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APPLICATIONS OF λ-TRUNCATIONS TO THE STUDY OF LOCAL AND GLOBAL SOLVABILITY OF NONLINEAR EQUATIONS

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Abstract. In this paper, we consider the equation $F(x) = y$ in a neighbourhood of a given point $\bar{x}$, where F is a given continuous mapping between finite-dimensional real spaces. We study a class of polynomial mappings and show that these polynomials satisfy certain regularity assumptions. We show that if a λ-truncation of F at $\bar{x}$ belongs to the considered class of polynomial mappings then for every y close to $F(\bar{x})$ there exists a solution to the equation $F(x) = y$ that is close to $\bar{x}$. For polynomial mappings satisfying the regularity conditions we study their stability to bounded continuous perturbations.

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1 Introduction

Let n and m be positive integers, $\bar{x} \in \mathbb{R}^n$ be a given point, $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping continuously differentiable in a neighbourhood of $\bar{x}$.

Numerous classical results of analysis allow to study properties of the mapping F when the point $\bar{x}$ is normal, i.e. the first derivative $F'(\bar{x}) : \mathbb{R}^n \to \mathbb{R}^m$ of the mapping F at the point $\bar{x}$ is a surjective linear operator. For instance, the classical inverse function theorem (see, for example, [4, Theorem 1F.6]) states that the equation

$$F(x) = y \tag{1.1}$$

with the unknown $x \in \mathbb{R}^n$ and the parameter $y \in \mathbb{R}^m$ has a solution $x(y)$ for y close to $F(\bar{x})$ such that $x(y) \to \bar{x}$ as $y \to F(\bar{x})$.

If the point $\bar{x}$ is abnormal, i.e. the linear operator $F'(\bar{x})$ is not surjective, then the study of the behavior of the mapping F in a neighbourhood of the point $\bar{x}$ becomes significantly more complicated. In this case, this problem is studied under certain conditions of non-degeneracy, formulated in terms of the first two derivatives $F'(\bar{x})$ and $F''(\bar{x})$ of the mapping F at the point $\bar{x}$, and under very general and natural assumptions. A detailed overview of the relevant results is given in [1]. The need for the investigation of equation (1.1) in the abnormal case is partly dictated by optimization problems with equality-type constraints that degenerate in one sense or another. For example, some topics related to numerical methods for investigation of optimization problems with abnormality were studied in [6]. Theoretical problems concerning degenerating constraints were discussed and studied in [1].

Another approach, meaningful also in the abnormal case, was proposed in [2]. Unlike the results using the first and second derivatives of the mapping F at the point $\bar{x}$, in [2], there were obtained solvability conditions that use the derivatives of higher orders. The corresponding results are applicable to equation (1.1) when the mentioned conditions of the nondegeneracy in terms of the first two derivatives $F'(\bar{x})$ and $F''(\bar{x})$ are violated. Note that in this case, the methods from [1] are fundamentally inaplicable.

The present work is a natural continuation of the research begun in [2]. In Section 1 we recall the definition of λ-truncation and the inverse function theorem from [2]. In Section 2 we give an example of a wide class of λ-truncations that have the regularity property. In Section 3 we study the stability of the surjectivity property of regular λ-truncations under continuous bounded perturbations.

2 Preliminaries

Below $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^n$ and $\mathbb{R}^m$ and $|\cdot|$ stands for the corresponding Euclidean norm in these spaces.

Let D be the set of all non-zero n-dimensional vectors $d = (d_1, ..., d_n)$ with the non-negative components, $\widehat{D} \subset D$ be the subset of all vectors with integer components.

For $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $s = (s_1, ..., s_n) \in \widehat{D}$ and $d = (d_1, ..., d_n) \in D$, we set

$$x^s := \prod_{j=1}^{n} x_j^{s_j}, \quad |x|^d := \prod_{j=1}^{n} |x_j|^{d_j}.$$

Here and below we assume that $x_j^0 = |x_j|^0 = 1$. The vector s is called the multi-index of the monomial x^s.

Given a vector $\lambda = (\lambda_1, ..., \lambda_n) \in D$ and nonempty finite sets $S_i \subset \widehat{D}$, $i \in \{1, ..., m\}$, assume that

$$\exists\, \alpha_i > 0 : \quad \langle \lambda, s \rangle = \alpha_i \quad \forall\, s \in S_i. \tag{2.1}$$

Given a collection of real numbers $p_{i,s} \neq 0$, $s \in S_i$, $i \in \{1, ..., m\}$, define the mapping $P = (P_1, ..., P_m) : \mathbb{R}^n \to \mathbb{R}^m$ by formula

$$P_i(x) = \sum_{s \in S_i} p_{i,s} x^s, \quad x = (x_1, ..., x_n) \in \mathbb{R}^n.$$

Note that the polynomials P_i have the property of quasihomogeneity, i.e.

$$P_i(t^{\lambda_1} x_1, ..., t^{\lambda_n} x_n) \equiv t^{\alpha_i} P_i(x), \quad x \in \mathbb{R}^n, \quad t > 0.$$

Moreover, since S_i are nonempty and $p_{i,s} \neq 0$ for all $s \in S_i$, all functions P_i are nonzero polynomials. Moreover, it is obvious that $P(0) = 0$.

