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ON APPROXIMATION OF SOLUTIONS OF SOME SEMI-ELLIPTIC EQUATIONS IN $\mathbb{R}^n$

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Abstract. It is proved that for some semi-elliptic equations the solution can be obtained as the limit when $\sigma \to \infty$, of the solution u_σ of a boundary value problem in the generalised ball $B_{\sigma,\mu}$. Also an estimatie at infinity for the tempered fundamental solution is obtained.

1 Introduction

Let $\mathbb{R}^n$ be the n-dimensional euclidean space of real vectors, N_0^n – the set of multi-indices, i.e., n-dimensional vectors $\alpha = (\alpha_1, ..., \alpha_n)$ with nonnegative integer components. Furthermore, let $\bar{m} = (m_1, ..., m_n)$ be a vector with natural components, $\mu = (\frac{1}{m_1}, ..., \frac{1}{m_n})$, $|\mu| = \sum\limits_{j=1}^{n} \mu_j$, $\mu_0 = \min\limits_{1\le j\le n} \mu_j$. If $x, \xi \in \mathbb{R}^n$, $\alpha \in N_0^n$ then we set

$$\xi^\alpha = \xi_1^{\alpha_1} \cdot ... \cdot \xi_n^{\alpha_n}, \quad D^\alpha = D_1^{\alpha_1}...D_n^{\alpha_n}, \quad D_j = \frac{1}{i}\frac{\partial}{\partial x_j} \quad (j = 1, ..., n),$$

$$(\mu, \alpha) = \sum_{j=1}^{n} \mu_j\, \alpha_j, \quad |x|_\mu = \left(\sum_{j=1}^{n} |x_j|^{\frac{2}{\mu_j}} \right)^{\frac{1}{2}}.$$

Let $P(D)$ be a linear differential operator of μ-order 2 with constant coefficients, i.e. the symbol of $P(D)$ is represented in the form $P(\xi) = \sum\limits_{(\mu,\alpha)\le 2} \gamma_\alpha \xi^\alpha$.

Definition. *The operator $P(D)$ is said to be semi-elliptic (see [3]), if there is a constant $\chi > 0$ such that*

$$\left| \sum_{(\mu,\alpha)=2} \gamma_\alpha \xi^\alpha \right| \ge \chi \left(|\xi_1|^{2/\mu_1} + ... + |\xi_n|^{2/\mu_n} \right) \quad \textit{for all} \quad \xi \in \mathbb{R}^n.$$

Let $P(D)$ be a semi-elliptic differential operator of μ-order 2 with constant coefficients and $Q(x, D)$ a differential operator of μ-order 2 with infinitely continuously differentiable coefficients which vanish for $|x|_\mu > a$.

In [4] the equation

$$[P(D) + Q(x, D)]\, u = f \quad \text{in} \quad \mathbb{R}^n,$$

has been considered when $P(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$. It has been shown that if for any $f \in L_{2,a,\mu}(\mathbb{R}^n)$ (i.e. $f \in L_2(\mathbb{R}^n)$, $f(x) = 0$ if $|x|_\mu > a$) there exists a solution u of the above equation, then this solution is unique in the class of functions vanishing at infinity.

Moreover, in [4] it has been proved that the boundary value problem in μ-ball $B_{\sigma,\mu} = \left\{ x \in \mathbb{R}^n; \quad |x|_\mu < \sigma \right\}$

$$P(D)\, u_\sigma = f \quad \text{in} \quad B_{\sigma,\mu},$$

$$u_\sigma \in \overset{\circ}{H}{}^{\bar{m}}(B_{\sigma,\mu}) \ ,$$

for sufficiently large $\sigma > 0$ has a unique solution u_σ in the anisotropic Sobolev space $H^{2\bar{m}}(B_{\sigma,\mu})$ and for any fixed compact K

$$\| u - u_\sigma \|_{H^{2\bar{m}}(K)} \leq c\, e^{-\gamma\, \sigma^{\mu_0}} \sigma^{M} \ \|f\|_{L_{2,a,\mu}(\mathbb{R}^n)} ,$$

where c, γ, μ_0, M are positive numbers which do not depend on f and σ.

The aim of the present paper is proving results of this type if $P(\xi) \neq 0$ for all $\xi \in R^n \setminus \{0\}$ but $P(0) = 0$. In Section 2 estimates for the tempered fundamental solution will be proved. In Section 3 the case $Q(x, D) \equiv 0$ will be considered and, finally, in Section 4 the main theorem for the general case will be proved.

Such results for elliptic equations have been obtained by L. Simon in [7].

2 Fundamental solution

It is clear that any linear differential operator $P(D)$ of μ-order 2 one can represent in the form

$$P(D) = \sum_{j=0}^{M} P_j(D) \equiv \sum_{j=0}^{M} \sum_{(\mu,\alpha)=d_j} \gamma_\alpha D^\alpha, \tag{1}$$

where $2 = d_0 > d_1 > ... > d_M \geq 0$. In this paper we suppose that operator (1) has the form

$$P(D) = \sum_{j=0}^{l} P_j(D) \tag{2}$$

where $l < M$ and its symbol satisfies the conditions

$$P(\xi) \neq 0 \ \text{ for all } \ \xi \in \mathbb{R}^n \setminus \{0\}\,. \tag{3}$$

Note that

$$P(0) = 0. \tag{4}$$

Theorem 1. *Let operator (2) satisfy the following conditions:*

a) $P(D)$ is semi-elliptic operator satisfying conditions (3), (4);

b) $P_l(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$;

c) $0 < d_l < |\mu|$.

Then $P(D)$ has a unique tempered fundamental solution E such that for any fixed α the estimate

$$D^\alpha E(x) \ = \ O\left(\frac{1}{|x|_\mu^{|\mu|-d_l+(\mu,\alpha)}}\right), \quad |x|_\mu \to \infty \tag{5}$$

holds.

Proof. By Theorem 1 of [5] we obtain

$$|\xi|_\mu^{2d_j} \ = \ O\left(P_j(\xi)^2\right), \ |\xi|_\mu \to 0\ , \ \ j = 0\,,\, 1\,,\, ...,\, l. \tag{6}$$

Thus

$$\frac{1}{P(\xi)} = O\left(\frac{1}{|\xi|_\mu^{d_l}}\right). \tag{7}$$

Since $d_l < |\mu|$ the function $\frac{1}{|\xi|_\mu^{d_l}}$ is integrable in a neighbourhood of zero (see [1], § 4), so $\frac{1}{P}$ is locally integrable in $\mathbb{R}^n$. By semi-ellipticity of P, $P_0(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Therefore $\frac{1}{P}$ is a tempered distribution and the inverse Fourier transform of $(2\pi)^{-\frac{n}{2}}\frac{1}{P}$, i.e.

$$E = (2\pi)^{-\frac{n}{2}}\, F^{-1}\left[\tfrac{1}{P}\right],$$

is a fundamental solution of $P(D)$.

