr{R}(\mathrm{F}, \Omega, C_N, \Lambda_N) \,|\, C_N, \Lambda_N\} \tag{1.2}$$

(which is N-th optimal error of numerical integration on the class F) and constructing a sequence $(C_N^*, \Lambda_N^* \mid N \in \mathbb{N})$ of weights and nodes such that the errors $\mathscr{R}(\mathrm{F}, \Omega, C_N^*, \Lambda_N^*)$ of cubature formulas (1.1) realize the order of optimal error (1.2). Cubature formulas $\mathscr{Q}(f, C_N^*, \Lambda_N^*)$ are called optimal in order.

A lot of works are devoted to the study of different formulations of problems of optimal numerical integration for various classes of smooth functions in several variables, see, for example, monographs [17], [19], [20, Chapter 6] and survey [7, Chapter 8] and the bibliographies therein. Comprehensive survey [7], monograph [20], papers [21], [11], [6], [3] show that the interest to problem of optimal numerical integration we will study here is unabated; a fairly detailed history of the issue and an extensive bibliographies can be found there as well.

In this paper, we give exact (in the sense of the order) estimates for quantity (1.2) in the case in which $\Omega = \mathbb{T}^m$ is the m-dimensional torus, F is the function class $\mathrm{B}_{pq}^{s\,\tau}(\mathbb{T}^m)$ of Nikol'skii – Besov type or $\mathrm{L}_{pq}^{s\,\tau}(\mathbb{T}^m)$ of Lizorkin – Triebel type, for some range of the parameters of these classes.

Let us introduce the notation that we will use throughout this article. Let $m \in \mathbb{N}$, $m \geq 2$, $\mathrm{z}_m = \{1, \ldots, k\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, +\infty)$. For $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, put $xy = x_1y_1 + \ldots + x_my_m$, $|x| = |x_1| + \ldots + |x_m|$, $|x|_\infty = \max(|x_\mu| : \mu \in \mathrm{z}_m)$; $x \leq y$ $(x < y) \Leftrightarrow x_\mu \leq y_\mu$ $(x_\mu < y_\mu)$ for all $\mu \in \mathrm{z}_m$. For $t \in \mathbb{R}$, $t_+ := \max\{0, t\}$.

Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^m)$ and $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^m)$ be the Schwartz spaces of test functions and tempered distributions, respectively; $\widehat{f} \equiv \mathcal{F}_m(f)$ and $\mathcal{F}_m^{-1}(f)$ direct and inverse Fourier transforms of $f \in \mathcal{S}'(\mathbb{R}^m)$; in particular, for $\varphi \in \mathcal{S}$,

$$\widehat{\varphi}(\xi) = \mathcal{F}_m(\varphi)(\xi) = \int_{\mathbb{R}^m} \varphi(x) e^{-2\pi i\,\xi x} dx, \quad \mathcal{F}_m^{-1}(\varphi)(\xi) = \int_{\mathbb{R}^m} \varphi(x) e^{2\pi i\,\xi x} dx, \quad \xi \in \mathbb{R}^m,$$

where $\xi x = \xi_1 x_1 + ... = \xi_m x_m$.

Let $\mathbb{T}^m = (\mathbb{R}/\mathbb{Z})^m$ be the m-dimensional torus; sometimes it will be convenient for us to identify $\mathbb{T}^m$ with the cube $Q_0 := [0,1)^m$ in $\mathbb{R}^m$. Further, we denote by $\widetilde{\mathcal{S}}' \equiv \mathcal{S}'(\mathbb{T}^m)$ the space of all distributions f from $\mathcal{S}'$ which are 1-periodic in each variable (i.e. such that $\langle f, \varphi(\cdot + \xi)\rangle = \langle f, \varphi\rangle$ for all $\varphi \in \mathcal{S}$ and any $\xi \in \mathbb{Z}^m$) and by $\widetilde{\mathcal{S}} := \widetilde{\mathcal{S}} := \mathcal{S}(\mathbb{T}^m)$ the space of all infinitely continuously differentiable functions on $\mathbb{T}^m$ endowed with the topology of uniform convergence of all derivatives over $\mathbb{T}^m$. Then the space $\mathcal{S}'(\mathbb{T}^m)$ is naturally identified with the space that is topologically dual to $\mathcal{S}(\mathbb{T}^m)$. It is well known that $f \in \widetilde{\mathcal{S}}'$ if and only if $\operatorname{supp} \widehat{f} \subset \mathbb{Z}^m$, i.e. distribution $\widehat{f}$ vanishes on the open set $\mathbb{R}^m \backslash \mathbb{Z}^m$.

For $0 < p \leq \infty$ and a measurable set $G \subset \mathbb{R}^m$, as usual, let $L_p(G)$ be the space of measurable functions $f : G \to \mathbb{C}$, which are Lebesgue integrable in p-th power (when $p = \infty$ essentially bounded) over G, endowed with the standard quasi-norm (norm if $p \geq 1$)

$$\| f \,|\, L_p(G) \| = \Big(\int_G |\, f(x) \,|^p dx \Big)^{\frac{1}{p}} \ (p < \infty), \ \| f \,|\, L_\infty(G) \| = \operatorname{ess\,sup}(|\, f(x) \,| : x \in G).$$

For $0 < q \leq \infty$, let $\ell_q := \ell_q(\mathbb{N}_0)$ be the space of all (complex) number sequences $(c_j) = (c_j : j \in \mathbb{N}_0)$ with finite standard quasi-norm (norm if $q \geq 1$) $\|(c_j) \,|\, \ell_q\|$.

Further, let $\ell_q(L_p(G))$ (respectively, $L_p(G; \ell_q)$) be the space of all function sequences $(g_j(x)) = (g_j(x) : k \in \mathbb{N}_0)$ $(x \in G)$ with finite standard quasi-norm (norm if $p, q \geq 1$)

$$\| \,(g_j(x)) \,|\, \ell_q(L_p(G)) \,\| = \| \,(\,\| \,g_j \,|\, L_p(G) \|) \,|\, \ell_q \,\|$$

(respectively,

$$\| \,(g_j(x)) \,|\, L_p(G; \ell_q) \,\| = \| \,\| \,(g_j(\cdot)) \,|\, \ell_q \,\| \,|\, L_p(G) \|).$$

In what follows we will often use the abbreviation $L_p := L_p(\mathbb{R}^m)$, $\widetilde{L}_p := L_p(\mathbb{T}^m)$, $\ell_q(L_p) := \ell_q(L_p(\mathbb{R}^m))$, $\ell_q(\widetilde{L}_p) := \ell_q(L_p(\mathbb{T}^m))$, $L_p(\ell_q) = L_p(\mathbb{R}^m; \ell_q)$, $\widetilde{L}_p(\ell_q) = L_p(\mathbb{T}^m; \ell_q)$.