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping that is continuous in a neighbourhood of the given point $\bar{x}$. The mapping P is said to be a *λ-truncation of the mapping F at the point $\bar{x}$*, if there exist nonempty finite sets $D_i \subset D$, $i \in \{1, ..., m\}$ such that the following properties are satisfied. Firstly, the strict inequalities

$$\langle \lambda, d \rangle > \alpha_i \quad \forall\, d \in D_i, \quad \forall\, i \in \{1, ..., m\}$$

hold. Secondly, the representation

$$F(x) \equiv F(\bar{x}) + P(x - \bar{x}) + \Delta(x - \bar{x}),$$

is valid, in which for the mapping $\Delta = (\Delta_1, ..., \Delta_m) : \mathbb{R}^n \to \mathbb{R}^m$ there exists const ≥ 0 such that

$$|\Delta_i(x - \bar{x})| \leq \text{const} \sum_{d \in D_i} |x - \bar{x}|^d \quad \forall\, i \in \{1, ..., m\}.$$

for every x from a neighbourhood of the point $\bar{x}$.

For a vector $h \in \mathbb{R}^n$ we say that the λ-truncation P is regular in the direction h, if

$$P(h) = 0, \quad P'(h)\mathbb{R}^n = \mathbb{R}^m, \tag{2.2}$$

and $h_j = 0$ for any j such that $\lambda_j = 0$.

In [2, Theorem 1], the following inverse function theorem is obtained. *Let $\lambda > 0$, P be a λ-truncation of the mapping F at $\bar{x}$ and P be regular in a direction $h \in \mathbb{R}^n$. Then there exists a neighbourhood O of the point $F(\bar{x})$ and a number* const > 0 *such that for every $y \in O$ there exists a solution $x(y)$ to equation* (1.1) *such that*

$$|x(y) - \bar{x}| \le \mathrm{const}|y - F(\bar{x})|^{\theta} \quad \forall\, y \in O. \tag{2.3}$$

Here

$$\theta = \left(\max_{i=\overline{1,m}} \alpha_i\right)^{-1} \min_{i=\overline{1,m}} \lambda_i. \tag{2.4}$$

In connection with the above introduced concept of the regularity in a direction h, the following question arises. Is it essential to assume that $h_j = 0$ for any j such that $\lambda_j = 0$, can the assumption $\lambda > 0$ in [2, Theorem 1] be replaced by the assumption $\lambda \ge 0$, $\lambda \ne 0$, and $h_j = 0$ for any j such that $\lambda_j = 0$? The following example gives a negative answer to this question.

Example 1. Let $n = 2$, $m = 1$, $\bar{x} = 0$, $F(x) = P(x) = x_1^2 + x_1^2 x_2$. We have

$$S_1 = \{(2,0),\ (2,1)\}.$$

Put $\lambda := (1,0)$ and $h := (1,1,1)$. Then (2.1) holds for $\alpha_1 = 2$ and equalities (2.2) take place. At the same time, for a solution (x_1, x_2) to the equation

$$P(x) = y$$

with the unknown $x \in \mathbb{R}^2$ we have the following. If $y < 0$, then we have $x_1^2 + x_1^2 x_2 < 0$. Therefore, $x_2 < -1$. So, in this example there exists no solution $x(y)$ to the equation $F(x) = y$ such that $x(y) \to \bar{x} = 0$ as $y \to F(\bar{x}) = 0$.

This example shows the essentiality of the assumption that $h_j = 0$ for any j such that $\lambda_j = 0$. It also shows that the assumption $\lambda > 0$ of [2, Theorem 1] is essential.

To apply [2, Theorem 1] to a continuous mapping F, we must first find an appropriate vector λ and a polynomial mapping P (if they exist) such that P is a λ-truncation of F in a neighbourhood of the point $\bar{x}$, and then verify that P is regular in some direction $h \in \mathbb{R}^n$.

3 On one type of regular λ-truncations

First, we present a class of mappings $F : \mathbb{R}^n \to \mathbb{R}^m$, for which the assumptions of [2, Theorem 1] are satisfied.