It is well known, that two different fundamental solutions of $P(D)$ differ from each other by a distribution with support in the set $\{\xi\,;\ P(\xi) = 0\}$. For the solutions u of the equation $P(D)u = 0$, $\operatorname{supp} F[u] = \{0\}$ (see (3), (4)), thus $F[u]$ has a unique representation of the form (see [8], § 8.4)

$$F[u] = \sum_{(\mu,\alpha)\le N} c_\alpha D^\alpha \delta$$

and so u is a polynomial. This implies the uniqueness of solutions of $P(D)u = \delta$ in the class of functions, vanishing at infinity.

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be such that $\psi = 1$ in a neighbourhood of zero. Then

$$E = E_1 + E_2 = (2\pi)^{-\frac{n}{2}} F^{-1}\left[\frac{\psi}{P}\right] + (2\pi)^{-\frac{n}{2}} F^{-1}\left[\frac{1-\psi}{P}\right]. \tag{8}$$

It is easy to verify, that for all $\alpha, \beta \in N_0^n$

$$\left|x^\beta\, D^\alpha E_2\right| \ = \ (2\pi)^{-\frac{n}{2}} \left|F^{-1}[D^\beta(\xi^\alpha \tfrac{1-\psi}{P})]\right|.$$

Note that exists $R > 0$ such that for all $\xi \in \mathbb{R}^n$ satisfying $|\xi|_\mu \geq R$

$$\frac{1}{P} = \frac{1}{P_0} - \frac{P-P_0}{P_0^2} + \cdots + (-1)^s \frac{(P-P_0)^s}{P_0^{s+1}} + (-1)^{s+1} \frac{1}{P} \frac{(P-P_0)^{s+1}}{P_0^{s+1}},$$

where s is an integer (see [6], § 12). We take s such that, when $|\xi|_\mu \to \infty$, the last term tends to zero quicker than $|\xi|_\mu^{-(|\mu|+2)}$. Then, as $D^\beta\left(\frac{\xi^\alpha}{P_0}\right)$ is a μ-homogeneous function of degree $(\mu,\alpha)-(\mu,\beta)-2$ (see [1], § 4), we obtain

$$D^\beta\left(\xi^\alpha \frac{1-\psi}{P}\right) = O\left(\frac{1}{|\xi|_\mu^{2+(\mu,\beta)-(\mu,\alpha)}}\right), \quad |\xi|_\mu \to \infty.$$

Therefore $D^\beta(\xi^\alpha \frac{1-\psi}{P}) \in L_1(\mathbb{R}^n)$ for $(\mu,\beta) > |\mu| - 2 + (\mu,\alpha)$. Thus for any $\alpha,\, \beta \in N_0^n$

$$(x^\beta\, D^\alpha E_2)(x) = O(1)\,, \quad |x|_\mu \to \infty. \tag{9}$$

Consider the term E_1 in (8). We have the equality

$$\left|x^\beta\, D^\alpha E_1\right| = (2\pi)^{-\frac{n}{2}} \left|F^{-1}[D^\beta(\xi^\alpha \frac{\psi}{P})]\right|. \tag{10}$$

The function $D^\beta(\xi^\alpha \frac{\psi}{P})$ is infinitely continuously differentiable in $\mathbb{R}^n \setminus \{0\}$ and has compact support. Moreover

$$D^\beta\left(\frac{1}{|\xi|_\mu^{d_l}}\right) = O\left(\frac{1}{|\xi|_\mu^{d_l+(\mu,\beta)}}\right), \quad |\xi|_\mu \to 0,$$

so

$$D^\beta(\xi^\alpha \frac{\psi}{P}) = O\left(\frac{1}{|\xi|_\mu^{d_l+(\mu,\beta)-(\mu,\alpha)}}\right), \quad |\xi|_\mu \to 0.$$

This implies that

$$D^\beta\left(\xi^\alpha \frac{\psi}{P}\right) \in L_1(\mathbb{R}^n) \;\text{ if }\; (\mu,\beta) < |\mu| - d_l + (\mu,\alpha),$$

hence by (10)

$$x^\beta\, (D^\alpha E_1)(x) = O(1)\,, \quad |x|_\mu \to \infty. \tag{11}$$

In order to show (11) for $(\mu,\beta) = |\mu| - d_l + (\mu,\alpha)$ it suffices to prove that the estimate

$$F^{-1}\left[D^\beta(\xi^\alpha \frac{\psi}{P})\right] = O(1)$$

holds at infinity. This follows since

$$D^\beta\left(\xi^\alpha \frac{\psi}{P}\right) = D^\beta\left(\xi^\alpha \frac{\psi}{P_l}\right) - D^\beta\left(\xi^\alpha \frac{\psi(P-P_l)}{P\,P_l}\right)$$

and

$$D^\beta(\xi^\alpha \tfrac{\psi(P-P_l)}{P\,P_l}) \in L_1(\mathbb{R}^n)$$

because at zero

$$\left[D^\beta(\xi^\alpha \tfrac{\psi(P-P_l)}{P\,P_l})\right](\xi) = O\left(\frac{1}{|\xi|_\mu^{2d_l - d_{l+1} + (\mu,\beta) - (\mu,\alpha)}}\right) = O\left(\frac{1}{|\xi|_\mu^{|\mu| - (d_{l+1} - d_l)}}\right).$$

Using integration by parts we get that for any test function φ in the Schwartz space S (see [4])

$$\left[D^\beta\left(\xi^\alpha \frac{\psi}{P_l}\right)\right](\varphi) = \lim_{\varepsilon\to+0} \int\limits_{R^n\setminus B_{\varepsilon,\mu}} \left[D^\beta\left(\xi^\alpha \frac{\psi}{P_l}\right)\right](\varphi)\,d\,\xi - c\,\varphi(0), \tag{12}$$

where c is a complex number depending only on P. For the functions

$$g_k(\xi) = \begin{cases} D^\beta\left(\xi^\alpha \frac{\psi}{P_l}\right), & |\xi|_\mu \geq \frac{1}{k} \\ 0, & |\xi|_\mu < \frac{1}{k} \end{cases}, \quad k = 1, 2, \ldots$$

it follows by equality (12) that the sequence g_k tends to the distribution $D^\beta(\xi^\alpha \frac{\psi}{P_l}) - c\delta$ in the sense of the space S' of tempered distributions. Thus the sequence $F^{-1}[g_k]$ tends to $F^{-1}[D^\beta(\xi^\alpha \frac{\psi}{P_l})] - c\,F^{-1}[\delta]$ in the sense of S'. Since $F^{-1}[\delta] = 1$, for the boundedness of $F^{-1}[D^\beta(\xi^\alpha \frac{\psi}{P_l})]$ it is suffices to prove that there exists a constant λ such that for any x

$$|(F^{-1}[g_k])(x)| \leq \lambda \quad \text{if } k \geq k_0(x).$$

By the definition of g_k

$$F^{-1}[g_k](x) = (2\pi)^{-n/2} \int\limits_{|\xi|_\mu \geq \frac{1}{k}} e^{i(x,\xi)} \left[D_\xi^\beta\left(\xi^\alpha \frac{\psi}{P_l}\right)\right] d\xi.$$