Let $\mathcal{Q}$ be the set of all half-open dyadic cubes in $\mathbb{R}^m$ of the form

$$Q = Q_{j\xi} = \{\, x \in \mathbb{R}^m \,:\, 2^j x - \xi \in [0,1)^m \,\} \ (j \in \mathbb{Z}, \xi \in \mathbb{Z}^m).$$

For a cube $Q = Q_{j\xi}$, we denote by $x_Q := 2^{-j} \cdot \xi$, $l(Q)(= 2^{-j})$, $j(Q) := j$ and $|Q|(= 2^{-jm})$ its "lower left" corner, side length, level and volume, respectively.

2 Definition of function spaces $\widetilde{B}^{s\,\tau}_{p\,q}$ and $\widetilde{F}^{s\,\tau}_{p\,q}$

First we choose a test function $\eta_0 \in \mathcal{S}$ such that

$$0 \le \widehat{\eta}_0(\xi) \le 1, \quad \xi \in \mathbb{R}^m; \quad \widehat{\eta}_0(\xi) = 1 \ \text{ if } |\xi|_\infty \le 1; \quad \operatorname{supp} \widehat{\eta}_0 = \{\xi \in \mathbb{R}^m \mid |\xi|_\infty \le 2\}.$$

Put $\widehat{\eta}(\xi) = \widehat{\eta}_0(2^{-1}\xi) - \widehat{\eta}_0(\xi)$, $\ \widehat{\eta}_j(\xi) := \widehat{\eta}_j(\xi) = \widehat{\eta}(2^{1-j}\xi)$, $\ j \in \mathbb{N}$. Then

$$\sum_{j=0}^{\infty} \widehat{\eta}_j(\xi) \equiv 1, \quad \xi \in \mathbb{R}^m,$$

i.e. $\{\widehat{\eta}_j(\xi) \,|\, j \in \mathbb{N}_0\}$ is a resolution of unity (by corridors) on $\mathbb{R}^m$. It is clear that

$$\eta(x) = 2^m \eta_0(2x) - \eta_0(x),\ \eta_j(x) := 2^{(j-1)m} \eta(2^{j-1}x),\ j \in \mathbb{N}. \tag{2.1}$$

Next we denote by Δ^η_j operators on $\mathcal{S}'$ defined as follows: for $f \in \mathcal{S}'$

$$\Delta^\eta_j(f, x) = f * \eta_j(x) = \langle f, \eta_j(x - \cdot)\rangle; \tag{2.2}$$

for the sake of convenience we put $\Delta^\eta_j(f, x) \equiv 0$ if $j < 0$.

We recall the definitions of two scales of the (inhomogeneous) smoothness spaces (on the whole Euclidean space) related to Morrey spaces.

Definition 1. Let $s, \tau \in \mathbb{R}$, $0 < p, q \le \infty$. Then

I. the Nikol'skii – Besov type space $B^{s\,\tau}_{p\,q} := B^{s\,\tau}_{p\,q}(\mathbb{R}^m)$ consists of all distributions $f \in \mathcal{S}'$, for which the quasi-norm

$$\|\, f \,|\, B^{s\,\tau}_{p\,q} \,\| \ = \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^\tau} \|(2^{sj} \Delta^\eta_j(f, x) \operatorname{sign}((j + 1 - j(Q))_+)) \,|\, \ell_q(L_p(Q))\|$$

is finite,

II. the Lizorkin – Triebel type space $F^{s\,\tau}_{p\,q} := F^{s\,\tau}_{p\,q}(\mathbb{R}^m)$ $(p < \infty)$ consists of all distributions $f \in \mathcal{S}'$, for which the quasi-norm

$$\|\, f \,|\, F^{s\,\tau}_{p\,q} \,\| \ = \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^\tau} \|(2^{sj} \Delta^\eta_j(f, x) \operatorname{sign}((j + 1 - j(Q))_+)) \,|\, L_p(Q; \ell_q)\|$$

is finite.

Remark 1. The inhomogeneous spaces $B^{s\,\tau}_{p\,q}$ and $F^{s\,\tau}_{p\,q}$ are introduced in [24] and thoroughly studied in [24], [15], [16], [22], [23]. We also note that (local) Morrey spaces and Nikol'skii – Besov – Morrey and Lizorkin – Triebel – Morrey spaces have been attracted a lot of attention, see, for instance, [24], [15], [16], [22], [23], [10], [9], [14] and bibliographies therein.

Let $g:\mathbb{R}^m\to\mathbb{C}$ be an arbitrary function, its periodization $\widetilde{g}:\mathbb{T}^m\to\mathbb{C}$ is defined as the (formal) sum of the series $\sum\limits_{\xi\in\mathbb{Z}^m} g(x+\xi)$.

By the Poisson summation formula (see, for example, [18, Chapter VII, Theorem 2.4]) it is easy to see that if $\varphi\in\mathcal{S}$ then $\widetilde{\varphi}\in\widetilde{\mathcal{S}}$, and, moreover, $\widetilde{\varphi}(x)=\sum_{\xi\in\mathbb{Z}^m}\widehat{\varphi}(\xi)e^{2\pi i\xi x}$.

Let

$$\widetilde{\mathcal{Q}}=\{Q\in\mathcal{Q}\,|\,Q\subset Q_0=[0,1)^m\}=\{Q_{j\xi}\,|\,j\in\mathbb{N}_0,\ \xi\in\mathbb{Z}^m:\mathbf{0}\le\xi<2^j\mathbf{1}\}\quad(\mathbf{0},\mathbf{1}\in\mathbb{R}^m).$$

Next we denote by $\widetilde{\Delta}_j^\eta$ the operators defined on $\widetilde{\mathcal{S}}'$ $(j\in\mathbb{N}_0)$, as follows: for $f\in\widetilde{\mathcal{S}}'$

$$\widetilde{\Delta}_j^\eta(f,x)=f*\widetilde{\eta}_j(x)=\langle f,\widetilde{\eta}_j(x-\cdot)\rangle=\sum_{\xi\in\mathbb{Z}^m}\widehat{\eta}_j(\xi)\widehat{f}(\xi)e^{2\pi i\,\xi x}. \tag{2.3}$$

Again, for the sake of convenience we put $\widetilde{\Delta}_j^\eta(f,x)\equiv 0$ if $j<0$.

In next definition we introduce two scales of the smoothness spaces (over $m-$dimensional torus) related to Morrey spaces.