Let n_i be positive integers such that $n = n_1 + ... + n_m + 1$, Q_i be given nonzero symmetric $n_i \times n_i$-matrices, b_i be given nonzero real numbers, $i \in \{1, ..., m\}$.

Consider m functions $P_i : \mathbb{R}^n \to \mathbb{R}$ defined by the formula

$$P_i(x) = \langle Q_i \chi_i, \chi_i \rangle + b_{i+1}(\chi_{i+1}^1)^3, \quad i \in \{1, ..., m\}. \tag{3.1}$$

Here $x = (\chi_1, ..., \chi_m, \chi_{m+1}^1) \in \mathbb{R}^n$, where $\chi_1 = (x_1, ..., x_{n_1}) \in \mathbb{R}^{n_1}$, $\chi_2 = (x_{n_1+1}, ..., x_{n_1+n_2}) \in \mathbb{R}^{n_2}$, etc.; χ_{i+1}^1 is the first component of the vector χ_{i+1}, $i \in \{1, ..., m-1\}$, i.e. $\chi_2^1 = x_{n_1+1}$, $\chi_3^1 = x_{n_1+n_2+1}$, etc.;

and $\chi^1_{m+1} = x_n$ is a real number, which for definiteness we will consider as the first coordinate of the one-dimensional vector χ_{m+1}.

So, for every $i \in \{1, ..., m\}$, the function P_i is the sum of two terms. The first one is a non-zero quadratic form in the variable χ_i, which is defined by the square $n_i \times n_i$-matrix Q_i. The second term is the non-zero cubic form $b_{i+1}(\chi^1_{i+1})^3$ in the variable χ_{i+1} with a given non-zero coefficient b_{i+1}.

Define an n-dimensional vector $\lambda > 0$ as follows. First, we put $\lambda_n = \widehat{\lambda}_{m+1} = \frac{1}{3}$. Then take $\widehat{\lambda}_i = \frac{1}{3}(\frac{3}{2})^{(m-i+1)}$, $i \in \{1, ..., m\}$. We define the remaining coordinates $\lambda_1, ..., \lambda_{n-1}$ of the vector λ so that in the places corresponding to the vector χ_i, all coordinates of the vector λ are equal to $\widehat{\lambda}_i$, $i \in \{1, ..., m\}$. As a result, we have

$$\lambda_1 = ... = \lambda_{n_1} = \frac{1}{3}\left(\frac{3}{2}\right)^m, \quad \lambda_{n_1+1} = ... = \lambda_{n_1+n_2} = \frac{1}{3}\left(\frac{3}{2}\right)^{m-1}, \quad ..., \quad \lambda_n = \frac{1}{3}. \tag{3.2}$$

Let S_i be the set of all the multi-indices of the monomials of P_i, $i \in \{1, ..., m\}$. So, each set S_i is a disjoint union of two finite nonempty subsets $S_{i,1} \sqcup S_{i,2}$.

The vectors $s \in S_{i,1}$ has two ones ore one two in the places corresponding to the vector χ_i, while the remaining components are zeros. The second subset $S_{i,2}$ consists of the only vector s with all the components equal to zero except the component corresponding to the variable χ^1_{i+1}. This component equals to three.

Let us show that (2.1) holds. Put

$$\alpha_i = \left(\frac{3}{2}\right)^{m-i}, \quad i \in \{1, ..., m\}.$$

Let us prove that $\langle \lambda, s\rangle = \alpha_i$ for every $s \in S_i$. Fix an arbitrary $i \in \{1, ..., m\}$. Let $s \in S_i$. If $s \in S_{i,1}$ then

$$\langle \lambda, s\rangle = 2\widehat{\lambda}_i = \frac{2}{3}\left(\frac{3}{2}\right)^{(m-i+1)} = \left(\frac{3}{2}\right)^{(m-i)} = \alpha_i.$$

If $s \in S_{i,2}$ then

$$\langle \lambda, s\rangle = 3\widehat{\lambda}_{i+1} = 3\frac{1}{3}\alpha_i = \alpha_i$$

for $i \le m-1$. Obviously, the last equality also holds for $i = m$. So, we have $\langle \lambda, s\rangle = \alpha_i$ for every $s \in S_i$. Hence, (2.1) holds.

So, it is shown that the polynomial mapping P is a λ-truncation of itself in a neighbourhood of zero.

Let us construct an n-dimensional vector h such that

$$P(h) = 0, \quad P'(h)\mathbb{R}^n = \mathbb{R}^m.$$

We divide the construction into several stages.

