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YESMUKHANBET SAIDAKHMETOVICH SMAILOV

Doctor of physical and mathematical sciences, Professor Smailov Esmuhanbet Saidakhmetovich passed away on May 24, 2024, at the age of 78 years.

Esmuhanbet Saidakhmetovich was well known to the scientific community as a high qualified specialist in science and education, and an outstanding organizer. Fundamental scientific articles and textbooks written in various fields of the theory of functions of several variables and functional analysis, the theory of approximation of functions, embedding theorems, and harmonic analysis are a significant contribution to the development of mathematics.

E.S. Smailov was born on October 18, 1946, in the village of Kyzyl Kesik, Aksuat district, Semipalatinsk region. In 1963, he graduated from high school with a silver medal, and in the same year he entered the Faculty of Mechanics and Mathematics of the Kazakh State University (Almaty) named after Kirov (now named after Al-Farabi). In 1971 he graduated from graduate school at the Institute of Mathematics and Mechanics.

He defended his PhD thesis in 1973 (supervisor was K.Zh. Nauryzbaev) and defended his doctoral thesis "Fourier multipliers, embedding theorems and related topics" in 1997. In 1993 he was awarded the academic title of professor.

E.S. Smailov since 1972 worked at the Karaganda State University named after E.A. Buketov as an associate professor (1972-1978), the head of the department of mathematical analysis (1978-1986, 1990-2000), the dean of the Faculty of Mathematics (1983-1987) and was the director of the Institute of Applied Mathematics of the Ministry of Education and Science of the Republic of Kazakhstan in Karaganda (2004 -2018).

Professor Smailov was one of the leading experts in the theory of functions and functional analysis and a major organizer of science in the Republic of Kazakhstan. He had a great influence on the formation of the Mathematical Faculty of the Karaganda State University named after E.A. Buketov and he made a significant contribution to the development of mathematics in Central Kazakhstan. Due to the efforts of Y.S. Smailov, in Karaganda an actively operating Mathematical School on the function theory was established, which is well known in Kazakhstan and abroad.

He published more than 150 scientific papers and 2 monographs. Under his scientific advice, 4 doctoral and 10 candidate theses were defended.

In 1999 the American Biographical Institute declared professor Smailov "Man of the Year" and published his biography in the "Biographical encyclopedia of professional leaders of the Millennium".

For his contribution to science and education, he was awarded the Order of "Kurmet" (="Honour").

The Editorial Board of the Eurasian Mathematical Journal expresses deep condolences to the family, relatives and friends of Esmuhanbet Saidakhmetovich Smailov.

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INVARIANT SUBSPACES IN NON-QUASIANALYTIC SPACES OF Ω-ULTRADIFFERENTIABLE FUNCTIONS ON AN INTERVAL

N.F. Abuzyarova, Z.Yu. Fazullin

Communicated by E.D. Nursultanov

Key words: Ω-ultradifferentiable function, Ω-ultradistribution, Fourier-Laplace transform, invariant subspace, spectral synthesis.

AMS Mathematics Subject Classification: 30D15, 42A38, 46F05.

Abstract. We consider and solve a weakened version of the classical spectral synthesis problem for differentiation operator in non-quasianalytic spaces of ultradifferentiable functions (UDF). Moreover, we deal with the widest class of UDF among all known ones. Namely, we study the spaces of Ω-ultradifferentiable functions introduced by Alexander Abanin in 2007-08. For subspaces of these spaces which are invariant under the differentiation operator we establish general conditions of weak spectral synthesis.

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1 Introduction

Let X be a locally convex space of infinitely differentiable functions on an interval $(a;b) \subseteq \mathbb{R}$ and $X \subset C^\infty(a;b)$ be a continuous embedding. Set $D = \frac{\mathrm{d}}{\mathrm{d}t}$,

$$e_{k,\lambda}(t) = t^k e^{\mathrm{i}\lambda t}, \quad t \in (a;b),\ k \in \mathbb{N}\bigcup\{0\},\ \lambda \in \mathbb{C}.$$

We assume that
1) D acts continuously in X;
2) X contains all functions of the form $e_{0,\lambda}, \quad \lambda \in \mathbb{C}$;
3) X is a non-quasianalytic function class.

Let $W \subset X$ be a closed subspace of X which is invariant under the differentiation operator: $D(W) \subset W$. Briefly, W is *D-invariant subspace*. By $\operatorname{Exp} W$ we denote the set of all exponential monomials $e_{k,\lambda}$ contained in W. Clearly, for any D-invariant subspace W we have the following implication:

$$e_{k,\lambda} \in W,\ k \geq 1 \Longrightarrow e_{j,\lambda} \in W, \quad j = 0, \dots, k-1.$$

A *classical spectral synthesis problem* for the differentiation operator in X is to find under what conditions D-invariant subspaces $W \subsetneq X$ are spanned by their sets $\operatorname{Exp} W$:

$$W = \overline{\operatorname{span} \operatorname{Exp} W}? \tag{1.1}$$

There are well-known results on this topic for D-invariant subspaces of holomorphic functions on a convex domain in $\mathbb{C}$ (see [15]–[17]), and for translation invariant subspaces in $C^\infty(\mathbb{R})$ (see [21]–[22]). However, as it has been noticed in [9], generally speaking, the classical spectral synthesis does not suit for description of D-invariant subspaces in $C^\infty(a;b)$. The matter is that there are non-trivial D-invariant subspaces in $C^\infty(a;b)$. containing no exponential functions. These are of the form

$$W_I = \{f \in C^\infty(a;b): \quad f = 0 \text{ on } I\}, \tag{1.2}$$

where I is any non-empty relatively closed subinterval of $(a;b)$.

In [9, Theorem 4.1], the authors also show that any non-trivial D-invariant subspace $W \subset C^\infty(a;b)$ contains maximal "residual subspace" of form (1.2). It implies that W has *residual interval* I_W defined as the smallest relatively closed subinterval of $(a;b)$ among all $I \subset (a;b)$ with the property $W_I \subset W$. We see that for D-invariant subspaces in $C^\infty(a;b)$ it is not enough to consider classical spectral synthesis (1.1). In [9], the authors have proposed another form of spectral synthesis problem. We call it the *problem of weak spectral synthesis*. The question is to know which non-trivial D-invariant subspaces W in $C^\infty(a;b)$ admit the representation

$$W = \overline{W_{I_W} + \operatorname{span} \operatorname{Exp} W}? \tag{1.3}$$

It is easy to see that the weakened form of spectral synthesis problem contains the classical one as a particular case. It corresponds to the case $I_W = (a;b)$. The problem of weak spectral synthesis (1.3) in $C^\infty(a;b)$ has been studied in papers [3]–[7].

Any non-quasianalytic function space $X \subsetneq C^\infty(a;b)$ contains D-invariant subspaces of form (1.2). For example,

$$W_c = \{f \in X: \ f^{(k)}(c) = 0: \ k = 0, 1, 2 \dots\}, \quad c \in (a;b).$$

It means that the problem of spectral synthesis in X should also be considered in its weakened form (1.3). Recently, we have studied this problem in the Beurling space of ultradifferentiable functions of normal type (see [8]). The dual approach we applied earlier in [3]–[4] for D-invariant subspaces in $C^\infty(a;b)$ turns out to be useful and effective in the space of ultradifferentiable functions. $X \subsetneq C^\infty(a;b)$.