Definition 2. $s,\tau\in\mathbb{R}$, $0<p,q\le\infty$. Then

I. the Nikol'skii – Besov type space $\widetilde{B}_{pq}^{s\,\tau}:=B_{pq}^{s\,\tau}(\mathbb{T}^m)$ consists of all distributions $f\in\widetilde{\mathcal{S}}'$, for which the quasi-norm

$$\|\,f\,|\,\widetilde{B}_{pq}^{s\,\tau}\,\|\;=\sup_{Q\in\widetilde{\mathcal{Q}}}\frac{1}{|Q|^\tau}\|(2^{sj}\widetilde{\Delta}_j^\eta(f,x)\mathrm{sign}((j+1-j(Q))_+))\,|\,\ell_q(L_p(Q))\|$$

is finite,

II. the Lizorkin – Triebel type space $\widetilde{F}_{pq}^{s\,\tau}:=F_{pq}^{s\,\tau}(\mathbb{T}^m)$ $(p<\infty)$ consists of all distributions $f\in\widetilde{\mathcal{S}}'$, for which the quasi-norm

$$\|\,f\,|\,\widetilde{F}_{pq}^{s\,\tau}\,\|\;=\sup_{Q\in\widetilde{\mathcal{Q}}}\frac{1}{|Q|^\tau}\|(2^{sj}\widetilde{\Delta}_j^\eta(f,x)\mathrm{sign}((j+1-j(Q))_+))\,|\,L_p(Q;\ell_q)\|$$

is finite.

We will call the unit balls $\widetilde{\mathrm{B}}_{pq}^{s\,\tau}:=\mathrm{B}_{pq}^{s\,\tau}(\mathbb{T}^m)$ and $\widetilde{\mathrm{F}}_{pq}^{s\,\tau}:=\mathrm{F}_{pq}^{s\,\tau}(\mathbb{T}^m)$ of those spaces the Nikol'skii-Besov and Lizorkin-Triebel classes, respectively.

Remark 2. Evidently the spaces $\widetilde{B}_{pq}^{s\,0}$ and $\widetilde{F}_{pq}^{s\,0}$ coincide with the well–known isotropic periodic Nikol'skii-Besov spaces $\widetilde{B}_{pq}^{s}$ and Lizorkin-Triebel spaces $\widetilde{F}_{pq}^{s}$ respectively (see, for instance, [13]). Furthermore, it is not hard to see that for any $\tau\le 0$ $\widetilde{B}_{pq}^{s\,\tau}=\widetilde{B}_{pq}^{s}$ and $F_{pq}^{s\,\tau}=\widetilde{F}_{pq}^{s}$ in the sense of equivalent quasi-norms unlike the spaces $B_{pq}^{s\,\tau}$ and $F_{pq}^{s\,\tau}$: as well known, $B_{pq}^{s\,\tau}=\{0\}$ and $F_{pq}^{s\,\tau}=\{0\}$ when $\tau<0$ (see [24, Chapter 2]).

We note that periodic Morrey spaces and Nikol'skii – Besov – Morrey and Lizorkin – Triebel – Morrey spaces (over $\mathbb{T}^m$) have been attracted increasing attention as well, see, for instance, [1], [12], [5] and bibliographies therein.

We will need φ – transform characterization for the spaces $\widetilde{B}_{pq}^{s\,\tau}$ and $\widetilde{F}_{pq}^{s\,\tau}$.

We choose test functions $\phi_0,\phi\in\mathcal{S}$ satisfying the following conditions :

$$\mathrm{supp}\,\widehat{\phi}_0\subset\{\xi\;:\;|\xi|_\infty\le 2\},\quad \mathrm{supp}\,\widehat{\phi}\subset\{\xi\;:\;1/2\le|\xi|_\infty\le 2\}, \tag{2.4}$$

$$|\widehat{\phi}_0(\xi)|\ge c>0\text{ when }|\xi|_\infty\le\frac{5}{3},\quad |\widehat{\phi}(\xi)|\ge c>0\text{ when }\frac{3}{5}\le|\xi|_\infty\le\frac{5}{3}. \tag{2.5}$$

Next we choose test functions $\psi_0, \psi \in \mathcal{S}$ satisfying conditions (2.4), (2.5) (with ψ instead of ϕ) and such that

$$\widehat{\breve{\varphi}}_0(\xi)\widehat{\psi}_0(\xi) + \sum_{j\in\mathbb{N}} \widehat{\breve{\phi}}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^m \tag{2.6}$$

($\breve{g}(x) \equiv \bar{g}(-x)$, $\bar{z}$ is the number complex conjugate to $z \in \mathbb{C}$). For $Q = Q_{j\lambda} \in \widetilde{\mathcal{Q}}$, we set (functions $\widetilde{\phi}_j$ are defined via (2.1) and the periodization)

$$\widetilde{\phi}_Q(x) \equiv |Q|^{1/2}\widetilde{\phi}_{j(Q)}(x - x_Q) = 2^{-jm/2}\widetilde{\phi}_j(x - 2^{-j}\lambda),$$

functions $\widetilde{\psi}_Q$ are defined analogously. Then in view of (2.6) it is not hard to show that for any $f \in \widetilde{\mathcal{S}}'$ we have the following decomposition (the convergence in the sense of $\widetilde{\mathcal{S}}'$)

$$f = \sum_{Q\in\widetilde{\mathcal{Q}}} \langle f, \widetilde{\phi}_Q\rangle\, \widetilde{\psi}_Q = \sum_{j\in\mathbb{N}_0}\sum_{j_Q=j} \langle f, \widetilde{\phi}_Q\rangle\, \widetilde{\psi}_Q. \tag{2.7}$$

Let us introduce (direct) φ – transform $\widetilde{S}_\varphi$ on $\widetilde{\mathcal{S}}'$ as follows

$$\widetilde{S}_\varphi \;:\; \widetilde{\mathcal{S}}' \ni f \mapsto \widetilde{S}_\varphi(f) \equiv (\langle f, \widetilde{\phi}_Q\rangle \,|\, Q \in \widetilde{\mathcal{Q}}),$$

and φ – transform $\widetilde{T}_\psi$ (formal left inverse to $\widetilde{S}_\varphi$) as follows

$$\widetilde{T}_\psi \;:\; (c_Q) = (c_Q \,|\, Q \in \widetilde{\mathcal{Q}}) \mapsto \widetilde{T}_\psi((c_Q)) = \sum_{Q\in\widetilde{\mathcal{Q}}} c_Q\, \widetilde{\psi}_Q.$$

Equality (2.7) means that the composition $\widetilde{T}_\psi \circ \widetilde{S}_\varphi$ is the identity on $\widetilde{\mathrm{S}}$.