First, we construct the vector h_1 by taking $h_1 := \widehat{x}_1$, where $\widehat{x}_1$ is an arbitrary vector such that $\langle Q_1\widehat{x}_1, \widehat{x}_1\rangle \neq 0$. This vector exists since $Q_1 \neq 0$. Now we choose the first component h^1_2 of the vector h_2 satisfying the equality

$$\langle Q_1 h_1, h_1\rangle + b_2(h^1_2)^3 = 0.$$

This real number h^1_2 exists since $b_2 \neq 0$. We have $h^1_2 \neq 0$ since $\langle Q_1\widehat{x}_1, \widehat{x}_1\rangle \neq 0$.

Now we construct the remaining coordinates of the vector h_2. If $n_2 = 1$ then we put $h_2 = h^1_2$. Since $h^1_2 \neq 0$ and $Q_2 \neq 0$, then $\langle Q_2 h_2, h_2\rangle \neq 0$. In this case, the construction of the vector h_2 is completed.

Assume now that $n_2 \geq 2$. Let us show that the already constructed first coordinate h_2^1 can be supplemented with real numbers $h_2^2, ..., h_2^{n_2}$ to the vector $h_2 \in \mathbb{R}^{n_2}$ so that $\langle Q_2 h_2, h_2 \rangle \neq 0$.

Consider the contrary, i.e. $\langle Q_2 \chi_2, \chi_2 \rangle = 0$ for every vector χ_2 such that its first component h_2^1 is not equal zero, i.e. $h_2^1 \neq 0$.

Denote by $M \subset \mathbb{R}^{n_2}$ the set of all vectors $\chi_2 \in \mathbb{R}^{n_2}$ whose first component is not zero.

Take an arbitrary $\chi_2 \in M$. Then there exists $t \neq 0$ such that the first component of $t\chi_2$ equals to h_2^1. Then the assumption made implies

$$t^2 \langle Q_2 \chi_2, \chi_2 \rangle = \langle Q_2 t\chi_2, t\chi_2 \rangle = 0.$$

Therefore, $\langle Q_2 \chi_2, \chi_2 \rangle = 0$. At the same time, the set M is everywhere dense in $\mathbb{R}^{n_2}$. So, since the quadratic form Q_2 is a continuous function, it vanishes over the entire space $\mathbb{R}^{n_2}$.

The latter contradicts the assumption $Q_2 \neq 0$. This means that the first coordinate h_2^1 can be supplemented with the real numbers $h_2^2, ..., h_2^{n_2}$ to the vector h_2 so that $\langle Q_2 h_2, h_2 \rangle \neq 0$. The construction of the vector h_2 in the case under consideration is completed.

Now we take the first component h_3^1 of the vector h_3 so that

$$\langle Q_2 h_2, h_2 \rangle + b_3 (h_3^1)^3 = 0.$$

Obviously, $h_3^1 \neq 0$.

We continue this procedure until the end. At the last stage, we take the vector h_m such that $\langle Q_m h_m, h_m \rangle \neq 0$ and take $h_{m+1}^1 \in \mathbb{R}$ such that

$$\langle Q_m h_m, h_m \rangle + (h_{m+1}^1)^3 = 0.$$

Obviously, $h_{m+1}^1 \neq 0$.

Define an n-dimensional vector h by the formula

$$h = (h_1, h_2, ..., h_m, h_{m+1}^1).$$

By construction we have $\langle Q_i h_i, h_i \rangle + b_{i+1}(h_{i+1}^1)^3 = 0$ for each $i \in \{1, ..., m\}$.

Let us show that the polynomial mapping P is regular in the constructed direction h, i.e. equalities (2.2) hold. The equality $P(h) = 0$ is satisfied due to the above constructions. Therefore, it suffices to verify that the rows of the matrix $P'(h)$ are linearly independent.

The i-th row $P_i'(h)$ of the matrix $P'(h)$ is

$$P_i'(h) = (0, \, ..., \, 2Q_i h_i, \, 3b_{i+1}(h_{i+1}^1)^2, \, 0, \, ..., \, 0), \quad i \in \{1, ..., m\}.$$

Here, $2Q_i h_i$ are on the places corresponding to χ_i and h_{i+1}^1 is the first component of the vector h_{i+1}.