In this paper, we study weak spectral synthesis problem (1.3) for a wide class of spaces of Ω-ultradifferentiable functions (briefly, Ω-UDF). General theory of Ω-UDF and Ω-ultradistributions is constructed in [1], [2] by Abanin. In particular, this theory includes all well-known spaces of UDF (Beurling-Börck spaces, Roumier-Komatsu ones, etc.) And we obtain new results on weak spectral synthesis in these general spaces of Ω-UDF.

2 Spectral synthesis

Let X be the space of Ω-UDF on an interval $(a;b)$ of the real line, that is $X = \mathcal{U}_\Omega(a;b)$, where $\Omega = \{\omega_n\}_{n=1}^\infty$ is a *regular* increasing (or decreasing) sequence of non-quasianalytic weights. For the explicit definition and main properties of such spaces see [1], [2].

Given a sequence of complex numbers Λ, we denote by exp^Λ the set of exponential monomials generated by this sequence. It means that for any λ, contained in Λ with the multiplicity $k \in \mathbb{N}$, set exp^Λ contains all functions $e^{-i\lambda t}, \dots, t^{k-1}e^{-i\lambda t}$.

Recall that *completeness radius* $r(\Lambda)$ of Λ equals the infimum of the set of all $r > 0$ such that the system exp^Λ is not complete in $C^\infty(-r;r)$ (or, equivalently, in each of spaces $C(-r;r)$, $L^2(-r;r)$).

By the well-known Beurling-Malliavin theorems (see, e.g., [13, Chapters X, XI]) Paley-Wiener-Schwartz-type theorem on strong dual space $\mathcal{U}'_\Omega(a;b)$ due to Abanin [1, Chapter 5], [2]), taking into account the property of non-quasianalyticity of weights ω_n, we get that the *function system* exp^Λ *is not complete in* $\mathcal{U}_\Omega(a;b)$ *if and only if* $r(\Lambda) < \frac{b-a}{2}$.

Let $I \subseteq (a;b)$ be a relatively closed interval, $|I|$ denote its length, and

$$W_I = \{f \in \mathcal{U}_\Omega(a;b): \quad f = 0 \text{ на } I\}. \tag{2.1}$$

To apply the dual scheme for studying weak spectral synthesis problem in $\mathcal{U}_\Omega(a;b)$, first, we sholud assure that every non-trivial D-invariant subspace W has a *residual interval* $I_W \subseteq (a;b)$ and a *residual* subspace W_{I_W}. In fact, we establish more general assertion.

Proposition 2.1. *For any closed subspace $L \subset \mathcal{U}_\Omega(a;b)$, there exists a relatively closed interval $I_L \subseteq (a;b)$ such that*

$$W_{I_L} \subset L, \quad W_I \setminus L \neq \emptyset \quad \forall I \subsetneq I_L.$$

Proof of this proposition is contained in Remark 3.

Consider a D-invariant subspace $W \subset \mathcal{U}_\Omega(a;b)$ with the residual interval $I_W \subseteq (a;b)$ and the supply of exponential monomials $\operatorname{Exp} W$. Let $\Lambda_W \subset \mathbb{C}$ be the sequence of exponents generating $\operatorname{Exp} W$, that is $\operatorname{Exp} W = exp^{\Lambda_W}$.

The spectrum of the restricted operator $D : W \to W$ is called a *spectrum of* W. We denote it by σ_W.

Proposition 2.2. 1) *For the spectrum of any non-trivial D-invariant subspace W, we have either $\sigma_W = \mathbb{C}$, or $\sigma_W = (-\mathrm{i}\Lambda_W)$.*
2) $r(\Lambda_W) > \frac{|I_W|}{2}$ *implies that $W = \mathcal{U}_\Omega(a;b)$.*

Remark 1. 1. It is not difficult to check that the spectrum of

$$\widetilde{W} = \overline{W_{I_W} + \operatorname{span} \operatorname{Exp} W}$$

equals $(-\mathrm{i}\Lambda_W)$. Particularly, it means that the relation $\sigma_W = (-\mathrm{i}\Lambda_W)$ is a necessary condition of the weak spectral synthesis for W.

2. Let $c,\ d \in (a;b)$, $W_{c,d} = \{f \in \mathcal{U}_\Omega(a;b) : \ f^{(k)}(c) = f^{(k)}(d) = 0,\ k = 0,1,2,\dots\}$, $W_{[c;d]}$ be defined by (2.1) with $I = [c;d]$. By the argument similar to one used in [9, §2], we get that $\sigma_{W_{c,d}} = \mathbb{C}$ and $\sigma_{W_{[c;d]}} = \emptyset$. At the same time, $\operatorname{Exp} W_{c,d} = \operatorname{Exp} W_{[c;d]} = \emptyset$. There may also be constructed generalisations with non-empty $\operatorname{Exp} W$.

Now, we formulate conditions of the weak spectral synthesis in $\mathcal{U}_\Omega(a;b)$.

Theorem 2.1. *Let $W \subsetneq \mathcal{U}_\Omega(a;b)$ be D-invariant subspace and $\sigma_W = -\mathrm{i}\Lambda_W$.*
If $r(\Lambda_W) < \frac{|I_W|}{2}$, then

$$W = \overline{W_{I_W} + \operatorname{span} exp^{\Lambda_W}}.$$

Corollary 2.1. *Let $W \subsetneq \mathcal{U}_\Omega(a;b)$ be D-invariant subspace and $\sigma_W = -\mathrm{i}\Lambda_W$.*
1) *If the residual interval I_W is not compact in $(a;b)$ then W admits weak spectral synthesis* (1.3).
2) *W admits classical spectral synthesis* (1.1) *if and only if $I_W = (a;b)$.*

Remark 2. It turns out that the sufficient condition in Theorem 2.1 coincides with the condition of admitting of weak spectral synthesis by D-invariant subspace in $C^\infty(a;b)$ (see [3, Theorem 2, Corollaries 2, 3, Remark 3] or [4, Theorem 5, Corollary 2], and [10, Theorems 1.1, 1.3]).

3 Preliminaries. Dual scheme

3.1 Spaces $\mathcal{U}_\Omega(a;b)$, $\mathcal{U}'_\Omega(a;b)$ and $\mathcal{P}$

Any Ω-UDF space is defined by a weight sequence $\Omega = \{\omega_n\}$ that may be increasing or decreasing:

$$\omega_n \le \omega_{n+1} \quad \forall n \in \mathbb{N} \quad \text{or} \quad \omega_n \le \omega_{n+1} \quad \forall n \in \mathbb{N}.$$

An element of Ω is a *weight function* $\omega_n : \mathbb{R} \to [0;\infty)$, which is Lebesque measurable and locally bounded in $\mathbb{R}$. Additionally, it must subject the requirements

$$\int_{\mathbb{R}} e^{\omega(t)} \mathrm{d}t < \infty, \tag{3.1}$$

$$\int_1^\infty \frac{\overline{\omega}(t)}{t^2} \mathrm{d}t < \infty, \tag{3.2}$$

where $\overline{\omega}(t) := \sup\{\omega(s) : \ |s| \le t\}$.