Definition 3. Let $0 < p, q \le \infty; s, \tau \in \mathbb{R}$. A number sequence $(c_Q) = (c_Q \,|\, Q \in \widetilde{\mathcal{Q}})$ belongs to the space $\widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}$, if $\|(c_Q) \,|\, \widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}\| < \infty$, where $\mathtt{A} \in \{\mathtt{B}, \mathtt{F}\}$ and

$$\|(c_Q) \,|\, \widetilde{\mathtt{B}}^{s\,\tau}_{p\,q}\| = \sup_{P\in\widetilde{\mathcal{Q}}} \frac{1}{|P|^\tau}\Big\{ \sum_{j=j(P)}^{\infty} 2^{j(s+\frac{m}{2}-\frac{m}{p})q}\Big[\sum_{Q\in\widetilde{\mathcal{Q}}:Q\subset P, j(Q)=j} |c_Q|^p\Big]^{q/p}\Big\}^{1/q},$$

$$\|(c_Q) \,|\, \widetilde{\mathtt{F}}^{s\,\tau}_{p\,q}\| = \sup_{P\in\widetilde{\mathcal{Q}}} \frac{1}{|P|^\tau}\Big\{ \int_P \Big[\sum_{j=j(P)}^{\infty} 2^{j(s+\frac{m}{2})q} \sum_{Q\in\widetilde{\mathcal{Q}}:Q, j(Q)=j} |c_Q|^q \chi_Q(x)\Big]^{p/q}\Big\}^{1/p} \quad (p < \infty).$$

(natural modification if $p = \infty$ and/or $q = \infty$)

(Here χ_Q is the characteristic function of Q.)

Theorem 2.1. *Let $(A, \mathtt{A}) \in \{(B, \mathtt{B}), (F, \mathtt{F})\}$, $0 < p, q \le \infty$, ($p < \infty$ if $(A, \mathtt{A}) = (F, \mathtt{F})$), $s \in \mathbb{R}, \tau \ge 0$. Then a distribution $f \in \widetilde{\mathcal{S}}'$ belongs to $\widetilde{A}^{s\tau}_{pq}$, if and only if the sequence $(\langle f, \widetilde{\phi}_Q\rangle \,|\, Q \in \widetilde{\mathcal{Q}})$ belongs to $\widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}$, moreover,*

$$\|\,(\langle f, \widetilde{\phi}_Q\rangle) \,|\, \widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}\,\| \;\approx^{1}\; \|\, f \,|\, \widetilde{A}^{s\,\tau}_{p\,q}\,\|.$$

Furthermore, the operators $\widetilde{S}_\varphi \,:\, \widetilde{A}^{s\,\tau}_{p\,q} \to \widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}$ and $\widetilde{T}_\psi \,:\, \widetilde{\mathtt{A}}^{s\,\tau}_{p\,q} \to \widetilde{A}^{s\,\tau}_{p\,q}$ are bounded and their composition $\widetilde{T}_\psi \circ \widetilde{S}_\varphi$ is the identity on $\widetilde{A}^{s\,\tau}_{p\,q}$.

[1]sign "≈" means that there exist positive constants C_1, C_2 independent of $f \in \widetilde{A}^{s\,\tau}_{p\,q}$ such that $C_1\|\, f \,|\, \widetilde{A}^{s\,\tau}_{p\,q}\,\| \le \|\,(\langle f, \widetilde{\phi}_Q\rangle) \,|\, \widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}\,\| \le C_2\|\, f \,|\, \widetilde{A}^{s\,\tau}_{p\,q}\,\|$.

Remark 3. The notion of φ – transform was invented by M. Frazier and B. Jawerth [8]. This theorem is the periodic analogue of Theorem 2.1 in [24] for the spaces $B^{s\,\tau}_{pq}$ and $F^{s\,\tau}_{pq}$. A special case of Theorem 2.1 for the isotropic spaces $\widetilde{B}^{s}_{pq}$ and $\widetilde{F}^{s}_{pq}$ was established in [4].

Theorem 2.2. *Let $A \in \{B, F\}$, $0 < p, q \le \infty$, ($p < \infty$ when $A = F$), $s \in \mathbb{R}, \tau \ge 0$. Then we have the following continuous embedding*

$$\widetilde{A}^{s\,\tau}_{pq} \hookrightarrow \widetilde{B}^{s+\tau m-\frac{m}{p}}_{\infty\infty}.$$

Moreover, if $\tau > \frac{1}{p}, 0 < q < \infty$ or $\tau \ge \frac{1}{p}, q = \infty$ we have

$$\widetilde{A}^{s\,\tau}_{pq} = \widetilde{B}^{s+\tau m-\frac{m}{p}}_{\infty\infty}$$

in the sense of equivalent quasi-norms.

Remark 4. The first statement of this theorem is an analogue of the results on the embedding of the spaces $A^{s\,\tau}_{pq}(\mathbb{R}^m)$ into the space $C_{ub}(\mathbb{R}^d)$ of uniformly continuous and bounded functions, see [24, Chapter 2, Section 2.2] and [16, Theorem 4.4]. Second statement is a direct periodic analogue of Theorem 2 in [22]. Note that for $s > 0$ the space $\widetilde{B}^{s}_{\infty\infty}$ coincides with the well-known Zygmund spaces $\mathcal{Z}^s(\mathbb{T}^m)$ (see details in [13, Chapter 3]).

3 Optimal error of numerical integration on classes $\widetilde{\mathrm{B}}^{s\,\tau}_{pq}$ and $\widetilde{\mathrm{L}}^{s\,\tau}_{pq}$

In this section, we formulate and discuss the main result of the paper on estimates exact in order for optimal errors of numerical integration on the Nikol'skii – Besov and Lizorkin – Triebel classes $\widetilde{\mathrm{B}}^{s\,\tau}_{pq} = \mathrm{B}^{s\,\tau}_{pq}(\mathbb{T}^m)$ and $\widetilde{\mathrm{F}}^{s\,\tau}_{pq} = \mathrm{F}^{s\,\tau}_{pq}(\mathbb{T}^m)$ under some condition on parameters s, p, q, τ, m ($s \in \mathbb{R}_+$, $1 \le p, q \le \infty$, $\tau \in [0, 1/p]$).

In what follows, we will use the signs $\ll$ and $\asymp$ of the ordinal inequality and equality: for functions $F : \mathbb{R}_+ \to \mathbb{R}_+$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ we write $F(u) \ll H(u)$ as $u \to \infty$, if there exists a constant $C = C(F, H) > 0$ such that the inequality $F(u) \le CH(u)$ holds true for $u \ge u_0 > 0$; $F(u) \asymp H(u)$ if $F(u) \ll H(u)$ and $H(u) \ll F(u)$ simultaneously.