Let the real numbers γ_i be such that

$$\sum_{i=1}^{m} \gamma_i P_i'(h) = 0. \tag{3.3}$$

The last component of the vectors $P_1'(h), ..., P_{m-1}'(h)$ corresponding to χ_{m+1} is zero, while the last component of $P_m'(h)$ equals to $3b_{m+1}(h_{m+1}^1)^2 \neq 0$. So, it follows from (3.3) that

$$\sum_{i=1}^{m-1} \gamma_i P_i'(h) = 0, \quad \gamma_m = 0. \tag{3.4}$$

Similarly, for vectors $P_1'(h), ..., P_{m-2}'(h)$ the second to last component corresponding to χ_{m-1} is zero, while the second to last component of $P_{m-1}'(h)$ equals $3b_m(h_m^1)^2 \neq 0$. So, it follows from (3.3) and (3.4) that

$$\sum_{i=1}^{m-2} \gamma_i P_i'(h) = 0, \quad \gamma_{m-1} = \gamma_m = 0.$$

Carrying out similar reasoning for $i = m-2$, $i = m-3$, etc. up to $i = 1$, as a result we obtain that $\gamma_1 = ... = \gamma_m = 0$. Therefore, the vectors $P_i'(h)$, $i \in \{1, ..., m\}$ are linearly independent.

So, we have shown that under the assumptions $Q_i \neq 0$ and $b_{i+1} \neq 0$ for $i \in \{1, ..., m\}$ the polynomial mapping P is a λ-truncation of itself in a neighbourhood of zero and it is regular in a direction h, i.e. (2.2) holds.

Thus, the following assertion is proved. Let λ be the above constructed n-dimensional vector, i.e. the components of λ are defined by formula (3.2). Let S_i be the finite set of all the multi-indices of the monomials of P_i, $i \in \{1, ..., m\}$, while P_i be defined by (3.1).

Theorem 3.1. *Let the mapping $P = (P_1, ..., P_m) : \mathbb{R}^n \to \mathbb{R}^m$ be defined by formula* (3.1)*, all the matrices Q_i and real numbers b_{i+1} be non-zero, $i \in \{1, ..., m\}$.*

Then for every $i \in \{1, ..., m\}$, equality (2.1) *holds and there exists a vector $h \in \mathbb{R}^n$ such that equalities* (2.2) *hold, i.e. $P(h) = 0$ and $P'(h)\mathbb{R}^n = \mathbb{R}^m$.*

The assumptions $Q_i \neq 0$ and $b_{i+1} \neq 0$ for every $i \in \{1, ..., m\}$ in Theorem 3.1 are essential even when $m = 1$. Let us show this.

Let us first assume that $b_2 \neq 0$ and $Q_1 = 0$. Then $P(x) \equiv b_2 x_n^3$. It is obvious that in this case at least one of the equalities in (2.2) is violated for every $h \in \mathbb{R}^n$. Assume now that $b_2 = 0$. Take a positive symmetric $(n-1) \times (n-1)$-matrix Q_1. We have $P(x) \equiv \langle Q_1\chi_1, \chi_1 \rangle \geq 0$ for every $x = (\chi_1, x_n) \in \mathbb{R}^n$. So, the classical inverse function theorem implies that if $P(h) = 0$ for some $h \in \mathbb{R}^n$ then $P'(h)\mathbb{R}^n = \{0\}$. Thus, relation (2.2) is violate for every $h \in \mathbb{R}^n$ in the second case too.

In the special case when $n = 3$, $m = 2$, Q_1 and Q_2 are 1×1-matrices and $b_2 = b_3 = 1$, the mapping $F = P$ was considered in [2, Example 7].

Let us now return to equation (1.1). The following assertion follows from Theorem 3.1 and [2, Theorem 1].

Theorem 3.2. *Let the mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ be continuous in a neighbourhood of the point $\bar{x}$, the mapping $P = (P_1, ..., P_m) : \mathbb{R}^n \to \mathbb{R}^m$, be defined by formula* (3.1) *and P be the λ-truncation of the mapping F in a neighbourhood of $\bar{x}$. Assume that all the matrices Q_i and the real numbers b_{i+1} are non-zero, $i \in \{1, ..., m\}$.*

Then there exists a neighbourhood O of the point $F(\bar{x})$ and a real const > 0 *such that for every $y \in O$ there exists a solution $x = x(y)$ to the equation $F(x) = y$ with the unknown x such that*

$$|x(y) - \bar{x}| \leq \mathrm{const}|y - F(\bar{x})|^\theta \quad \forall\, y \in O.$$

Here $\theta = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^{m-1}$.

Proof. Theorem 3.1 implies that there exists a vector $h \in \mathbb{R}^n$ such that (2.2). Moreover, P is the λ-truncation of the mapping F in a neighbourhood of the point $\bar{x}$. So, applying [2, Theorem 1] we obtain that there exists a neighbourhood O of the point $F(\bar{x})$ and a real number const > 0 such that for every $y \in O$ there exists a solution $x = x(y)$ to the equation (1.1) satisfying the inequality (2.3). Computing the value of θ by formula (2.4) we obtain that $\theta = \frac{1}{3} \cdot (\frac{2}{3})^{m-1}$. □

Note that for the mapping F in the above theorem the equality $F'(\bar{x}) = 0$ holds. Therefore, the classical inverse function theorem is not applicable to the mapping F. In addition, since each matrix Q_i is only non-zero and can be sign-definite, the results from the survey [1] are not applicable as well.