It should also be assumed that all weights $\omega_n \in \Omega$ and the sequence Ω itself satisfy some additional restrictions in order to guarantee that $\mathcal{U}_\Omega(a;b)$ is continuousy embedded into $C^\infty(a;b)$ and invariant under the differentiation operator. In this case, $\mathcal{U}_\Omega(a;b)$ is a locally convex space of (M^*)-type if Ω is increasing or, respectovely is a locally convex space of (LN^*)-type if Ω is decreasing. Particularly, in both cases, $\mathcal{U}_\Omega(a;b)$ is a complete reflexive Hausdorff space, the open mapping theorem and the closed graph theorem are true in this space. Moreover, $\mathcal{U}_\Omega(a;b)$ contains all polynomials, all exponential functions e^{-itz}, $z \in \mathbb{C}$, and it is a toplogical module over the ring $\mathbb{C}[t]$. The differentiation $D = \frac{\mathrm{d}}{\mathrm{d}t}$ is a continuous operator in $\mathcal{U}_\Omega(a;b)$.

Recall that given a sequence Ω, by $\mathcal{D}_\Omega(a;b)$ we denote the space of all *test* Ω*-UDF*, that are compactly supported in $(a;b)$. Because of (3.2), this space is non-trivial. Ω-ultradistributions are defined to be elements of the strong dual space $\mathcal{D}'_\Omega := \mathcal{D}'_\Omega(\mathbb{R})$ (see [1, Chapter 2,3]). It is known that any classical distribution also is an Ω-ultradistributions, that is $\mathcal{D}' \subsetneq \mathcal{D}_\Omega(\mathbb{R})$.

All basic notions of the classical distribution theory are extended to Ω-ultradistributions. In particular, it is true for the notion of "support" of an Ω-ultradistribution and the meaning of the phrase "Ω-ultradistribution equals zero on an open set". If supports of an Ω-ultradistribution S and a test Ω-UDF f have no common point then $S(f) = 0$. For $S \in \mathcal{D}'_\Omega \bigcap \mathcal{D}'_{\widetilde{\Omega}}$, where Ω and $\widetilde{\Omega}$ are two different weight sequences, the support of S as an Ω-ultradistribution equals its support if we think of S as an $\widetilde{\Omega}$-ultradistribution.

According to Theorem 5.2.2 in [1], *the strong dual space* $\mathcal{U}'_\Omega(a;b)$ *is formed by all* Ω*-ultradistributions that are compactly supported in* $(a;b)$.

For the technical convinience, now we consider a symmetric interval $(-a;a)$ instead of an arbitrary one $(a;b)$.

Given a weight ω, we recall that its continuation to $\mathbb{C}$ is defined by the formula

$$H_\omega(x + \mathrm{i}y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\omega(x + \xi y)}{1 + \xi^2} \mathrm{d}\xi$$

(see [1, 1.4]). This is a non-negative function in $\mathbb{C}$, $H_\omega(z) = H_\omega(\bar{z})$, $z \in \mathbb{C}$. It is also harmonic in the upper half-plane and in the lower one.

For every $\omega_n \in \Omega$, and every $k \in \mathbb{N}$, we set

$$\mathcal{P}_{n,k} = \left\{ \varphi \in H(\mathbb{C}) : \ \|\varphi\|_{n,k} := \sup_{z \in \mathbb{C}} \frac{|\varphi(z)|}{\exp\left(c_k |\operatorname{Im} z| + H_{\omega_n}(-z)\right)} < \infty \right\}, \tag{3.3}$$

where $0 < c_k \nearrow a$. Clearly, $\mathcal{P}_{n,k}$ is a Banach space.

Set

$$\mathcal{P}_{(\Omega),a} = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{P}_{n,k}$$

if Ω is increasing, and

$$\mathcal{P}_{\{\Omega\},a} = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \mathcal{P}_{n,k}$$

if Ω is decreasing.

By $\mathcal{P}$ we denote an arbitrary one of these two spaces, $\mathcal{P}_{(\Omega),a}$ or $\mathcal{P}_{\{\Omega\},a}$. Locally convex space $\mathcal{P}$ is a complete Hausdorff reflexive and bornological. It is also important to notice that $\mathcal{P}$ is a topological module over the ring of all polynomials $\mathbb{C}[z]$.

Recall that for every $S \in \mathcal{U}'_\Omega(-a;a)$, its Fourier-Laplace transform is defined by formula

$$S \mapsto \mathcal{F}(S)(z) := S(e^{-\mathrm{i}tz}), \quad z \in \mathbb{C},$$

and

$$\mathcal{F} : \mathcal{U}'_\Omega(-a;a) \to H(\mathbb{C}).$$

Theorem A. *Fourier-Laplace transform $\mathcal{F}$ is a linear and topological isomorphism between spaces $\mathcal{U}'_\Omega(-a;a)$ and $\mathcal{P}$.*

For the regular weight sequences Ω, Theorem A was established in [1, Theorem 5.4.2], [2]. The author also proved in [1] that the norm $\|\varphi\|_{n,k}$ defined by (3.3) may be replaced by the following one:

$$\|\varphi\|_{n,k} = \sup_{z\in\mathbb{C}} \frac{|\varphi(z)|}{\exp\left(c_k|\operatorname{Im} z| + \omega_n(-\operatorname{Re} z)\right)}.$$

This change leads to the same locally convex space $\mathcal{P}$.

Notice that all above defintions and facts are true for an arbitrary interval $(a;b) \subseteq \mathbb{R}$, not only for the symmetric one. In further presentation, we denote by $\mathcal{P}$ the space $\mathcal{F}(\mathcal{U}'_\Omega(a;b))$.

For a closed subspace $W \subset \mathcal{U}_\Omega(a;b)$, its *annihilator subspace* W^0 is defined to be

$$W^0 = \{S \in \mathcal{U}'_\Omega(a;b) : \ S(f) = 0 \ \forall f \in W\}.$$

Because of $\mathcal{U}_\Omega(a;b)$ is reflexive, by Khan-Banach theorem and Theorem A, we obtain

Proposition 3.1. (**General duality principle.**) *There is one-to-one correspondence between the set $\{W\}$ of all closed subspaces of $\mathcal{U}_\Omega(a;b)$ and the set $\{\mathcal{J}\}$ of all closed subspaces of $\mathcal{P}$:*

$$W \longleftrightarrow \mathcal{J} \Longleftrightarrow \mathcal{J} = \mathcal{F}(W^0).$$

Now, we list some properties of elements of $\mathcal{P}$.