By the first part of the proof it follows that the functions $F^{-1}[g_k] - (2\pi)^{-\frac{n}{2}} h_k(x)$, where

$$h_k(x) = \int\limits_{|\xi|_\mu \geq \frac{1}{k}} e^{i(x,\xi)}\,\psi(\xi)\,D^\beta\left(\frac{\xi^\alpha}{P_l}\right) d\xi,$$

are uniformly bounded, so it suffices to show that there exists a number $\lambda > 0$ such that

$$|\,h_k(x)\,| \leq \lambda \ \text{ if } k \geq k_0(x). \tag{13}$$

Let the function ψ have the special form: $\psi(\xi) = \psi_0\left(|\xi|_\mu\right)$ and suppose that $\psi_0 \geq 0$, $\psi_0(r) = 0$ if $r > b$. Then applying μ-spherical (generalized spherical) transformation of coordinates [1], we obtain

$$h_k(x) = \int\limits_{1/k}^{b} \left[\int\limits_{|\theta|_\mu = 1} e^{i(x,r^\mu\theta)}\psi_0(r) \left[D_\xi^\beta\left(\frac{\xi^\alpha}{P_l}\right)\Big|_{\xi = r^\mu\,\theta}\right] r^{|\mu|-1} \sum_{i=1}^{n} \mu_i^2\theta_i\,d\theta\right] dr =$$

$$= \int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{1/k}^{b} \frac{e^{i(x,r^\mu\theta)}}{r} \psi_0(r)\, dr \right] d\theta, \tag{14}$$

where

$$g(\theta) = D^\beta \left(\frac{\xi^\alpha}{P_l}\right)\Big|_{\xi=r^\mu\theta} \sum_{i=1}^{n} \mu_i^2 \theta_i\, r^{|\mu|}.$$

The function $g(\theta)$ does not depend on r, since $D^\beta\left(\frac{\xi^\alpha}{P_l}\right)$ is a $\mu-$homogeneous function of degree $(\mu, \alpha) - (\mu, \beta) - d_l = -|\mu|$.

Formula (14) implies that

$$h_k(x) = \int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{1/k}^{b} \frac{e^{i\,(x\,,\,r^\mu\theta)} - 1}{r} \psi_0(r)\, dr \right] d\theta +$$

$$+ \left[\int\limits_{|\theta|_\mu=1} g(\theta)\, d\theta \right] \left[\int\limits_{1/k}^{b} \frac{1}{r} \psi_0(r)\, dr \right].$$

The first term in the right-hand side and h_k are convergent in S' as $k \to \infty$ so the second term is also convergent in S' which implies that

$$\int\limits_{|\theta|_\mu=1} g(\theta)\, d\theta = 0. \tag{15}$$

We shall consider h_k for the case of $x = (0, ..., 0, x_n)$, since the general case can be reduced to this one by a simple transformation of the coordinates. In this case

$$h_k(x) = \int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{1/k}^{b} \frac{e^{i\,x_n\,\cdot\, r^{\mu_n}\theta_n} - 1}{r} \psi_0(r)\, dr \right] d\theta =$$

$$= \int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{\frac{(x_n\theta_n)^{m_n}}{k}}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}} - 1}{\sigma} \psi_0\left(\frac{\sigma}{(x_n\theta_n)^{m_n}}\right) d\sigma \right] d\theta.$$

Thus

$$h_k(x) = \int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{\frac{(x_n\theta_n)^{m_n}}{k}}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}} - 1}{\sigma} (\psi_0 - 1)\left(\frac{\sigma}{(x_n\theta_n)^{m_n}}\right) d\sigma \right] d\theta +$$

$$+ \int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{\frac{(x_n\theta_n)^{m_n}}{k}}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}}-1}{\sigma}\,d\sigma \right] d\theta. \tag{16}$$

The first term in the right-hand side of (16) is uniformly bounded, because there exists a number c_1 such that

$$(\psi_0 - 1)\left(\frac{\sigma}{(x_n\theta_n)^{m_n}}\right) = 0 \quad \text{if} \quad \left|\frac{\sigma}{(x_n\theta_n)^{m_n}}\right| < c_1.$$

Using the inequalities

$$\frac{1}{|\sigma|} \;\le\; \frac{1}{c_1\left|\frac{\sigma}{(x_n\theta_n)^{m_n}}\right|}\;, \quad \left|(e^{i\sigma^{\mu_n}}-1)(\psi_0-1)\right| \le c_2,$$

we get

$$\left| \int\limits_{\frac{(x_n\theta_n)^{m_n}}{k}}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}}-1}{\sigma}\,(\psi_0-1)\left(\frac{\sigma}{(x_n\theta_n)^{m_n}}\right) d\sigma \right| \le$$

$$\le\; c_2(b-\frac{1}{k})\;|(x_n\theta_n)^{m_n}|\cdot\frac{1}{c_1\,|(x_n\theta_n)^{m_n}|}\;\le\;\frac{c_2}{c_1}\,b, \tag{17}$$

where $c_2 > 0$ is independent of σ.

The second term in the right-hand side of (16) can be written in the form

$$\int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{\frac{(x_n\theta_n)^{m_n}}{k}}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}}-1}{\sigma}\,d\sigma \right] d\theta \;=$$

$$= \int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{\frac{(x_n\theta_n)^{m_n}}{k}}^{\operatorname{sgn}(\,(x_n\theta_n)^{m_n})} \frac{e^{i\,\sigma^{\mu_n}}-1}{\sigma}\,d\sigma \right] d\theta \;+$$

$$+ \int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{\operatorname{sgn}((x_n\theta_n)^{m_n})}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}}-1}{\sigma}\,d\sigma \right] d\theta. \tag{18}$$

Since $g(\theta)$ is bounded and

$$\left| \int\limits_{\frac{(x_n\theta_n)^{m_n}}{k}}^{\operatorname{sgn}((x_n\theta_n)^{m_n})} \frac{e^{i\,\sigma^{\mu_n}}-1}{\sigma}\,d\sigma \right| \;\le\; \sup_{|\sigma|\le 1} \left|\frac{e^{i\,\sigma^{\mu_n}}-1}{\sigma}\right| \quad \text{if } k \ge (x_n\theta_n)^{m_n},$$

we have the estimate of the form

$$\left| \int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{\frac{(x_n\theta_n)^{m_n}}{k}}^{\operatorname{sgn}(\,(x_n\theta_n)^{m_n})} \frac{e^{i\,\sigma^{\mu_n}}-1}{\sigma}\,d\sigma \right] d\theta \right| \le \lambda \quad if\;\; k \ge |x_n^{m_n}|\,. \tag{19}$$

Finally we have

$$\int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{\operatorname{sgn}((x_n\theta_n)^{m_n})}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}}-1}{\sigma}\,d\sigma \right] d\theta =$$

$$= \int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{\operatorname{sgn}((x_n\theta_n)^{m_n})}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}}}{\sigma}\,d\sigma \right] d\theta -$$

$$- \int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{\operatorname{sgn}((x_n\theta_n)^{m_n})}^{b\,(x_n\theta_n)^{m_n}} \frac{1}{\sigma}\,d\sigma \right] d\theta. \tag{20}$$

Equality (15) implies that the second term in the right-hand side of (20) is bounded.