4 The stability of λ-truncations to nonlocal perturbations

Theorem 1 from [2] is local in nature. The main idea of the proof of this theorem is to replace the original equation $F(x) = y$ with the equivalent equation

$$P(h+\xi) + \Phi(t,\xi) = \widetilde{y}(t,\eta)$$

for η from a neighbourhood of zero. Here P is a λ-truncation of the mapping F in a neighbourhood of zero which is regular in a direction h, t and η are parameters, ξ is an unknown variable from a neighbourhood of zero, $\widetilde{y}$ is an auxiliary function. This leads to the problem of the global solvability of the equation

$$P(x) + \Phi(x) = y \tag{4.1}$$

for all $y \in \mathbb{R}^m$ and all the continuous bounded mappings $\Phi : \mathbb{R}^n \to \mathbb{R}^m$.

The solvability of equation (4.1) for all specified Φ and y can be interpreted as the stability of the solvability property of the equation $P(x) = 0$ under the perturbations Φ and y.

Let the following be given: a vector $\lambda = (\lambda_1, ..., \lambda_n) \in D$ nonempty finite sets $S_i \subset \widehat{D}$, $i \in \{1, ..., m\}$ satisfying (2.1), and real numbers $p_{i,s}$, $s \in S_i$, $i \in \{1, ..., m\}$. Define the mapping $P = (P_1, ..., P_m) : \mathbb{R}^n \to \mathbb{R}^m$ by the formula

$$P_i(x) = \sum_{s \in S_i} p_{i,s} x^s, \quad x = (x_1, ..., x_n) \in \mathbb{R}^n.$$

For an arbitrary bounded continuous mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, we denote

$$\|\Phi\| := \sup_{x \in \mathbb{R}^n} |\Phi(x)|.$$

Denote by B^m_δ the closed ball in $\mathbb{R}^m$ centred at zero with the radius $\delta \geq 0$, i.e.

$$B^m_\delta := \{y \in \mathbb{R}^m : |y| \leq \delta\}.$$

Theorem 4.1. *Assume that P satisfies (2.1), $\lambda > 0$ and there exists a vector $h \in \mathbb{R}^n$ such that P is regular in the direction h, i.e. equalities (2.2) hold.*

Then there exists a real number const > 0 such that for every continuous bounded mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ and for every vector $y \in \mathbb{R}^m$ there exists a solution $x = (x_1, ..., x_n) \in \mathbb{R}^n$ to equation (4.1) such that

$$|x_j| \leq \mathrm{const}\left(\max_{i=\overline{1,m}}\left((\|\Phi\| + |y|)^{1/\alpha_i}\right)\right)^{\lambda_j}, \quad j \in \{1, ..., n\}. \tag{4.2}$$

Proof. Take an arbitrary bounded continuous mapping $\Phi = (\Phi_1, ..., \Phi_m) : \mathbb{R}^n \to \mathbb{R}^m$ and an arbitrary vector $y = (y_1, ..., y_m) \in \mathbb{R}^m$. As it is mentioned above, $P(0) = 0$ and so if both $\Phi = 0$ and $y = 0$ then $x = 0$ is the desired solution to equation (4.1). So, we assume that either $\Phi \neq 0$ or $y \neq 0$. Hence,

$$\|\Phi\| + |y| \neq 0.$$

We apply the classical inverse function theorem to the mapping P at the point h. Since the equalities (2.2) hold, i.e.

$$P(h) = 0 \quad \text{and} \quad P'(h)\mathbb{R}^n = \mathbb{R}^m,$$

this theorem implies that there exist reals $\mu > 0$ and $\delta > 0$ and a continuous mapping $g : B^m_\delta \to \mathbb{R}^n$ such that

$$P(h + g(z)) = z, \quad |g(z)| \leq \mu|z| \quad \forall\, z \in B^m_\delta. \tag{4.3}$$

Without loss of generality we will assume that a positive $\delta < 1$.