Because of (3.2), all functions in $\mathcal{P}$ belong to the Cartwright class C of entire functions. In particular, any $\varphi \in \mathcal{P}$ is an entire function of completely regular growth with respect to the order 1 having exponential type less that $\frac{b-a}{2}$. Multiplying φ by a suitable function of the form $e^{-\mathrm{t}_\varphi z}$, $t_\varphi \in \mathbb{R}$, we get an entire function ψ with the indicator function

$$h_\psi(\theta) = \pi\Delta_\varphi|\sin\theta|, \qquad \Delta_\varphi < \frac{b-a}{2},$$

where $2\Delta_\varphi$ denotes the density of zero sequence $\mathcal{Z}_\varphi$ of φ. An indicator diagram of ψ equals

$$\mathrm{i}[-h_\psi(-\pi/2); h_\psi(\pi/2)].$$

Finally, because of the well-known Beurlig-Malliavin results (cf. [13]), from relation (3.2) we derive that given a complex sequence Λ, $r(\Lambda) < \frac{b-a}{2}$ is equivalent to $\Lambda \subset \mathcal{Z}_\varphi$ for some $\varphi \in \mathcal{P}$. In this case, the inequality $r(\Lambda) \le \pi\Delta_\varphi$ is also true. And $\Lambda = \mathcal{Z}_\varphi$ implies $r(\Lambda) = \pi\Delta_\varphi$.

3.2 Dual scheme

According to **the general duality principle** (Proposition 3.1), there is a one-to-one correspondence between closed subspaces $W \subset \mathcal{U}_\Omega(a;b)$ and closed subspaces $\mathcal{J} \subset \mathcal{P}$. It is not difficult to check that W is D-invariant if and only if $z\mathcal{J} \subset \mathcal{J}$, that is $\mathcal{J}$ is a *closed submodule* in $\mathcal{P}$ (over the ring $\mathbb{C}[z]$). In further presentation, we consider only *closed* submodules in $\mathcal{P}$ and write "submodule" instead of "closed submodule".

For an arbitrary submodule $\mathcal{J} \subset \mathcal{P}$ its *zero set* $\mathcal{Z}_\mathcal{J}$ is defined by

$$\mathcal{Z}_\mathcal{J} = \bigcap_{\varphi \in \mathcal{J}} \mathcal{Z}_\varphi,$$

where $\mathcal{Z}_\varphi$ is zero set of φ.

Indicator segment of $\mathcal{J}$ is denoted by

$$[c_\mathcal{J}; d_\mathcal{J}] \subset \overline{\mathbb{R}}, \tag{3.4}$$

where $c_\mathcal{J} = -\sup\limits_{\varphi \in \mathcal{J}} h_\varphi(-\pi/2)$, $d_\mathcal{J} = \sup\limits_{\varphi \in \mathcal{J}} h_\varphi(\pi/2) \in \overline{\mathbb{R}}$, and h_φ is the indicator function of φ.

Proposition 3.2. (**Special duality principle.**) *There is one-to-one correspondence between the set $\{W\}$ of all D-invariant subspaces in $\mathcal{U}_\Omega(a;b)$ and the set $\{\mathcal{J}\}$ of all submodules in $\mathcal{P}$:*

$$W \longleftrightarrow \mathcal{J} \Longleftrightarrow \mathcal{J} = \mathcal{F}(W^0),$$

where $W^0 = \{S \in \mathcal{U}'_\Omega(a;b) :\ S(f) = 0\ \forall f \in W\}$. In addition,

$$I_W = [c_\mathcal{J}; d_\mathcal{J}] \bigcap (a;b), \quad \operatorname{Exp} W = Exp^{\mathcal{Z}_\mathcal{J}}. \tag{3.5}$$

Proof. We need only to check the first relation in (3.5).

Set $I_0 = (a;b) \bigcap [c_\mathcal{J}; d_\mathcal{J}]$. Notice that

$$f \mapsto f(\cdot + y), \quad f \mapsto f(\cdot - y), \quad y > 0,$$

are continuous operators acting in $\mathcal{U}_\Omega(a;+\infty)$ and $\mathcal{U}_\Omega(-\infty;b)$, respectively.

For a function $f \in W_{I_0} \subset \mathcal{U}_\Omega(a;b)$, we can write

$$f = f_- + f_+, \quad f_- \in W_{I_-}, \quad f_+ \in W_{I_+},$$

where $I_- = (-\infty; d_\mathcal{J}]$, $I_+ = [c_\mathcal{J}; +\infty)$, $W_{I_-} \subset \mathcal{U}_\Omega(a;+\infty)$, $W_{I_+} \subset \mathcal{U}_\Omega(-\infty;b)$.

Further, for $S \in \mathcal{F}^{-1}(\mathcal{J})$ we have

$$\operatorname{supp} g(\cdot - y) \bigcap \operatorname{supp} S = \emptyset, \quad g \in W_{I_-},\ y > 0,$$

$$\operatorname{supp} \tilde{g}(\cdot + y) \bigcap \operatorname{supp} S = \emptyset, \quad \forall\, \tilde{g} \in W_{I_+},\ y > 0.$$

It follows that

$$S(f) = S(f_- + f_+) = \lim_{y \to 0+} \left(S(f_-(x - y)) + S(f_+(x + y))\right) = 0$$

for any Ω-ultradistribution $S \in \mathcal{F}^{-1}(\mathcal{J})$. By the general duality principle, we get that $W_{I_0} \subset W$.

Now, let us consider an arbitrary interval $I' \subsetneq I_0$ respectively closed in $(a;b)$. From the definition of $c_{\mathcal{J}}$ and $d_{\mathcal{J}}$ and general theory of Ω-UDF and Ω-ultradistributions, we derive that for every $c' \in (c_{\mathcal{J}}; d_{\mathcal{J}}) \setminus I'$ there exist $S \in \mathcal{F}^{-1}(\mathcal{J})$, $f \in \mathcal{U}_\Omega(a;b)$ and $\delta > 0$ such that

$$S(f) \neq 0, \quad \operatorname{supp} f \subset (c' - \delta; c' + \delta) \subset (c_{\mathcal{J}}; d_{\mathcal{J}}) \setminus I'.$$

Hence, by the duality principle, $f \notin W$. On the other hand, we have $f \in W_{I'}$. It means that interval I_0 is the smallest one among all respectively closed in $(a;b)$ intervals I for which $W_I \subset W$.

So, we get the relation $I_W = I_0$ and finish the proof.

□

Remark 3. The notion of the indicator segment may be defined for an arbitrary closed subspace $\mathcal{J} \subset \mathcal{P}$. Applying the argument used in the proof of the first relation in (3.5) to an arbitrary closed subspace $W \subset \mathcal{U}_\Omega(a;b)$ and $\mathcal{J} = \mathcal{F}(W^0)$, we easily get Proposition 2.1.

We call a submodule $\mathcal{J} \subset \mathcal{P}$ *weakly localisable* if it contains *all* functions $\varphi \in \mathcal{P}$ satisfying the conditions

$$\mathcal{Z}_{\mathcal{J}} \subset \mathcal{Z}_\varphi \quad \text{and} \quad [-h_\varphi(-\pi/2); h_\varphi(\pi/2)] \subset [c_{\mathcal{J}}; d_{\mathcal{J}}]$$

Submodule $\mathcal{J} \subset \mathcal{P}$ is called *localisable* (*ample*) if it contains *all* functions $\varphi \in \mathcal{P}$ with the property $\mathcal{Z}_{\mathcal{J}} \subset \mathcal{Z}_\varphi$. In other words, the localisable submodule is a weakly localisable one with the indicator segment equaled to $[a;b] \subset \overline{\mathbb{R}}$.