Indeed

$$\left| \int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{\operatorname{sgn}((x_n\theta_n)^{m_n})}^{b\,(x_n\theta_n)^{m_n}} \frac{1}{\sigma}\,d\sigma \right] d\theta \right| =$$

$$= \left| \int\limits_{|\theta|_\mu=1} g(\theta)\,\ln\left| b\,|x_n^{m_n}| \left(\frac{x_n^{m_n}}{|x_n^{m_n}|} \cdot \theta_n^{m_n} \right) \right| d\theta \right| \le$$

$$\le \left| \int\limits_{|\theta|_\mu=1} g(\theta)\,\ln\left| \frac{x_n^{m_n}}{|x_n^{m_n}|} \cdot \theta_n^{m_n} \right| d\theta \right| \le \sup|g| \left| \int\limits_{|\theta|_\mu=1} \ln\left| \frac{x_n^{m_n}}{|x_n^{m_n}|} \cdot \theta_n^{m_n} \right| d\theta \right|, \tag{21}$$

where $\int\limits_{|\theta|_\mu=1} \ln\left| \frac{x_n^{m_n}}{|x_n^{m_n}|} \cdot \theta_n^{m_n} \right| d\theta$ is bounded.

Furthermore

$$\int\limits_{|\theta|_\mu=1} g(\theta) \left[\int\limits_{\operatorname{sgn}((x_n\theta_n)^{m_n})}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}}}{\sigma}\,d\sigma \right] d\theta =$$

$$= \int\limits_{|(x_n\theta_n)^{m_n}|\,\ge 1/b} g(\theta) \left[\int\limits_{\operatorname{sgn}((x_n\theta_n)^{m_n})}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}}}{\sigma}\,d\sigma \right] d\theta +$$

$$+ \int\limits_{|(x_n\theta_n)^{m_n}|<1/b} g(\theta) \left[\int\limits_{\operatorname{sgn}((x_n\theta_n)^{m_n})}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}}}{\sigma}\, d\sigma \right] d\theta \,. \tag{22}$$

The terms in the right-hand side of (22) can be estimated as follows: if $|(x_n\theta_n)^{m_n}| \geq 1/b$, i.e. $|b(x_n\theta_n)^{m_n}| \geq 1$, then

$$\int\limits_{\operatorname{sgn}((x_n\theta_n)^{m_n})}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}}}{\sigma}\, d\sigma \;=\; m_n \left[\frac{e^{i\,b\,x_n\,\theta_n}}{i\,x_n\,\theta_n} \;-\; \frac{e^{i\,\operatorname{sgn}((x_n\theta_n)^{m_n})}}{\operatorname{sgn}\left((x_n\theta_n)^{m_n}\right)} \right] \;+$$

$$+ \int\limits_{\operatorname{sgn}((x_n\theta_n)^{m_n})}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}}}{i\sigma^{\mu_n+1}}\, d\sigma, \tag{23}$$

which is uniformly bounded since $\int\limits_1^\infty \frac{1}{\sigma^{1+\mu_n}}\, d\sigma < \infty$. Moreover

$$\int\limits_{|(x_n\theta_n)^{m_n}|\,<\,1/b} g(\theta) \left[\int\limits_{\operatorname{sgn}((x_n\theta_n)^{m_n})}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}}}{\sigma}\, d\sigma \right] d\theta \;=$$

$$= \int\limits_{|(x_n\theta_n)^{m_n}|\,<\,1/b} g(\theta) \left[\int\limits_{\operatorname{sgn}((x_n\theta_n)^{m_n})}^{b\,(x_n\theta_n)^{m_n}} \frac{e^{i\,\sigma^{\mu_n}}-1}{\sigma}\, d\sigma \right] d\theta +$$

$$+ \int\limits_{|(x_n\theta_n)^{m_n}|\,<\,1/b} g(\theta) \left[\ln b \;+\; \ln\left| \frac{x_n^{m_n}}{|x_n^{m_n}|}\theta_n^{m_n} \right| \;+\; \ln|x_n^{m_n}| \right] d\theta \;, \tag{24}$$

where the first term in the right-hand side and the function $g(\theta)\ln b$ are bounded and the integral

$$\int\limits_{|(x_n\theta_n)^{m_n}|\,<\,1/b} g(\theta)\, \ln\left| \frac{x_n^{m_n}}{|x_n^{m_n}|}\theta_n^{m_n} \right| d\theta$$

is uniformly bounded, because the integral

$$\int\limits_{|\theta|_\mu=1} \ln\left| \frac{x_n^{m_n}}{|x_n^{m_n}|} \cdot \theta_n^{m_n} \right| d\theta$$

is finite and does not depend on x.

Finally the expression

$$\ln|x_n^{m_n}| \int\limits_{\left|\frac{x_n^{m_n}}{|x_n^{m_n}|}\theta_n^{m_n}\right| \,<\, \frac{1}{b\,|x_n^{m_n}|}} g(\theta)\; d\theta \;=\; \ln|x_n^{m_n}| \;\cdot\; O\left(\frac{1}{|x_n^{m_n}|}\right)$$

is also uniformly bounded.

Therefore formulas (14)–(24) imply inequality (11) for $(\mu, \beta) = |\mu| - d_l + (\mu, \alpha)$, which together with inequality (9) completes the proof of the theorem. □

Remark 1. *Here is an example of differential operator satisfying the conditions of Theorem 1:*

$$P(D) = D_1^8 + D_2^4 + D_3^4 + D_1^4 - D_2^2 - D_3^2.$$

The following theorem can be proved similarly.

Theorem 2. *Let* $P(D) = \sum\limits_{(\mu,\alpha)\leq 2} \gamma_\alpha D^\alpha$ *be a semi-elliptic operator with constant coefficients.*

Then, for any tempered fundamental solution E *of* $P(D)$*, the distributional derivatives* $D^\alpha E$ *are locally integrable functions in* $\mathbb{R}^n$ *if* $(\mu, \alpha) < 2$ *and the estimate*

$$D^\alpha E(x) = O\left(\frac{1}{|x|_\mu^{|\mu|-2+(\mu,\alpha)}}\right), \quad |x|_\mu \to 0$$

holds.