Main result of the paper is the following

Theorem 3.1. *Assume that $\mathrm{A} \in \{\mathrm{B}, \mathrm{F}\}$, $1 \le p, q \le \infty$, $s > 0$, $\tau \ge 0$ ($p < \infty$ if $\mathrm{A} = \mathrm{F}$). Then the relation*

$$\mathscr{R}_N(\widetilde{\mathrm{A}}^{s\,\tau}_{pq}) \asymp N^{-\frac{s}{m}-(\tau-\frac{1}{p})_+} \quad \text{as} \quad N \to \infty$$

holds true.

Remark 5. By Theorem 2.2 the hypotheses of Theorem 3.1 guarantee the continuous embedding $\widetilde{A}^{s\,\mathtt{m}}_{pq} \hookrightarrow C(\mathbb{T}^m)$, which is required in problems of numerical integration ($A \in \{B, F\}$).

Remark 6. As mentioned in Introduction, there is an extensive literature devoted to optimal cubature formulas for classes of functions of several variables. Here we discuss results directly related to Theorem 3.1, namely, results on function classes on the torus included in the Nikol'skii – Besov and Lizorkin – Triebel scales from Definition 1.

For $s > m/p, 1 \le p \le \infty$, the estimates of $\mathscr{R}_N(\widetilde{\mathrm{F}})$, exact in order, for the isotropic Sobolev and Nikol'skii classes ($\widetilde{\mathrm{F}} = \widetilde{\mathrm{W}}^s_p$ and $\widetilde{\mathrm{F}} = \widetilde{\mathrm{H}}^s_p \equiv \widetilde{\mathrm{B}}^s_{p\infty}$) are given in [20, Chapter 3] (in fact, the anisotropic case is also considered there):

$$\mathscr{R}_N(\widetilde{\mathrm{W}}^s_p) \asymp \mathscr{R}_N(\widetilde{\mathrm{H}}^s_p) \asymp N^{-\frac{s}{m}} \quad \text{as} \quad N \to \infty$$

and the simplest sequence of "parallelepipedal" cubature formulas

$$\mathscr{Q}^{\star}_N(f) := \sum_{\xi\in\mathbb{Z}^m:0\le\xi_\mu<M(N),\mu\in z_m} \frac{1}{M(N)^m} f\Big(\frac{\xi}{M(N)}\Big),$$

$$M(N)\in\mathbb{N} \,:\, M(N)^m \le N < (M(N)+1)^m,$$

can be taken as optimal one.

We recall that $\widetilde{H}^s_\infty = \mathcal{Z}^s(\mathbb{T}^m)$ and for $1<p<\infty$ we have $\widetilde{W}^s_p = \widetilde{F}^s_{p2}$ in the sense of equivalent norms (see details in [13, Chapter 3]).

Further, in [3] the following sharp estimates are obtained: for $A\in\{B,F\}$, $1\le p,q\le\infty$ ($p<\infty$ if $A=F$), $s>m/p$ if $A=B$ and $s>\max\{m/p,m/q\}$ if $A=F$ we have

$$\mathscr{R}_N(\widetilde{\mathrm{A}}^s_{pq}) \asymp N^{-\frac{s}{m}} \qquad \text{as} \qquad N\to\infty.$$

In [3], to prove upper estimates, the well-known Frolov's cubature formulas are used because there it is studied general case of the function spaces of product type, in particular, the function spaces with mixed smoothness. But it is easy to see that for isotropic classes $\widetilde{\mathrm{A}}^s_{pq}$ the sequence of "parallelepipedal" cubature formulas $\mathscr{Q}^{\star}_N(f)$ can be taken as optimal one as well.

Thus, in view of Theorem 2.2 it remains to prove the theorem for the case $0<\tau\le 1/p$.

4 Proof of Theorem 3.1

By Definition 1 it is evident that the quasi-norms of both scales $\widetilde{B}^{s\,\tau}_{pq}$ and $\widetilde{F}^{s\,\tau}_{pq}$ are monotonic with respect to parameter τ: for any $\tau_1<\tau_2$ we have $\|\cdot\mid\widetilde{A}^{s\,\tau_1}_{pq}\|\le\|\cdot\mid\widetilde{A}^{s\,\tau_2}_{pq}\|$. Hence, the elementary embedding $\widetilde{A}^{s\,\tau_2}_{pq}\hookrightarrow\widetilde{A}^{s\,\tau_1}_{pq}$ holds ($A\in\{B,F\}$). From here and Remark 6, it follows that the upper estimates

$$\mathscr{R}_N(\widetilde{\mathrm{A}}^{s\,\tau}_{pq}) \ll \mathscr{R}_N(\widetilde{\mathrm{A}}^s_{pq}) \asymp N^{-\frac{s}{m}} \qquad \text{as} \qquad N\to\infty$$

hold for any $\tau>0$.

Now we turn to proving the matching lower estimates.

Taking into account the monotonicity of norms $\|\cdot\mid\widetilde{A}^{s\,\tau}_{pq}\|$ (with respect to τ) as well as Jensen's inequality ($\|\cdot\mid\ell_{q_1}\|\ge\|\cdot\mid\ell_{q_2}\|$ if $1\le q_1<q_2\le\infty$), we get the following simple inclusions $\widetilde{\mathrm{B}}^{s\,\tau}_{p\,q}\supset\widetilde{\mathrm{B}}^{s\,\frac{1}{p}}_{p\,1}$ and $\widetilde{\mathrm{F}}^{s\,\tau}_{p\,q}\supset\widetilde{\mathrm{F}}^{s\,\frac{1}{p}}_{p\,1}$ if $1\le q\le\infty$ and $\tau\le 1/p$.

Since the estimates in Theorem 3.1 do not depend on p, q and $\tau\le 1/p$, in view of inclusions mentioned above, it suffices to prove the required lower estimates for the classes $\widetilde{\mathrm{B}}^{s\,\frac{1}{p}}_{p\,1}$ and $\widetilde{\mathrm{F}}^{s\,\frac{1}{p}}_{p\,1}$. Moreover, for $\widetilde{\mathrm{B}}^{s\,\frac{1}{p}}_{p\,1}$, we can restrict ourselves to the case $1\le p<\infty$ because the required estimate for $\widetilde{\mathrm{B}}^{s\,0}_{\infty\,1}\equiv\widetilde{\mathrm{B}}^{s}_{\infty\,1}$ is known (see Remark 6).

To this end, we apply Bakhvalov's method to obtain those lower bounds for optimal error $\mathscr{R}_N(\mathrm{F},\Omega)$. This method was proposed by N.S. Bakhvalov [2]. Its idea is for a given N and any cubature formula (1.1) to construct a "bad" function g_{Λ_N}, $\|g_{\Lambda_N}\mid F\|=1$, vanishing at all nodes, in the form of a sum with positive coefficients of special shifted dilations, a suitable fixed smooth bump function for which

$$\mathscr{R}(g_{\Lambda_N},\Omega,C_N,\Lambda_N) = \int_\Omega g_{\Lambda_N}(x)dx = \|g_{\Lambda_N}\mid\widetilde{L}_1\|$$

has the required order.