Weakly localisable submodule $\mathcal{J}$ is the biggest one among all submodules $\widetilde{\mathcal{J}}$ such that

$$\mathcal{Z}_{\widetilde{\mathcal{J}}} = \mathcal{Z}_{\mathcal{J}} \quad \text{and} \quad [c_{\widetilde{\mathcal{J}}}; d_{\widetilde{\mathcal{J}}}] = [c_{\mathcal{J}}; d_{\mathcal{J}}].$$

By **special duality principle**, we obtain

Proposition 3.3. *D-invatiant subspace $W \subset \mathcal{U}_\Omega(a;b)$ admits weak spectral synthesis if and only if its annihilator submodule $\mathcal{J} = \mathcal{F}(W^0)$ is weakly localisable.*

Proposition 3.3 is the basis of the *dual scheme*: it reduces spectral synthesis problem to the equivalent dual one dealing with the question of local description of submodules of entire functions. This dual scheme goes back to I.F. Krasichkov-Ternovskii and L. Ehrenpreis.

4 Stability, saturation and weak localisability

As it was shown in [18]–[20] due to Krasichkov-Ternovskii, studying of (weak) localisability of submodules is equivalent to exploring their *stability* and *saturation* properties. We use notions and notations from [18]–[20].

From the definition and topological properties of $\mathcal{P}$, it follows that $\mathcal{P}$ is *b-stable*, that is *for any bounded set $B\subset\mathcal{P}$, the set defined by*

$$B' := \left\{\frac{\varphi}{z-\lambda}:\ \varphi\in B,\ \lambda\in\mathbb{C},\ \varphi(\lambda)=0\right\}$$

is also bounded in $\mathcal{P}$.

Notice that $\mathcal{P}$ is b-stable bornological space. It implies that $\mathcal{P}$ is *pointwise stable*:
for every $\lambda\in\mathbb{C}$ and any neighbourhood of zero $U\subset\mathcal{P}$, there exists a neighbourhood of zero U'_λ such that

$$\varphi\in U'_\lambda,\quad \varphi(\lambda)=0\Longrightarrow\frac{\varphi}{z-\lambda}\in U$$

(see [19, § 4]).

Submodule $\mathcal{J}\subset\mathcal{P}$ is *stable at a point* $\lambda\in\mathbb{C}$ if for any $\varphi\in\mathcal{J}$ vanishing at λ with the multiplicity exceeding the multiplicity of λ as a zero of $\mathcal{J}$ implies that $\frac{\varphi}{z-\lambda}\in\mathcal{J}$. Submodule $\mathcal{J}$ is *stable* if it is stable at every point $\lambda\in\mathbb{C}$.

From Propositions 4.2–4.6 in [19] and pointwise stability of $\mathcal{P}$, it follows that stability of $\mathcal{J}$ at one point implies its stability at all points in $\mathbb{C}$.

Because of pointwise stability of $\mathcal{P}$, a weak localizable submodule is necessarily stable. However, in general, the inverse is not true (see [18], [19]).

Recall some notions and facts from these papers that will be required in further presentation. We cite all them for a space of scalar entire functions.

A separable locally convex space $\mathcal{P}\subset H(\mathbb{C})$ is called *b-stable* if for any bounded set $B\subset\mathcal{P}$, the set of all *entire* functions ψ of the form

$$\psi=\frac{\varphi}{z-\lambda},\ \lambda\in\mathbb{C},\ \varphi\in B,$$

is contained and bounded in $\mathcal{P}$.

The space $\mathcal{P}$ is *analytically condensed* if for any finite set of functions $\varphi_1,\dots,\varphi_m\in\mathcal{P}$, the set

$$B=\{\psi\in H(\mathbb{C}):\ |\psi(z)|\le|\varphi_1(z)|+\dots+|\varphi_m(z)|,\ z\in\mathbb{C}\}$$

is contained and bounded in $\mathcal{P}$.

A subset $\mathcal{J}\subset\mathcal{P}$ is *b-saturated with respect to* $\varphi\in\mathcal{P}$ if there exists a bounded set $B\subset\mathcal{P}$ for which the following implication holds: if an entire function ν satisfies the inequality

$$|\nu(z)\psi(z)|\le|\psi(z)|+|\varphi(z)|,\quad z\in\mathbb{C},$$

for every $\psi\in B\bigcap\mathcal{J}$, then $\nu=const.$

A closed subspace $\mathcal{J}\subset\mathcal{P}$ is called a *submodule* in $\mathcal{P}$ if the implication

$$\varphi\in\mathcal{J},\ p\in\mathbb{C}[z],\ p\varphi\in\mathcal{P}\Longrightarrow p\varphi\in\mathcal{J}$$

holds. Notice that in this definition the space $\mathcal{P}$ must not be a module over $\mathbb{C}[z]$. *Stability* and *zero set* $\mathcal{Z}_\mathcal{J}$ for $\mathcal{J}$ are also defined in the same way in this case.

For bornological b-stable spaces, the following assertion holds.

Theorem C. [18] (**Bornological version of individual theorem.**) *Let $\mathcal{J}$ be a stable submodule in a Hausdorff bornological b-stable space $\mathcal{P}$, $\psi\in\mathcal{P}$ and $\mathcal{Z}_\mathcal{J}\subset\mathcal{Z}_\psi$.*

Then, $\psi\in\mathcal{J}$ if and only if $\mathcal{J}$ be b-saturated with respect to ψ.

Now, we obtain a sufficient condition of b-saturation suitable for applications.

Proposition 4.1. *Let $\mathcal{P}$ be an analytically condensed Hausdorff b-stable locally convex space of entire functions, $\mathcal{J} \subset \mathcal{P}$, $\varphi \in \mathcal{P}$. For a function $\psi \in \mathcal{J}$, we set*

$$B_{\varphi,\psi} := \left\{ \Psi \in \mathcal{P} : \ \frac{\Psi}{\psi} \in H(\mathbb{C}), \ |\Psi(z)| \le |\varphi(z)| + |\psi(z)|, \ z \in \mathbb{C} \right\}. \tag{4.1}$$

If $B_{\varphi,\psi} \subset \mathcal{J}$, then $\mathcal{J}$ is b-saturated with respect to φ.

Proof. Define

$$B = \{\Phi \in H(\mathbb{C}) : \ |\Phi(z)| \le |\varphi(z)| + |\psi(z)|, \ z \in \mathbb{C}\}.$$

This set is bounded in $\mathcal{P}$.

Let ν be an entire function satisfying the inequality

$$|\nu(z)\Phi(z)| \le |\varphi(z)| + |\Phi(z)|, \quad \forall z \in \mathbb{C}, \tag{4.2}$$

with any $\Phi \in B \bigcap \mathcal{J}$.