Remark 2. *It is known (see e.g. [2]) that on* $\mathbb{R}^n \backslash \{0\}$ E *is an infinitely continuously differentiable function in the classical sense.*

3 Equations with constant coefficients

For any domain $\Omega \subset \mathbb{R}^n$ denote by $H^{\bar{m}}(\Omega)$ the anisotropic Sobolev space of functions with the finite norm

$$\| u \|_{\bar{m},\Omega} = \| u \|_\Omega + \sum_{i=1}^n \| D_i^{m_i} u \|_\Omega , \tag{25}$$

where $\| \cdot \|_\Omega$ is the norm of the space $L_2(\Omega)$. The closure of the set $C_0^\infty(\Omega)$ in norm (25) we denote by $\overset{\circ}{H}{}^{\bar{m}}(\Omega)$.

Let

$$B_{\sigma,\mu} = \Big\{ x \in \mathbb{R}^n ; \quad |x|_\mu < \sigma \Big\}, \; S_{\sigma,\mu} = \Big\{ x \in \mathbb{R}^n ; \; |x|_\mu = \sigma \Big\},$$

$$L_{2,a,\mu}(\mathbb{R}^n) = \Big\{ f \in L_2(\mathbb{R}^n); \quad f(x) = 0 \;\text{ if, }\; |x|_\mu > a \Big\}.$$

Lemma 1. *Suppose that the operator* $P(D)$ *satisfies the conditions of Theorem 2 and* $f \in L_{2,a,\mu}(\mathbb{R}^n)$.

Then the equation

$$P(D)\, u = f \quad in\ \mathbb{R}^n \tag{26}$$

has a unique solution $u \in H^{2\bar{m}}(\mathbb{R}^n)$*, tending to zero at infinity. Moreover*

$$u(x) = O\left(\frac{1}{|x|_\mu^{|\mu|-d_l}}\right).$$

This solution u can be representedas $u = E * f$, where E is the fundamental solution in Theorem 1 and for any compact $K \subset \mathbb{R}^n$ the inequality

$$\| u \|_{2\bar{m},K} \;\le\; c(K)\; \| f \|_{L_{2,a,\mu}(\mathbb{R}^n)} \tag{27}$$

holds, where $c(K) > 0$ is independent of f.

Proof. We have to prove only estimate (27), since the other statements of Lemma 1 follow by Theorem 1. Let $(\mu, \alpha) < 2$. In virtue of Theorem 2 $D^\alpha E$ is locally integrable in $\mathbb{R}^n$, so

$$D^\alpha u(x) = \int\limits_{B_{a,\mu}} f(y) D^\alpha E(x-y)dy.$$

Thus by Young's inequality

$$|\, D^\alpha u(x)\,|^2 \;\le\; \int\limits_{B_{a,\mu}} |f(y)|^2\; |D^\alpha E(x-y)|\, dy \;\;\cdot \int\limits_{B_{a,\mu}} |D^\alpha E(x-y)|\, dy.$$

For any $b > 0$ we have

$$\int\limits_{B_{b,\mu}} |\, D^\alpha u(x)\,|^2\; dx \;\le c_1 \int\limits_{B_{a,\mu}} \left(|f(y)|^2 \int\limits_{B_{b,\mu}} |D^\alpha E(x-y)|\; dx \right) dy \;\le$$

$$\le\; c_2 \int\limits_{B_{a,\mu}} |f(y)|^2\, dy\,,$$

where c_1 and $c_2 = c_2(b)$ are positive constants. Therefore, for $(\mu, \alpha) < 2$ and any $b > 0$,

$$\|D^\alpha u\|_{B_{b,\mu}} \;\le\; c(b)\; \|f\|_{L_{2,a,\mu}(\mathbb{R}^n)}\,. \tag{28}$$

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 in a neighbourhood of compact K, $\mathrm{supp}\psi \subset B_{b,\mu}$. Then

$$\begin{aligned} \|u\|_{2\bar{m},K} \;&\le\; \|\psi\, u\|_{2\bar{m},B_{b,\mu}} \;\le\; c_2(b)\,[\,\|P(D)(\psi\, u)\|_{B_{b,\mu}} \;+\; \|\psi\, u\|_{B_{b,\mu}}\,] \;\le \\ &\le\; c_3\,[\,\|f\|_{L_{2,a,\mu}(\mathbb{R}^n)} + \|\, u\,\|'_{2\bar{m},B_{b,\mu}}\;]\,, \end{aligned} \tag{29}$$

where $\|u\|'_{2\bar{m},B_{b,\mu}} = \sum\limits_{(\mu,\alpha)<2} \|D^\alpha u\|_{B_{b,\mu}}$. From (28) and (29) we get inequality (27). □

For $|\mu| > 2d_l$ another estimate can be proved for the solution of equation (26).

Lemma 2. *Suppose that the operator $P(D)$ satisfies the conditions of Theorem 1, $|\mu| > 2d_l$ and $f \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$.*

Then equation (26) has a unique solution $u \in H^{2\bar{m}}(\mathbb{R}^n)$ and the following estimate is valid

$$\|\, u\,\|_{2\bar{m},\mathbb{R}^n} \le c \left[\, \|f\|_{L_1(\mathbb{R}^n)} + \|f\|_{L_2(\mathbb{R}^n)} \right], \tag{30}$$

where $c > 0$ is independent of f.

Proof. Since $u = E * f = F^{-1}[\frac{1}{P}F[f]]$ is a solution of equation (26), by Lemma 1 we have only to prove that $u \in H^{2\bar{m}}(\mathbb{R}^n)$ and estimate (30) holds.