To construct those "bad" functions, we will use the so-called atomic decomposition of the spaces $\widetilde{A}^{s\,\tau}_{pq}$.

We need some notions and notation.

For $s, t \in \mathbb{R}$, $0 < p, q \leq \infty$, we define the numbers: $\lfloor t \rfloor$ (the integer part of t), $t_* = t - \lfloor t \rfloor$, $p \wedge q = \min\{p, q\}$, $\sigma_p = m(1/p - 1)_+$, $\sigma_{pq} = m(1/(p \wedge q) - 1)_+$. Further, $\tau_{sp} = 1/p + (1 - (\sigma_p + m - s)_*)/m$ if $s \leq \sigma_p$, and $\tau_{sp} = 1/p + (s - \sigma_p)/m$ if $s > \sigma_p$, $\tau_{spq} = 1/p + (1 - (\sigma_{pq} + m - s)_*)/m$ if $s \leq \sigma_{pq}$, and $\tau_{spq} = 1/p + (s - \sigma_{pq})/m$ if $s > \sigma_p$.

Let $Q \in \mathcal{Q}$. A function $a_Q : \mathbb{T}^m \to \mathbb{C}$ is called a smooth atom ("with a support close to Q") if the following conditions are satisfied:

$$\mathrm{supp}\,(a_Q) \subset \widetilde{3Q},\ |\partial^\alpha a_Q(x)| \leq |Q|^{-1/2 - |\alpha|/m},\ |\alpha| \leq \max\{\lfloor s + \tau m + 1 \rfloor, 0\}.$$

(Here $3Q$ is the dilation of Q with the same center, $\widetilde{D}$ is " the periodic continuation" of a set $D \subset Q_0$, i.e.

$$\widetilde{D} = \mathbb{Z}^m + D = \cup_{\xi \in \mathbb{Z}^m} (\xi + D), \quad \xi + D = \{\xi + x \,|\, x \in D\}.)$$

Then we call the sequence $(a_Q | Q \in \mathcal{Q})$ a family of (smooth) atoms for $\widetilde{A}^{s,\tau}_{p,q}$.

Theorem 4.1. *Let* $(A, \mathtt{A}) \in \{(B, \mathtt{B})(F, \mathtt{F})\}$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$. *Assume that* $0 \leq \tau < \tau_{sp}$ *if* $A = B$ *and* $0 \leq \tau < \tau_{spq}$, $p < \infty$ *if* $A = F$. *Then* $f \in \widetilde{A}^{s\,\tau}_{p\,q}$ *if and only if there exist* $(a_Q \,|\, Q \in \widetilde{\mathcal{Q}})$, *a family of atoms for* $\widetilde{A}^{s\,\tau}_{p\,q}$, *and a sequence* $(c_Q \,|\, Q \in \widetilde{\mathcal{Q}}) \in \widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}$ *such that*

$$f = \sum_{Q \in \widetilde{\mathcal{Q}}} c_Q\, a_Q \qquad (\text{convergence in } \widetilde{\mathcal{S}}') \tag{4.1}$$

and

$$\|\, f \,|\, \widetilde{A}^{s\,\tau}_{p\,q}\| \approx \inf \|(c_Q \,|\, Q \in \widetilde{\mathcal{Q}})) \,|\, \widetilde{\mathtt{A}}^{s\,\tau}_{p\,q}\|, \tag{4.2}$$

where inf *is taken over all representations* (4.1).

Remark 7. This theorem is a direct periodic analog of Theorem 3.3 from [24] for the spaces $\widetilde{A}^{s\,\tau}_{p\,q}$. Notice that in [3] we use an analog of Theorem 4.1 for product spaces, which includes as special case atomic characterizations for isotropic function spaces $\widetilde{B}^{s\,\tau}_{p\,q}$ and $\widetilde{F}^{s\,\tau}_{p\,q}$ (with the restriction $p < \infty$ in the case of $F-$spaces). Up to now for function spaces $\widetilde{F}^{s}_{\infty\,q}(0 < q < \infty)$, atomic decomposition remained unproven. Theorem 4.1 completes this gap because we have the coincidence $\widetilde{F}^{s}_{\infty\,q} = \widetilde{F}^{s\,1/p}_{p\,q}(0 < p < \infty, 0 < q \leq \infty)$ in the sense of equivalent quasi-norms. In non-trivial case $0 < p, q < \infty$, the coincidence $F^{s}_{\infty\,q}(\mathbb{R}^m) = F^{s\,\frac{1}{p}}_{p\,q}(\mathbb{R}^m)$ is shown in [24, Chapter 2], arguing in periodic settings is the same.

Remark 8. Here we recall a very important (correct and constructive) definition of the Lizorkin – Triebel spaces $F^{s}_{\infty\,q}(\mathbb{R}^m)(0 < q < \infty)$ invented by M. Frazier and B. Jawerth [8] : for $s \in \mathbb{R}$, $0 < q \leq \infty$, the Lizorkin – Triebel space $F^{s}_{\infty\,q} := F^{s}_{\infty\,q}(\mathbb{R}^m)$ consists of all distributions $f \in \mathcal{S}'$, for which the quasi-norm

$$\|\, f \,|\, F^{s}_{\infty\,q} \,\| = \|\Delta^{\eta}_0(f) \,|\, L_\infty\| + \Big(\sup_{Q \in \mathcal{Q}: j(Q) \geq 1} \frac{1}{|Q|} \int_Q \sum_{j = j(Q)}^{\infty} |2^{sj} \Delta^{\eta}_j(f, x)|^q dx \Big)^{1/q}$$

is finite.

Moreover, in [8] the following quasi-norm

$$\|\, f \,|\, F^{s}_{\infty\,q} \,\|_\star = \Big(\sup_{Q \in \mathcal{Q}: j(Q) \geq 0} \frac{1}{|Q|} \int_Q \sum_{j = j(Q)}^{\infty} |2^{sj} \Delta^{\eta}_j(f, x)|^q dx \Big)^{1/q}$$

is defined which is equivalent to the original one.