Setting $\Phi = \psi \in \mathcal{J} \bigcap B$, we get

$$|\nu(z)\psi(z)| \le |\varphi(z)| + |\psi(z)|, \quad z \in \mathbb{C}.$$

In what folows that

$$\Phi_1 = \nu\psi \in \mathcal{J} \bigcap B,$$

and

$$|\nu(z)\Phi_1(z)| \le |\varphi(z)| + |\Phi_1(z)|, \ z \in \mathbb{C}.$$

This leads us to the inequality

$$\left|\frac{1}{2}\nu^2(z)\psi(z)\right| \le |\varphi(z)| + |\psi(z)|, \ z \in \mathbb{C},$$

which means that $\frac{1}{2}\nu^2\psi \in B \bigcap \mathcal{J}$.

Continuing to argue in a similar way, we obtain that

$$\frac{1}{2^{k-1}}\nu^k\psi \in B \bigcap \mathcal{J}, \quad k = 2, 3, \ldots$$

Hence, we have

$$\frac{|\nu(z)|^k}{2^{k+1}} \le \frac{|\varphi(z)|}{|\psi(z)|} + 1, \ z \in \mathbb{C}, \ k = 2, 3, \ldots$$

These inequalities imply that $\nu = \text{const}$. Because ν is an *arbitrary* entire function satisfying (4.2), we conclude that $\mathcal{J}$ b-saturated with respect to φ. □

Remark 4. If it is additionaly known that submodule $\mathcal{J}$ is stable and $\mathcal{Z}_{\mathcal{J}} \subset \mathcal{Z}_{\varphi}$, then the sufficient condition in Proposition 4.1 is also necessary one. Indeed, because of Theorem C, we have $\varphi \in \mathcal{J}$. Setting $\psi = \varphi$, we obtain the required assertion.

Given a function $\varphi \in \mathcal{P}$, we denote by $\mathcal{J}(\varphi)$ the submodule consisting of all functions $\psi \in \mathcal{P}$ of the form $\psi = \omega\varphi$, where ω is an entire function of minimal type with respect to the order 1. Clearly, $\mathcal{J}(\varphi)$ is weakly localisable submodule.

Proposition 4.2. *Let $\mathcal{J} \subset \mathcal{P}$ be a stable submodule.*
If $\varphi \in \mathcal{P}_a$ satisfies the conditions

$$\mathcal{Z}_{\mathcal{J}} \subset \mathcal{Z}_{\varphi}, \quad [h_{\varphi}(-\pi/2); h_{\varphi}(\pi/2)] \subset (c_{\mathcal{J}}; d_{\mathcal{J}}),$$

then $\mathcal{J}(\varphi) \subset \mathcal{J}$.

Proof. Consider an arbitrary function $\psi \in \mathcal{J}(\varphi)$. Because $c_{\mathcal{J}} < c_\varphi$ and $d_{\mathcal{J}} > d_\varphi$, taking into account the definitions of $c_{\mathcal{J}}$ and $d_{\mathcal{J}}$ (see (3.4)), we derive that there exist φ_1, $\varphi_2 \in \mathcal{J}$ for which

$$c_{\mathcal{J}} \le c_{\varphi_1} < c_\varphi, \quad d_\varphi < d_{\varphi_2} \le d_{\mathcal{J}}.$$

Set $\varphi_B = \varphi_1 + \varphi_2$. This function has completely regular growth with respect to the order 1. Notice that the indicator diagram of $\psi \in \mathcal{J}(\varphi)$ equals $\mathrm{i}[c_\varphi; d_\varphi]$. Hence, it is a compact subset of the indicaor diagram of φ_B, that implies

$$\frac{\psi(z)}{\varphi_B(z)} \to 0, \quad z = re^{i\theta} \tag{4.3}$$

as $r \to \infty$ outside some subset of $(0; +\infty)$ of zero relative measure.

Moreover, relation (4.3) holds unformly with respect to all

$$\theta \in \{|\pi/2 - \theta| < \delta\} \bigcup \{|-\pi/2 - \theta| < \delta\},$$

where $\delta > 0$ is small enough.

Show that $\mathcal{J}$ is b-saturated with respect to ψ. For this purpose, we set $B = \{\varphi_B\}$ and consider an entire function ρ satisfying

$$|\rho(z)\varphi_B(z)| \le |\psi(z)| + |\varphi_B(z)|, \quad z \in \mathbb{C}. \tag{4.4}$$

By the theorem on summation of indicator functions, we derive that ρ has minimal type with respect to the order 1. Moreover, by the maximum modulus principle, from (4.3) we get that ρ is bounded along the imaginary axis. Hence, $\rho = \mathit{const}$. So, we conclude that the stable submodule $\mathcal{J}$ is b-saturated with respect to ψ. Finally, by the bornologiacal version of individual theorem (Theorem C), we obtain that $\psi \in \mathcal{J}$. $\square$

Now, we can prove the criterion of weak localizability for stable submodules in $\mathcal{P}$.

Theorem 4.1. *A stable submodule $\mathcal{J} \subset \mathcal{P}$ is weakly localizable if and only if there exists $\varphi \in \mathcal{J}$ such that*

$$\mathcal{J}(\varphi) \subset \mathcal{J}.$$

Proof. Clearly, we need to prove only the assertion on sufficiency.

1) First, we assume that $\mathcal{J}(\varphi) \subset \mathcal{J}$ and the indicator diagram of φ equals $\mathrm{i}\,[c_{\mathcal{J}}; d_{\mathcal{J}}]$. Notice that the case when $c_{\mathcal{J}} = d_{\mathcal{J}}$ is also non-trivial.

Let $\psi \in \mathcal{P}$ be such that

$$\mathcal{Z}_\psi \supset \mathcal{Z}_{\mathcal{J}}, \quad \mathrm{i}\,[c_\psi; d_\psi] \subset \mathrm{i}\,[c_{\mathcal{J}}; d_{\mathcal{J}}].$$

For $\mathcal{P}$, all conditions of Proposition 4.1 are satisfied. From $\mathcal{J}(\varphi) \subset \mathcal{J}$ and conditions on the indicator diagrams of φ and ψ, it follows that the set defined by them in (4.1) is a subset of $\mathcal{J}$ (it equals $\mathcal{J}(\varphi)$). By Proposition 4.1, we derive that $\mathcal{J}$ is b-saturated with respect to ψ. By Theorem C, it means that $\psi \in \mathcal{J}$. Because ψ is an arbitrary function, we conclude that $\mathcal{J}$ is weakly localisable.

2) Now, assume that

$$\mathcal{J}(\varphi) \subset \mathcal{J}, \quad [c_\varphi; d_\varphi] \subsetneq [c_{\mathcal{J}}; d_{\mathcal{J}}] \subset (a; b).$$

Then, value of at least one of the expressions

$$\delta_1 := c_\varphi - c_{\mathcal{J}} \quad \text{or} \quad \delta_2 := d_{\mathcal{J}} - d_\varphi$$

is strictly positive. Consider in detail the case, when $\delta_1 > 0$ and $\delta_2 > 0$.