Parceval's equality and the equivalence

$$\sum_{(\mu,\alpha)\leq 2} |\,\xi^{\alpha}| \quad \sim \quad 1 + |\xi|_{\mu}^{2}$$

imply that

$$\|\, u \,\|_{2\,\bar{m},\mathbb{R}^n} \;\sim\; \left\| (1 + |\xi|_{\mu}^{2})\, F[u](\xi) \right\|_{\mathbb{R}^n}. \tag{31}$$

Formula (7) and the semi-ellipticity of $P(D)$ imply that

$$\frac{1 + |\xi|_{\mu}^{2}}{|P(\xi)|^{2}}\, |F\,[f]\,|^{2} \;\leq c_1\, |F\,[f]\,|^{2} \quad if \quad |\xi|_{\mu} > 1$$

and

$$\frac{(1 + |\xi|_{\mu})^{2}}{|P(\xi)|^{2}}\, |F[f]|^{2} \;\leq\; \frac{c_2}{|\xi|_{\mu}^{2d_l}}\, |F[f]|^{2} \;\leq\; \frac{c_2}{|\xi|_{\mu}^{2d_l}}\, |\sup(F[f])|^{2} \;\leq\; \frac{c_3}{|\xi|_{\mu}^{2d_l}}\, \|f\|_{L_1(\mathbb{R}^n)}^{2}$$

if $|\xi|_{\mu} < 1$. Since $|\mu| > 2d_l$, estimate (30) follows by these inequalities and (31). □

For any $\sigma > 0$ consider the following boundary value problem of variational type

$$P(D)\, u_{\sigma} = f \quad \text{in} \quad B_{\sigma,\mu}\,, \tag{32}$$

$$u_{\sigma} \in \overset{\circ}{H}{}^{\bar{m}}(B_{\sigma,\mu}). \tag{33}$$

Theorem 3. *Suppose that the operator $P(D)$ satisfies the conditions of Theorem 1 and the polynomial $P(\xi)$ has the form*

$$P(\xi) = \sum_{\substack{(\mu,\alpha)\leq 1\\(\mu,\beta)\leq 1}} \gamma_{\alpha,\beta}\xi^{\alpha+\beta},$$

where $\gamma_{\alpha,\beta} \in \mathbb{R}$, and for any sequence of complex numbers $(\zeta_0, ..., \zeta_{\alpha}, ...) \neq 0$ the following inequalities

$$\sum_{\substack{(\mu,\alpha)=1\\(\mu,\beta)=1}} \gamma_{\alpha,\beta}\, \zeta_{\alpha}\overline{\zeta_{\beta}} > 0, \qquad \sum_{\substack{(\mu,\alpha)<1\\(\mu,\beta)<1}} \gamma_{\alpha,\beta}\, \zeta_{\alpha}\overline{\zeta_{\beta}} \geq 0 \tag{34}$$

hold.

Then for arbitrary $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $\sigma > 0$ problem (32)–(33) has a unique solution $u_{\sigma} \in \overset{2\bar{m}}{H}(B_{\sigma,\mu})$.

Moreover, if $|\mu| > 2d_l$ and $2\mu_0 > \mu_i\,(i = 1, ..., n)$, then for every fixed compact $K \subset \mathbb{R}^n$ and $(\mu,\tau) \leq 2$

$$\sup_{K}\, |\, D^{\tau}\, u \;-\; D^{\tau}\, u_{\sigma}\, | \;\leq\; \frac{c(K)}{\sigma^{\frac{|\mu|}{2}-2-d_l+(\mu,\tau)}} \;\left(\|f\|_{L_1(\mathbb{R}^n)} \;+\; \|f\|_{L_2(\mathbb{R}^n)} \right). \tag{35}$$

where $c(K) > 0$ is independent of f and σ.

Proof. The function $u_\sigma \in H^{2\bar{m}}(B_{\sigma,\mu})$ is a solution of problem (32)–(33) if and only if function $\tilde{u}_\sigma$, defined by $\tilde{u}_\sigma(y) = u_\sigma(y\,\sigma^\mu)$ satisfies the problem

$$\sum_{\substack{(\mu,\alpha)\le 1\\(\mu,\beta)\le 1}} \frac{\gamma_{\alpha,\beta}}{\sigma^{(\mu,\alpha)+(\mu,\beta)}}\, D^{\alpha+\beta}\, \tilde{u}_\sigma \;=\; \tilde{f}_\sigma \quad \text{in} \quad B_{1,\mu}, \tag{36}$$

$$u_\sigma \in \overset{\circ}{H}{}^{\bar{m}}(B_{1,\mu}) \tag{37}$$

where $\tilde{f}_\sigma(y) \;=\; f(y\,\sigma^\mu)$. Conditions (34) imply the uniqueness of the solution of (36)–(37), because multiplying equation (36) by $\overline{\tilde{u}}$ we get the estimate

$$\left\|\tilde{f}_\sigma\right\|_{B_{1,\mu}} \|\tilde{u}_\sigma\|_{B_{1,\mu}} \;\ge\; \int\limits_{B_{1,\mu}} \tilde{f}_\sigma \overline{\tilde{u}}_\sigma \;=\; \sum_{\substack{(\mu,\alpha)\le 1\\(\mu,\beta)\le 1}} \gamma_{\alpha,\beta} \int\limits_{B_{1,\mu}} \frac{D^\alpha \tilde{u}_\sigma}{\sigma^{(\mu,\alpha)}}\,\overline{(\frac{D^\beta \tilde{u}_\sigma}{\sigma^{(\mu,\beta)}})} \;\ge$$

$$\ge\; \frac{1}{\sigma^2} \int\limits_{B_{1,\mu}} \sum_{\substack{(\mu,\alpha)=1\\(\mu,\beta)=1}} \gamma_{\alpha,\beta}(D^\alpha \tilde{u}_\sigma)(\overline{D^\beta \tilde{u}_\sigma}) \;\ge\; \frac{c_1}{\sigma^2} \int\limits_{B_{1,\mu}} \sum_{(\mu,\alpha)=1} (D^\alpha \tilde{u}_\sigma)^2 \;\ge$$

$$\ge\; \frac{c_2}{\sigma^2}\, \|\tilde{u}_\sigma\|^2_{\bar{m},B_{1,\mu}}\,, \tag{38}$$

where c_1 and c_2 are positive constants.

Therefore for some constant $c_3 > 0$

$$\|\,\tilde{u}_\sigma\,\|_{B_{1,\mu}} \;\le\; c_3\sigma^2 \left\|\tilde{f}_\sigma\right\|_{B_{1,\mu}}. \tag{39}$$

Thus for any $\tilde{f}_\sigma \in L_2(B_{1,\mu})$ there exists a unique function $\tilde{u}_\sigma \in \overset{\circ}{H}{}^{\bar{m}}(B_{1,\mu})$ such that

$$\sum_{\substack{(\mu,\alpha)\le 1\\(\mu,\beta)\le 1}} \frac{\gamma_{\alpha,\beta}}{\sigma^{(\mu,\alpha)+(\mu,\beta)}} \int\limits_{B_{1,\mu}} (D^\alpha \tilde{u}_\sigma)(\,\overline{D^\beta v}\,) = \int\limits_{B_{1,\mu}} \tilde{f}_\sigma \bar{v},$$

for all $v \in \overset{\circ}{H}{}^{\bar{m}}(B_{1,\mu})$, which means that we proved the existence of the solution $\tilde{u}_\sigma \in H^{2\bar{m}}(B_{1,\mu})$ of problem (36)–(37).