Denote

$$t := \max_{i=\overline{1,m}} \left(\left(\frac{\|\Phi\| + |y|}{\delta} \right)^{1/\alpha_i} \right). \tag{4.4}$$

Note that if $\|\Phi\| + |y| \le \delta$ then $t = \left(\frac{\|\Phi\|+|y|}{\delta}\right)^{1/\alpha}$, where $\alpha = \max\{\alpha_1, ..., \alpha_m\}$. If $\|\Phi\| + |y| > \delta$ then $t = \left(\frac{\|\Phi\|+|y|}{\delta}\right)^{1/\alpha}$, where $\alpha = \min\{\alpha_1, ..., \alpha_m\}$.

The inequality

$$t^{-\alpha_i} \le \frac{\delta}{\|\Phi\| + |y|} \quad \forall\, i \in \{1, ..., m\} \tag{4.5}$$

takes place. Indeed, fix an arbitrary $i \in \{1, ..., m\}$. It follows from (4.4) that $t \ge \big((\|\Phi\| + |y|)/\delta\big)^{1/\alpha_i}$. Therefore, we have $t^{\alpha_i} \ge (\|\Phi\| + |y|)/\delta$. Hence, inequalities (4.5) take place.

Denote

$$x(\xi) := (t^{\lambda_1}(h_1 + \xi_1), ..., t^{\lambda_n}(h_n + \xi_n)), \quad \xi \in \mathbb{R}^n.$$

Define the mapping $\Gamma = (\Gamma_1, ..., \Gamma_m) : \mathbb{R}^n \to \mathbb{R}^m$ by the formula

$$\Gamma_i(\xi) = t^{-\alpha_i}(y_i - \Phi_i(x(\xi))), \quad \xi \in \mathbb{R}^n, \quad i \in \{1, ..., m\}.$$

Obviously, the mapping Γ is continuous, since $x(\cdot)$ is an affine one by the definition.

For every ξ, we have

$$|\Gamma(\xi)| = \sqrt{\sum_{i=1}^m t^{-2\alpha_i}(y_i - \Phi_i(x(\xi)))^2} \le \sqrt{\sum_{i=1}^m \left(\frac{\delta}{\|\Phi\| + |y|}\right)^2 (y_i - \Phi_i(x(\xi)))^2} =$$

$$= \frac{\delta}{\|\Phi\| + |y|}\sqrt{\sum_{i=1}^m (y_i - \Phi_i(x(\xi)))^2} = \frac{\delta}{\|\Phi\| + |y|}|\Phi(x(\xi)) - y| \le \frac{\delta}{\|\Phi\| + |y|}(\|\Phi\| + |y|) = \delta.$$

Here, the first equality follows from the definition of Γ, the first inequality follows from inequalities (4.5), and the last inequality is the triangle inequality.

The obtained estimate implies that $|\Gamma(\xi)|$ is sufficiently small, i.e. $|\Gamma(\xi)| \le \delta\ \forall\ \xi$. Hence, the composition $g(\Gamma(\xi))$ is well-defined for all $\xi \in B^n_\mu$. Moreover, the inequality

$$|g(\Gamma(\xi))| \overset{(4.3)}{\le} \mu\delta < \mu \quad \forall\, \xi \in B^n_\mu$$

takes place. Therefore, the composition $\xi \mapsto g(\Gamma(\xi))$, $\xi \in B^n_\mu$ of continuous mappings is a continuous self-mapping of the ball B^n_μ. So, Brouwer's fixed-point theorem (see, for example, [7, Theorem 1.6.2]) implies that there exists a point $\widetilde{\xi} = (\widetilde{\xi}_1, ..., \widetilde{\xi}_n) \in B^n_\mu$ such that

$$\widetilde{\xi} = g(\Gamma(\widetilde{\xi})).$$

Let us show that the point $x := x(\widetilde{\xi})$ is the desired solution to equation (4.1).

At first, let us verify the equality $P(x) + \Phi(x) = y$. For every $i \in \{1, ..., m\}$, we have

$$P_i(x(\widetilde{\xi})) = P_i(t^{\lambda_1}(h_1 + \widetilde{\xi}_1), ..., t^{\lambda_n}(h_n + \widetilde{\xi}_n)) = \sum_{s \in S_i} p_{i,s} \prod_{j=1}^n (t^{\lambda_j}(h_j + \widetilde{\xi}_j))^{s_j} =$$

$$= \sum_{s\in S_i} p_{i,s} t^{\langle\lambda,s\rangle} \prod_{j=1}^{n} (h_j + \widetilde{\xi}_j)^{s_j} = t^{\alpha_i} \sum_{s\in S_i} p_{i,s} \prod_{j=1}^{n} (h_j + \widetilde{\xi}_j)^{s_j} = t^{\alpha_i} P_i(h + \widetilde{\xi}).$$

Here, the first equality follows from the definition of $x(\widetilde{\xi})$, the second equality follows from the definition of P_i, the second to last equalities follow from (2.1), and the last equality follows from the definition of P_i.