By Proposition 4.2, for all $\delta' \in [0;\delta_1)$ and $\delta'' \in [0;\delta_2)$, we have

$$\mathcal{J}(e^{i\delta' z}\varphi) \subset \mathcal{J}, \quad \mathcal{J}(e^{-i\delta'' z}\varphi) \subset \mathcal{J}.$$

Particularly,

$$e^{i\delta' z}\varphi, \quad e^{-i\delta'' z}\varphi \in \mathcal{J}, \quad \delta' \in [0;\delta_1), \ \delta'' \in [0;\delta_2). \tag{4.5}$$

Set $\Phi = (e^{i\delta_1 z} + e^{-i\delta_2 z})\varphi$. Because the relations

$$\lim_{\delta' \to \delta_1} e^{i\delta' z}\varphi = e^{i\delta_1 z}\varphi, \quad \lim_{\delta'' \to \delta_2} e^{-i\delta'' z}\varphi = e^{-i\delta_2 z}\varphi$$

hold with respect to the topology of $\mathcal{P}$, taking into account (4.5), we obtain that $\Phi \in \mathcal{J}$.

Any function $\Psi \in \mathcal{J}(\Phi)$ can be represented as

$$\Psi = \rho\Phi = \rho(e^{i\delta_1 z} + e^{-i\delta_2 z})\varphi,$$

where ρ is an entire function of zero exponential type.

It is not difficult to check that $\rho\varphi \in \mathcal{P}$. By Proposition 4.2, we get

$$\rho\varphi \in \mathcal{J}, \quad \Psi_{\delta'} = e^{i\delta' z}\rho\varphi \in \mathcal{J}, \quad \forall\, \delta' \in (0;\delta_1),$$

$$\Psi_{\delta''} = e^{-i\delta'' z}\rho\varphi \in \mathcal{J}, \quad \forall\, \delta'' \in (0;\delta_2).$$

From

$$\Psi = \lim\,(\Psi_{\delta'} + \Psi_{\delta''}) \quad \text{as} \quad \delta' \to \delta_1, \quad \delta'' \to \delta_2,$$

it follows that $\Psi \in \mathcal{J}$. Because Ψ is an arbitrary function in $\mathcal{J}(\Phi)$, the relation $\mathcal{J}(\Phi) \subset \mathcal{J}$ holds.

We have established that our submodule $\mathcal{J}$ contains the submodule $\mathcal{J}(\Phi)$ generated by the function Φ which indicator diagram equals $\mathrm{i}[c_{\mathcal{J}}; d_{\mathcal{J}}]$. Together with the first part of the proof, this leads us to the conclusion that $\mathcal{J}$ is a weakly localizable submodule.

3) It remains to consider the case, in which $c_{\mathcal{J}} = a$ or (and) $d_{\mathcal{J}} = b$.

Let $\Psi \in \mathcal{P}_a$ and $\mathrm{i}[c_\Psi; d_\Psi] \subset \mathrm{i}[c_{\mathcal{J}}; d_{\mathcal{J}}]$, $\mathcal{Z}_\Psi \supset \mathcal{Z}_{\mathcal{J}}$. To check that $\Psi \in \mathcal{J}$ we fix a segment $[c'; d']$ satisfying the relations

$$[c'; d'] \subset (a;b) \bigcap [c_{\mathcal{J}}; d_{\mathcal{J}}], \quad [c_\Psi; d_\Psi] \subset [c'; d'], \quad [c_\varphi; d_\varphi] \subset [c'; d']. \tag{4.6}$$

Denote by $\mathcal{J}'$ a weakly localizable submodule with the indicator segment $[c'; d']$ and $\mathcal{Z}_{\mathcal{J}'} = \mathcal{Z}_{\mathcal{J}}$. It is easy to see that $\widetilde{\mathcal{J}} = \mathcal{J} \bigcap \mathcal{J}'$ is a closed stable submodule with the indicator segment $[c'; d']$ and $\mathcal{Z}_{\widetilde{\mathcal{J}}} = \mathcal{Z}_{\mathcal{J}}$.

By (4.6) it follows that $\mathcal{J}(\varphi) \subset \widetilde{\mathcal{J}}$. Further, by two previous parts of the proof, we get $\widetilde{\mathcal{J}} = \mathcal{J}'$. Taking into account (4.6) one more time, we obtain that

$$\Psi \in \widetilde{\mathcal{J}} \subset \mathcal{J}.$$

□

By the Beurling-Malliavin radius of completeness and multiplier theorems (see, e.g., [13, X-XI]), we derive

Proposition 4.3. *Submodule $\mathcal{J} \subset \mathcal{P}$ contains non-zero functions if and only if the relation*

$$d_{\mathcal{J}} - c_{\mathcal{J}} \geq 2(\mathcal{Z}_{\mathcal{J}}) \tag{4.7}$$

holds.

By Proposition 4.3, we see that the weak localizability problem is non-trivial only for submodules satisfying (4.7). It turns out that there may be two essentially different cases:

$$d_{\mathcal{J}} - c_{\mathcal{J}} = 2r(\mathcal{Z}_{\mathcal{J}})$$

and

$$d_{\mathcal{J}} - c_{\mathcal{J}} > 2r(\mathcal{Z}_{\mathcal{J}}). \tag{4.8}$$

In the first case, there exist stable submodules, that are not weakly localizable. There also exist weakly localizable ones (cf. [4], [6]). And vice versa, any stable submodule satisfying (4.8) is weakly localizable.

Theorem 4.2. *Let $\mathcal{J} \subset \mathcal{P}$ be a stable submodule.*
If

$$d_{\mathcal{J}} - c_{\mathcal{J}} > 2r(\mathcal{Z}_{\mathcal{J}}) \tag{4.9}$$

then $\mathcal{J}$ is non-trivial and weakly localizable.

Proof. By the Beurling-Malliavin theorems and Theorem A, taking into account the properties of weights in Ω, we obtain that there exists non-zero function $\varphi_0 \in \mathcal{P}$ such that

$$\mathcal{Z}_{\mathcal{J}} \subset \mathcal{Z}_{\varphi_0}, \quad [h_{\varphi_0}(-\pi/2); h_{\varphi_0}(\pi/2)] \subset (c_{\mathcal{J}}; d_{\mathcal{J}}).$$

According to Proposition 4.2, the inclusion $\mathcal{J}(\varphi_0) \subset \mathcal{J}$ holds. By Theorem 4.1, we get the required assertion.

□

Corollary 4.1. *Let $\mathcal{J} \subset \mathcal{P}$ be a stable submodule and its indicator segment be not compact in $(a; b)$. Then, $\mathcal{J}$ is a weakly localizable submodule. In particular, the stable submodule $\mathcal{J} \subset \mathcal{P}_a$ is localizable if and only if*

$$c_{\mathcal{J}} = -a, \quad d_{\mathcal{J}} = a.$$

Remark 5. Notice that we work with dual scheme using two famous Beurling-Malliavin theorems. This is one more example of applying them for solving problems which concern with completeness of exponential systems and exponential bases (cf. [3], [4], [7], [9],[10], [12]).

5 Solving weak spectral synthesis problem in $\mathcal{U}_\Omega(a; b)$

5.1 Spectrum of D-invariant subspace

From the previous section, taking into account Proposition 2.1 we see that a necessary condition of weak spectral synthesis for a D-invariant subspace $W \subset \mathcal{U}_\Omega(a; b)$ is the property of stability of its annihilator submodule. It follows that we need to know an equivalent dual requirement for the subspace itself.