Equation (36) can be written in the form

$$\sum_{\substack{(\mu,\alpha)\le 1\\(\mu,\beta)\le 1}} \gamma_{\alpha,\beta}\, \sigma^{2-(\mu,\alpha)-(\mu,\beta)}\, D^{\alpha+\beta}\, \tilde{u}_\sigma \,+\, \sigma^2\, \tilde{u}_\sigma = \sigma^2\, \tilde{f}_\sigma \;+\; \sigma^2\, \tilde{u}_\sigma. \tag{40}$$

For the characteristic polynomial of the differential operator in (40)

$$P(\xi) \,+\, \sigma^2 \,>0\,, \quad \text{if } \xi \in R^n\,,$$

hence in virtue of the results of [4] we get the estimate

$$\left\{ \sum_{k=0}^{2m_0} \sigma^{2k\,m_0} \left(\sum_{(\mu,\alpha)\le 2-\,k\mu_0} \|\, D^\alpha\, \tilde{u}_\sigma \,\|^2_{B_{1,\mu}} \right) \right\}^{1/2} \;\le\; c\,\sigma^2 \left[\left\|\tilde{f}_\sigma\right\|_{B_{1,\mu}} \;+\; \|\,\tilde{u}_\sigma\|_{B_{1,\mu}} \right].$$

where c is a positive constant. Inequality (39) implies that

$$\sum_{(\mu,\alpha)\leq 2-k\mu_0} \| D^\alpha \tilde{u}_\sigma \|_{B_{1,\mu}} \leq c\, \sigma^{4-k\mu_0} \left\| \tilde{f}_\sigma \right\|_{B_{1,\mu}}$$

and so if $(\mu,\alpha) \leq 2-\mu_0$

$$\|D^\alpha u_\sigma\|_{L_2(S_{\sigma,\mu})} \leq \sigma^{\frac{|\mu|-\mu_0}{2} - (\mu,\alpha)} \| D^\alpha \tilde{u}_\sigma\|_{L_2(S_{1,\mu})} \leq$$

$$\leq c_4\, \sigma^{\frac{|\mu|-\mu_0}{2} - (\mu,\alpha)} \sigma^{2+(\mu,\alpha)+\mu_0} \left\| \tilde{f}_\sigma \right\|_{B_{1,\mu}} \leq c_5\, \sigma^{\frac{\mu_0}{2}+2} \| f \|_{B_{\sigma,\mu}}. \qquad (41)$$

where c_4, c_5 are positive constants.

Let $x_0 \in B_{\sigma,\mu}$. Since $2\mu_0 > \mu_i \, (i = 1, ..., n)$ by applying Green's formula from [4] for the difference of the solutions $\vartheta_\sigma = u - u_\sigma$ we get

$$|\vartheta_\sigma(x_0)| = \left| \sum_{\mu_0 \leq (\mu,\alpha)+(\mu,\beta) \leq 2-\mu_0} \int_{S_{\sigma,\mu}} g_{\alpha,\beta}(\frac{x}{|x|_\mu}) D^\alpha \vartheta_\sigma(x) D^\beta E(x_0 - x)\, dS \right| \leq$$

$$\leq \sum_{\mu_0 \leq (\mu,\alpha)+(\mu,\beta) \leq 2-\mu_0} \int_{S_{\sigma,\mu}} \left| g_{\alpha,\beta}(\frac{x}{|x|_\mu}) \right| |D^\alpha u(x)| \left|D^\beta E(x_0 - x)\right| dS +$$

$$+ \sum_{\mu_0 \leq (\mu,\alpha)+(\mu,\beta) \leq 2-\mu_0} \int_{S_{\sigma,\mu}} \left| g_{\alpha,\beta}(\frac{x}{|x|_\mu}) \right| |D^\alpha u_\sigma(x)| \left|D^\beta E(x_0 - x)\right| dS \leq$$

$$\leq c_6 \sum_{\mu_0 \leq (\mu,\alpha)+(\mu,\beta) \leq 2-\mu_0} \left[\left(\|D^\alpha u\|_{L_2(S_{\sigma,\mu})} + \|D^\alpha u_\sigma\|_{L_2(S_{\sigma,\mu})} \right) \left\|D^\beta E(x_0 - x)\right\|_{L_2(S_{\sigma,\mu})} \right], \qquad (42)$$

where c_6 does not depend on x_0, since the coefficients $g_{\alpha,\beta}$ are bounded functions.

Let $K \subset \mathbb{R}^n$ be a fixed compact, $x_0 \in K$. Then by Theorem 1 for $(\mu,\beta) \leq 2-\mu_0$

$$D^\beta E(x_0 - x) = O\left(\frac{1}{|x|_\mu^{|\mu|-d_l+(\mu,\beta)}} \right),$$

hence

$$\left\|D^\beta E(x_0 - x)\right\|_{S_{\sigma,\mu}} = O\left(\frac{1}{\sigma^{\frac{|\mu|+\mu_0}{2} - d_l + (\mu,\beta)}} \right). \qquad (43)$$

Moreover, Lemma 2 implies that for $(\mu,\alpha) \leq 2-\mu_0$

$$\|D^\alpha u\|_{L_2(S_{\sigma,\mu})} \leq c_7 \sigma^{\frac{\mu_0}{2}} \|u\|_{2\bar{m},B_{1,\mu}} \leq c_7 \sigma^{\frac{\mu_0}{2}} \| u \|_{2\bar{m},\mathbb{R}^n} \leq$$

$$\leq c_8 \sigma^{\frac{\mu_0}{2}} \left[\|f\|_{L_1(\mathbb{R}^n)} + \|f\|_{L_2(\mathbb{R}^n)} \right], \qquad (44)$$

where $c_8 > 0$ is independent of f.

By estimates (40)–(44) we get inequality (35) in the case $\tau = 0$. Similarly inequality (35) can be proved for arbitrary multi-index τ satisfying $(\mu,\tau) \leq 2$. $\square$

4 Equations with variable coefficients

In this paragraph the equation of the form

$$A(x, D)u = f, \tag{45}$$

will be considered, where $A(x, D) \equiv P + \lambda Q$, $P = P(D)$ is the operator in Theorem 1, $Q = Q(x, D)$ is a linear diifferential operator of μ - order not higher than 2 with infinitely continuously differentiable coefficients, vanishing for $|x|_\mu \geq a$, $f \in L_{2,a,\mu}(\mathbb{R}^n)$, and λ is a complex number.

It is clear, that the operator $P + \lambda Q$ is semi-elliptic when $|\lambda|$ is sufficiently small. Denote by Λ the set of all λ such that the operator $P+\lambda Q$ is semi-elliptic. Obviously, Λ is an open set. Denote by Λ_0 the set of all connected components of Λ, which contain the point $\lambda = 0$. It is easy to see that if μ-order of $Q(x, D)$ is less than 2, then Λ_0 is the whole complex plane.

Denote by $P^{-1}\omega$ the (unique) solution of the equation $Pu = \omega$, which vanishes at infinity, i.e. $u = E * \omega$, where E is the fundamental solution in Theorem 1.

Using the method of [4] we get the following lemma.