In [4], we studied the spaces (there different notation was used) $\widetilde{F}^{s}_{\infty\, q}$ which are defined as follows : for $s \in \mathbb{R}, 0 < q < \infty$, the Lizorkin – Triebel space $\widetilde{F}^{s}_{\infty\, q} := F^{s}_{\infty\, q}(\mathbb{T}^m)$ consists of all distributions $f \in \widetilde{\mathcal{S}}'$, for which the quasi-norm

$$\| \, f \, | \, \widetilde{F}^{s}_{\infty\, q} \, \| \; = \Big(\sup_{Q \in \mathcal{Q}: j(Q) \geq 0} \frac{1}{|Q|} \int_Q \sum_{j=j(Q)}^{\infty} |2^{sj} \Delta^{\eta}_j (f, x)|^q dx \Big)^{1/q}$$

is finite.

Proof of Theorem 3.1. Now we turn directly to constructing the "bad" functions mentioned above.

We pick a function $h \in \mathcal{S}$ such that

$$\mathrm{supp}\,(h) = [0,1]^m, \;\; \widehat{h}(0) > 0, \;\; \max\{|\partial^{\alpha} h(x)| : x \in [0,1]^m, \alpha \leq \lfloor s + \tau m + 1 \rfloor\} = 1.$$

For $Q \in \widetilde{\mathcal{Q}}$, we define

$$h_Q(x) := |Q|^{-1/2} h(2^{j(Q)} \cdot (x - x_Q)) := 2^{j(Q)m/2} h(2^{j(Q)} \cdot (x - x_Q))$$

and their periodizations $\widetilde{h}_Q(x)$. It is clear that the sequence $(\widetilde{h}_Q \, | \, Q \in \widetilde{\mathcal{Q}})$ is a family of atoms for all $\widetilde{A}^{s\,\tau}_{pq}$.

For a sequence $\mathrm{c} := (c_Q \, | \, Q \in \widetilde{\mathcal{Q}})$ (which will be specified later), we consider a function

$$\widetilde{H}_{\mathrm{c}}(x) := \sum_{Q \in \widetilde{\mathcal{Q}}} c_Q \widetilde{h}_Q(x).$$

First we evaluate the integral $\int_{Q_0} \widetilde{H}_{\mathrm{c}}(x) dx$:

$$\int_{Q_0} \widetilde{H}_{\mathrm{c}}(x) dx = \widehat{h}(0) \sum_{Q \in \widetilde{\mathcal{Q}}} c_Q |Q|^{1/2}. \tag{4.3}$$

In view of Theorem 4.1 (see (4.2)) we get the inequality

$$\| \widetilde{H}_{\mathrm{c}} \, | \, \widetilde{A}^{s\,\tau}_{pq} \| \; \ll \; \| \mathrm{c} \, | \, \widetilde{\mathtt{A}}^{s\,\tau}_{pq} \|. \tag{4.4}$$

Next we write down the norms $\| \mathrm{c} \, | \, \widetilde{\mathtt{B}}^{s\,\frac{1}{p}}_{p1} \|$ and $\| \mathrm{c} \, | \, \widetilde{\mathtt{F}}^{s\,1}_{11} \|$ (in view of Remark 7 and Theorem 2.1 the last norm is equivalent to $\| \mathrm{c} \, | \, \widetilde{\mathtt{F}}^{s\,\frac{1}{p}}_{p1} \|$) :

$$\| \mathrm{c} \, | \, \widetilde{\mathtt{B}}^{s\,\frac{1}{p}}_{p1} \| = \sup_{P \subset \widetilde{\mathcal{Q}}} \frac{1}{|P|^{1/p}} \sum_{j=j(P)}^{\infty} 2^{j(s + \frac{m}{2} - \frac{m}{p})} \Big(\sum_{Q \subset P: j(Q) = j} |c_Q|^p \Big)^{1/p} =: \sup_{P \subset \widetilde{\mathcal{Q}}} J(P) \tag{4.5}$$

and from the coincidence of the spaces $\widetilde{\mathtt{B}}^{s\,1}_{11}$ and $\widetilde{\mathtt{F}}^{s\,1}_{11}$ and the equality of their norms $\| \cdot \, | \, \widetilde{\mathtt{B}}^{s\,1}_{11} \| = \| \cdot \, | \, \widetilde{\mathtt{F}}^{s\,1}_{11} \|$ we get

$$\| \mathrm{c} \, | \, \widetilde{\mathtt{F}}^{s\,1}_{11} \| = \| \mathrm{c} \, | \, \widetilde{\mathtt{B}}^{s\,1}_{11} \| = \sup_{P \subset \widetilde{\mathcal{Q}}} \frac{1}{|P|} \sum_{j=j(P)}^{\infty} 2^{j(s - \frac{m}{2})} \sum_{Q \subset P: j(Q) = j} |c_Q|. \tag{4.6}$$

Let $N \in \mathbb{N}$ be an arbitrary number and $\mathscr{Q}(\,\cdot\,, C_N, \Lambda_N)$ be an arbitrary cubature formula of form (1.1), $\Lambda_N := (\lambda(1), \ldots, \lambda(N)) \subset \Omega^N$ its grid of nodes. We choose the natural number j_N such that $2^{(j_N - 2)m} \leq N < 2^{(j_N - 1)m}$.

Further, we denote $\widetilde{\mathscr{Q}}^j := \{Q \in \widetilde{\mathscr{Q}} \,|\, j(Q) = j\}$. It is clear that in the collection $\widetilde{\mathscr{Q}}^{j_N}$ consisting of $2^{j_N m}$ cubes there exist at least $2^{(j_N-1)m}$ cubes $Q(1), \dots, Q(2^{(j_N-1)m})$ which are free of nodes belonging to Λ_N. We put $\bar{Q}(\Lambda_N) = Q(1) \cup \dots \cup Q(2^{(j_N-1)m})$.

Now we are in position to define the required sequence of coefficients $\mathrm{c}^\star = (c^\star_Q \,|\, Q \in \widetilde{\mathscr{Q}})$:

$$c^\star_Q = 0 \ \text{ if } \ Q \cap \bar{Q}(\Lambda_N) = \varnothing, \quad c^\star_Q = c_j = 2^{-jt} \ \text{ if } \ Q \in \widetilde{\mathscr{Q}}^j \ \text{ and } \ Q \subset \bar{Q}(\Lambda_N),$$

here the real number $t > s + m/2$ is fixed. Then, it is not hard to verify that for any $\lambda \in \Lambda_N$ we have $\widetilde{H}_{\mathrm{c}^\star}(\lambda) = 0$. Therefore,

$$\mathscr{Q}(\widetilde{H}_{\mathrm{c}^\star}, C_N, \Lambda_N) = 0, \quad \mathscr{R}(\widetilde{H}_{\mathrm{c}^\star}, Q_0, C_N, \Lambda_N) = \int_{Q_0} \widetilde{H}_{\mathrm{c}^\star}(x)dx. \tag{4.7}$$