The first step on this way belongs to A. Aleman and B. Korenblum. In [9], the notion of *spectrum* of D-invariant subspace in $C^\infty(a; b)$ was introduced. Namely, the spectrum σ_W was defined as a complement of $\mathbb{C}$ to the set of all *regular points* of the restricted operator $D : W \to W$. Here, $\mu \in \mathbb{C}$ is *regular* if $(D - \mu\,\mathbf{id})$ is a bijective map in W. For any regular point $\mu \in \mathbb{C}$, there exists a linear and continuous inverse operator

$$(D - \mu\,\mathbf{id})^{-1} : W \to W.$$

A. Aleman and B. Korenblum proved the following two assertions (see [9, Theorem 2.1, Proposition 3.1]):
1) the spectrum of D-invariant subspace $W \subset C^\infty(a;b)$ is either equal to the whole complex plane, or equal to a finite or denumerable (may be, empty) set of multiple points in $\mathbb{C}$ with the unique possible limit point at infinity;
2) the relation $\sigma_W \neq \mathbb{C}$ implies that $\mathcal{J} = \mathcal{F}(W^0)$ is stable at any point $\lambda \notin \mathrm{i}\sigma_W$ (hence, as we have noticed above, the annihilator submodule is stable).

We should mention that the initial form of the second assertion in [9] is a different one, because its authors used other techniques, not the dual scheme.

Our purposes are to establish the same assertion like the first cited one for D-invariant subspaces $W \subset \mathcal{U}_\Omega(a;b)$ and to prove that σ_W is discrete if and only if the corresponding annihilator submodule $\mathcal{J} = \mathcal{F}(W^0)$ is stable.

Proposition 5.1. *Let $W \subset \mathcal{U}_\Omega(a;b)$ be a D-invariant subspace, $\mathcal{J}$ be its annihilator submodule.*

A point $\mu \in \mathbb{C}$ is regular for the restricted operator $D : W \to W$ if and only if both following conditions hold: 1) $\mathrm{i}\mu \notin \mathcal{Z}_\mathcal{J}$; 2) *submodule $\mathcal{J}$ is stable at $\lambda = \mathrm{i}\mu$.*

Proof. Necessity. 1) Because $\mu \notin \sigma_W$ implies that $e^{\mu t} \notin W$, according to the duality principle, we get $\mathrm{i}\mu \notin \mathcal{Z}_\mathcal{J}$.
2) Without loss of generality, assume that $\mu = 0$.

Let $\varphi \in \mathcal{J}$ be such that $\varphi(0) = 0$, and set

$$S = \mathcal{F}^{-1}(\varphi), \quad \widetilde{S} = \mathrm{i}\mathcal{F}^{-1}\Big(\frac{\varphi}{z}\Big).$$

Denote by D^* a generalized differentiation operator acting in $\mathcal{U}'_\Omega(a;b)$. This is an adjoint operator to D. It is not difficult to check that

$$\mathcal{F}(D^*(\widetilde{S})) = \varphi.$$

This is equivalent to the relation

$$D^*(\widetilde{S}) = S.$$

For any $f \in W$, there exists $g \in W$ such that $Dg = f$. Therefore, it follows that

$$\widetilde{S}(f) = \widetilde{S}(Dg) = D^*(\widetilde{S})(g) = S(g) = 0.$$

Hence, we conclude that

$$\widetilde{S} \in W^0, \qquad \frac{\varphi}{z} \in \mathcal{J}.$$

Sufficiency. Without loss of generality, we assume that $\mu = 0$.

Let A be an inverse-shift operator acting in $\mathcal{P}$, that is

$$A(\psi)(z) = \frac{\psi(z) - \psi(0)}{z}.$$

The space $\mathcal{U}_\Omega(a;b)$ may be considered as strong dual space to $\mathcal{P}'$. Then, we see that the "lifting" $\widehat{A}$ of A^* acts in $\mathcal{U}_\Omega(a;b)$ and satisfies the relation

$$D\widehat{A}(f) = -\mathrm{i}f, \quad f \in \mathcal{U}_\Omega(a;b). \tag{5.1}$$

Similarly, for the "lifting" $\widehat{D}$ of D^*, we have

$$A\widehat{D}(\varphi) = -\mathrm{i}\varphi, \tag{5.2}$$

and $\widehat{D}$ acts in $\mathcal{P}$.

Let $\mathcal{J}$ be a stable submodule, $0 \notin \mathcal{Z}_{\mathcal{J}}$.

First, we consider the case $\mathcal{J} = \mathcal{J}_\varphi$, $\varphi(0) = 1$. Setting $S = \mathcal{F}^{-1}(\varphi)$, we write

$$W = W_S = \{f \in \mathcal{E}_a : \ S(D^k f) = 0, \ k = 0, 1, 2, \dots\}. \tag{5.3}$$

For any $g \in W_S$, set

$$f = \mathrm{i}\widehat{A}(g) - S(\mathrm{i}\widehat{A}(g)). \tag{5.4}$$

Clearly, $S(f) = 0$. Because of (5.1), we have

$$Df = g,$$

Hence,

$$S(D^k f) = 0, \ k = 0, 1, 2, \dots,$$

and $f \in W_S$.

We have shown that $D : W_S \to W_S$ is a surjective operator. Further, because of $\varphi(0) = 1$, the only solution of equation $Df = 0$ in W_S is zero. It follows that $D : W_S \to W_S$ is a bijection.

Now, consider an arbitrary D-invariant subspace W. Let $\mathcal{J}$ be its annihilator submodule. There exists $\varphi \in \mathcal{J}$ such that $\varphi(0) = 1$. For an arbitrary $\psi \in \mathcal{J}$, we have

$$\psi = z\frac{\psi - \psi(0)\varphi}{z} + \psi(0)\varphi.$$

This relation and the stability of $\mathcal{J}$ imply that

$$\mathcal{J} = z\mathcal{J} + \mathcal{J}_\varphi.$$

By the duality principle, $W = W_1 \bigcap W_S$, where W_1 is the D-invariant subspace whose annihilator submodule equals $z\mathcal{J}$, and W_S is defined by formula (5.3) for $S = \mathcal{F}^{-1}(\varphi)$.

For any $g \in W$, we define function f by formula (5.4). Then, as above, we have $f \in W_S$. Taking into account (5.1) and (5.2), we also get $f \in W_1$. Finally, we obtain $f \in W$ and conclude that $D : W \to W$ is a surjection. Clearly, this operator is also injective. $\square$

Corollary 5.1. *For the spectrum of a D-invariant subspace $W \subset \mathcal{E}_a$ we have either $\sigma_W = -\mathrm{i}\mathcal{Z}_{\mathcal{J}}$, where $\mathcal{J} = \mathcal{F}(W^0)$ if the annihilator submodule is stable,*
or $\sigma_W = \mathbb{C}$ if the annihilator submodule is not stable.

Now we are ready to prove all new propositions formulated in this paper.

Assertions of Proposition 2.2 follow from Proposition 5.1 and Proposition 4.3 by the duality argument.

Theorem 2.1 and Corollary 2.1 are dual propositions to Theorem 4.2 and Corollary 4.1, respectively.

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