Lemma 3. *Let the operator P satisfy the conditions of Theorem 1, and $Q(x, D)$ satisfy the above conditions.*

Then for any $\lambda \in \Lambda_0$ a function u is a solution of (45), vanishing at infinity, if and only if $\omega = Pu$ is a solution of the equation

$$\omega + \lambda\, Q\, P^{-1}\omega = f \tag{46}$$

in $L_{2,a,\mu}(\mathbb{R}^n)$.

Consider next the following problem

$$A(x, D)\, u_\sigma = f \quad \text{in} \quad B_{\sigma,\mu}, \tag{47}$$

$$u_\sigma \in \overset{\circ}{H}{}^{\bar{m}}(B_{\sigma,\mu}). \tag{48}$$

Theorem 4. *Suppose that the conditions of Theorem 1 and Lemma 3 are satisfied and $\lambda \in \Lambda_0$ is a fixed number.*

Then there exists $\sigma_0 > 0$ such that for all $\sigma \geq \sigma_0$ and $f \in L_{2,a,\mu}(\mathbb{R}^n)$ problem (47)–(48) has a unique solution $u_\sigma \in H^{2\bar{m}}(B_{\sigma,\mu})$ and if $|\mu| > 2d_l$, $m_0 < 2\, m_i\, (i = 1, ..., n)$, then for any compact $K \subset \mathbb{R}^n$

$$\| u - u_\sigma \|_{2\bar{m},K} \leq \frac{c_1(K)}{\sigma^{\frac{|\mu|}{2} - 2 - d_l}} \, \|f\|_{L_{2,a,\mu}(\mathbb{R}^n)}\,, \tag{49}$$

where $c_1(K) > 0$ is independent of f and σ.

Proof. In virtue of Theorem 3 for any $f \in L_{2,\,a,\mu}(\mathbb{R}^n)$ the problem

$$P\, v_\sigma = f \text{ in } B_{\sigma,\mu}, \tag{50}$$

$$v_\sigma \in \overset{\circ}{H}{}^{\bar{m}}(B_{\sigma,\mu}). \tag{51}$$

has a unique solution $v_\sigma \in H^{2\bar{m}}(B_{\sigma,\mu})$. Denote the solution of (50)–(51) by $P_\sigma^{-1}f$. Then $P_\sigma^{-1} : L_{2,a,\mu}(\mathbb{R}^n) \to H^{2\bar{m}}(B_{\sigma,\mu})$ is a bounded linear operator and by estimate (35) it follows that for any compact $K \subset \mathbb{R}^n$

$$\left\|(P_\sigma^{-1} - P^{-1})f\right\|_{2\bar{m},K} \le \frac{c_2}{\sigma^{\frac{|\mu|}{2}-2-d_l}} \|f\|_{L_{2,a,\mu}(\mathbb{R}^n)}, \tag{52}$$

where $c_2 > 0$ is independent of f and σ.

Consequently, the difference of the operators

$$G = I + \lambda_0 Q P^{-1}, \quad G_\sigma = I + \lambda_0 Q P_\sigma^{-1}$$

mapping $L_{2,a,\mu}(\mathbb{R}^n)$ into itself (I – denotes the identity operator in $L_{2,a,\mu}(\mathbb{R}^n)$), can be estimated as follows

$$\| G_\sigma - G \| \le \left\| \lambda_0 Q(P_\sigma^{-1} - P^{-1}) \right\| \le \frac{c_3}{\sigma^{\frac{|\mu|}{2}-2-d_l}}. \tag{53}$$

By Lemma 3 it follows that the inverse of G exists and $G^{-1} : L_{2,a,\mu}(\mathbb{R}^n) \to L_{2,a,\mu}(\mathbb{R}^n)$ is a bounded linear operator. Therefore estimate (53) implies that for sufficiently large σ also the inverse G_σ^{-1} exists and $G_\sigma^{-1} : L_{2,a,\mu}(\mathbb{R}^n) \to L_{2,a,\mu}(\mathbb{R}^n)$ is a bounded linear operator. Since the equality $u_\sigma = P_\sigma^{-1}\omega_\sigma$ defines an one-to-one mapping between the solutions $u_\sigma \in H^{2\bar{m}}(B_{\sigma,\mu})$ of (47)–(48) and the solutions $\omega_\sigma \in L_{2,a,\mu}(\mathbb{R}^n)$ of the equation $G_\sigma\omega_\sigma = f$, for sufficiently large σ and arbitrary $f \in L_{2,a,\mu}(\mathbb{R}^n)$ problem (47)–(48) has a unique solution.

The difference of the solutions $u_\sigma = P_\sigma^{-1}G_\sigma^{-1}f$ and $u = P^{-1}G^{-1}f$ can be estimated as follows. By (53) the number $\sigma_0 > 0$ can be chosen such that for $\sigma > \sigma_0$

$$\| G_\sigma - G \| \le \frac{1}{2\,\|G^{-1}\|} \text{ and so } \| G^{-1}(G_\sigma - G) \| \le \frac{1}{2}.$$

Hence

$$\|G_\sigma^{-1}\| - \left\| \left\{ G \left[I + G^{-1}(G_\sigma - G)\right] \right\}^{-1} \right\| =$$
$$= \left\| \left[I + G^{-1}(G_\sigma - G)\right]^{-1} G^{-1} \right\| \le 2 \, \|G^{-1}\|$$

and

$$\| G_\sigma^{-1} - G^{-1} \| = \| G_\sigma^{-1}(G - G_\sigma)G^{-1}\| \le 2 \, \|G^{-1}\|^2 \, \| G_\sigma - G \|.$$

This estimate and (53) imply that

$$\left\| G_\sigma^{-1} - G^{-1} \right\| \le \frac{c_4}{\sigma^{\frac{|\mu|}{2}-2-d_l}},$$

where $c_4 > 0$ is independent of σ.

Therefore by (52) we get (49). Indeed

$$\|u - u_\sigma\|_{2\bar{m},K} = \left\|\left[\left(P_\sigma^{-1} - P^{-1}\right) G_\sigma^{-1} + P^{-1}\left(G_\sigma^{-1} - G^{-1}\right)\right] f\right\|_{2\bar{m},K} \le$$
$$\le \frac{c_2}{\sigma^{\frac{|\mu|}{2}-2-d_l}} \left\| G_\sigma^{-1} f \right\|_{L_{2,a,\mu}(\mathbb{R}^n)} + c_5 \left\|\left(G_\sigma^{-1} - G^{-1}\right) f\right\|_{L_{2,a,\mu}(\mathbb{R}^n)} \le$$
$$\le \frac{c_6}{\sigma^{\frac{|\mu|}{2}-2-d_l}} \|f\|_{L_{2,a,\mu}(\mathbb{R}^n)},$$

where $c_6 > 0$ is independent of f and σ. □

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