From (4.5) and the definition of $\mathrm{c}^\star$ it follows that for any P with $j(P) < j_N$

$$\mathrm{J}(P) = \frac{1}{|P|^{1/p}} \sum_{j=j_N}^{\infty} 2^{j(s+\frac{m}{2}-\frac{m}{p})} c_j \Big(\sum_{Q \subset P \cap \bar{Q}(\Lambda_N): j(Q)=j} 1 \Big)^{1/p} \le$$

$$\le 2^{j(P)\frac{m}{p}} \sum_{j=j_N}^{\infty} 2^{j(s+\frac{m}{2}-\frac{m}{p})} c_j 2^{(j-j(P))\frac{m}{p}} = \sum_{j=j_N}^{\infty} 2^{j(s+\frac{m}{2})} c_j \ll 2^{j_N(s+\frac{m}{2}-t)},$$

further, for any P with $j(P) \ge j_N$ such that $P \cap \bar{Q}(\Lambda_N) = \varnothing$ obviously we have $\mathrm{J}(P) = 0$. Finally, for any P with $j(P) \ge j_N$ such that $P \subset \bar{Q}(\Lambda_N) = \varnothing$ we get

$$\mathrm{J}(P) = \frac{1}{|P|^{1/p}} \sum_{j=j(P)}^{\infty} 2^{j(s+\frac{m}{2}-\frac{m}{p})} c_j \Big(\sum_{Q \subset P: j(Q)=j} 1 \Big)^{1/p} =$$

$$= 2^{j(P)\frac{m}{p}} \sum_{j=j(P)}^{\infty} 2^{j(s+\frac{m}{2}-\frac{m}{p})} c_j 2^{(j-j(P))\frac{m}{p}} = \sum_{j=j(P)}^{\infty} 2^{j(s+\frac{m}{2})} c_j \ll 2^{j(P)(s+\frac{m}{2}-t)} \le 2^{j_N(s+\frac{m}{2}-t)},$$

Hence, taking into account (4.4) we obtain

$$\|\widetilde{H}_{\mathrm{c}^\star} \,|\, \widetilde{B}^{s\,\frac{1}{p}}_{p1}\| \ll \|\mathrm{c}^\star \,|\, \widetilde{\mathrm{B}}^{s\,\frac{1}{p}}_{p1}\| \ll 2^{j_N(s+\frac{m}{2}-t)},$$

in particular,

$$\|\widetilde{H}_{\mathrm{c}^\star} \,|\, \widetilde{F}^{s\,1}_{11}\| \ll \|\mathrm{c}^\star \,|\, \widetilde{\mathrm{F}}^{s\,1}_{11}\| \ll 2^{j_N(s+\frac{m}{2}-t)},$$

From (4.3) and the definition of $\mathrm{c}^\star$ it follows that

$$\int_{Q_0} \widetilde{H}_{\mathrm{c}^\star}(x)dx = \widehat{h}(0) \sum_{Q \in \widetilde{\mathscr{Q}}} c^\star_Q |Q|^{1/2} = \widehat{h}(0) 2^{(j_N-1)m} \sum_{Q \in Q(1)} c^\star_Q |Q|^{1/2} = \widehat{h}(0) 2^{(j_N-1)m} \times$$

$$\times \sum_{j=j_N}^{\infty} c_j 2^{-jm/2} \sum_{Q \in Q(1): j(Q)=j} 1 = \widehat{h}(0) 2^{(j_N-1)m} \sum_{j=j_N}^{\infty} c_j 2^{-jm/2} 2^{(j-j_N)m} \asymp 2^{j_N(m/2-t)}.$$

Therefore, for an arbitrary cubature formula $\mathscr{Q}(\,\cdot\,, C_N, \Lambda_N)$ and functions

$$\widetilde{h}^{\mathrm{c}^\star} := \frac{\widetilde{H}_{\mathrm{c}^\star}}{\|\widetilde{H}_{\mathrm{c}^\star} \,|\, \widetilde{B}^{s\,1/p}_{p1}\|} \in \widetilde{\mathrm{B}}^{s\,1/p}_{p1}, \quad \widetilde{g}^{\mathrm{c}^\star} := \frac{\widetilde{H}_{\mathrm{c}^\star}}{\|\widetilde{H}_{\mathrm{c}^\star} \,|\, \widetilde{F}^{s\,1}_{11}\|} \in \widetilde{\mathrm{F}}^{s\,1/p}_{p1}$$

we get

$$\mathscr{R}(\widetilde{\mathrm{B}}^{s\,1/p}_{p1}, C_N, \Lambda_N) \geq \mathscr{R}(\widetilde{h}^{\mathrm{C}^\star}, C_N, \Lambda_N) \gg \int_{Q_0} \widetilde{H}_{\mathrm{C}^\star}(x)dx/\|\mathrm{C}^\star \,|\, \widetilde{\mathtt{B}}^{s\,\frac{1}{p}}_{p1}\| \gg 2^{-sj_N} \asymp N^{-\frac{s}{m}}$$

and

$$\mathscr{R}(\widetilde{\mathrm{F}}^{s\,1/p}_{p1}, C_N, \Lambda_N) \geq \mathscr{R}(\widetilde{g}^{\mathrm{C}^\star}, C_N, \Lambda_N) \gg \int_{Q_0} \widetilde{H}_{\mathrm{C}^\star}(x)dx/\|\mathrm{C}^\star \,|\, \widetilde{\mathtt{F}}^{s\,1}_{11}\| \gg 2^{-sj_N} \asymp N^{-\frac{s}{m}}$$

From the last two inequalities it follows that

$$\mathscr{R}_N(\widetilde{\mathrm{F}}^{s\,1/p}_{p1}) \gg N^{-\frac{s}{m}}, \qquad \mathscr{R}_N(\widetilde{\mathrm{B}}^{s\,1/p}_{p1}) \gg N^{-\frac{s}{m}} \qquad \text{as} \quad N \to \infty.$$

Thus, the required lower estimates

$$\mathscr{R}_N(\widetilde{\mathrm{A}}^{s\,\tau}_{pq}) \gg N^{-\frac{s}{m}} \qquad \text{as} \quad N \to \infty.$$

are established, which completes the proof of Theorem 3.1. □

Remark 9. Here we emphasize the most important special case of Theorem 3.1 ($1 \leq q < \infty$)

$$\mathscr{R}_N(\widetilde{\mathrm{F}}^{s}_{\infty q}) \asymp N^{-\frac{s}{m}} \qquad \text{as} \quad N \to \infty,$$

which completes investigation of optimal numerical integration on isotropic function spaces of both Nikol'skii – Besov and Lizorkin – Triebel scales.

Remark 10. Proofs of Theorem 2.1, Theorem 2.2 and Theorem 4.1 will be published elsewhere.

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