Moreover, for each $i \in \{1, ..., m\}$, we have

$$P_i(h + \widetilde{\xi}) = P_i(h + g(\Gamma(\widetilde{\xi}))) = \Gamma_i(\widetilde{\xi}) = t^{-\alpha_i}(y_i - \Phi_i(x(\widetilde{\xi}))).$$

Here, the first equality holds since $\widetilde{\xi} = g(\Gamma(\widetilde{\xi}))$, the second equality follows from the identity in (4.3) since $\Gamma(\widetilde{\xi}) \in B^m_\delta$, and the last equality follows from the definition of the mapping Γ_i.

So, it follows from the obtained equalities that

$$P_i(x(\widetilde{\xi})) = t^{\alpha_i} P_i(h + \widetilde{\xi}) = t^{\alpha_i} t^{-\alpha_i}(y_i - \Phi_i(x(\widetilde{\xi}))) = y_i - \Phi_i(x(\widetilde{\xi})) \quad \forall\, i \in \{1, ..., m\}.$$

Hence, for the vector $x = x(\widetilde{\xi})$ we have $P(x) + \Phi(x) = y$.

Let us prove that the desired inequalities hold for the components of the constructed vector x. For each $j \in \{1, ..., n\}$, we have

$$|x_j(\widetilde{\xi})| = t^{\lambda_j} |h_j + \widetilde{\xi}_j| = \left(\max_{i=\overline{1,m}} \left(\left(\frac{\|\Phi\| + |y|}{\delta} \right)^{1/\alpha_i} \right) \right)^{\lambda_j} |h_j + \widetilde{\xi}_j| \leq$$

$$\leq \frac{|h| + \mu}{(\min\limits_{i=\overline{1,m}} (\delta^{1/\alpha_i}))^{\lambda_j}} \left(\max_{i=\overline{1,m}} \left((\|\Phi\| + |y|)^{1/\alpha_i} \right) \right)^{\lambda_j}.$$

Here, the first equality follows from the definition of $x(\widetilde{\xi})$, the second equality follows from (4.4), and the inequality holds since $\widetilde{\xi} \in B^n_\mu$. Denoting $\dfrac{|h| + \mu}{(\min\limits_{i=\overline{1,m}} (\delta^{1/\alpha_i}))^{\lambda_j}}$ by const we obtain that estimate (4.2) is proved. Thus, since $x := x(\widetilde{\xi})$ is the desired solution to equation (4.1) we complete the proof. $\square$

In the special case when $\Phi(x) \equiv 0$, Theorem 4.1 implies the following assertion on surjectivity of λ-truncations.

Corollary 4.1. *Assume that P satisfies (2.1), $\lambda > 0$ and there exists a vector $h \in \mathbb{R}^n$ such that P is regular in the direction h, i.e. equalities (2.2) hold.*

Then there exists a real number const > 0 such that for every vector $y \in \mathbb{R}^m$ there exists a solution $x = (x_1, ..., x_n) \in \mathbb{R}^n$ to the equation $P(x) = y$ such that

$$|x_j| \leq \mathrm{const} \left(\max_{i=\overline{1,m}} \left(|y|^{1/\alpha_i} \right) \right)^{\lambda_j}, \quad j \in \{1, ..., n\}.$$

Let us briefly discuss the problem on stability under set-valued perturbation. Recall that a set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a mapping that corresponds to each $x \in \mathbb{R}^n$ a non-empty closed subset of $\mathbb{R}^m$. This mapping is called bounded if there exists $R > 0$ such that $\Phi(x) \subset B^m_R$ for every $x \in \mathbb{R}^n$. A set-valued mapping Φ is called convex-valued if $\Phi(x)$ is convex for each $x \in \mathbb{R}^n$. A set-valued mapping Φ is called lower semicontinuous if for every $x \in \mathbb{R}^n$, for every open set $W \subset \mathbb{R}^k$ such that $W \cap \Phi(x) \neq \emptyset$ there exists a neighbourhood $V \subset \mathbb{R}^n$ of x such that $W \cap \Phi(\chi) \neq \emptyset$ for each $\chi \in V$.

For an arbitrary bounded set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we denote

$$\|\Phi\| := \sup\{|y| :\, x \in \mathbb{R}^n, \quad y \in \Phi(x)\}.$$

Corollary 4.2. *Assume that P satisfies* (2.1), $\lambda > 0$ *and there exists a vector $h \in \mathbb{R}^n$ such that P is regular in the direction h, i.e. equalities* (2.2) *hold.*

Then there exists a real number $\mathrm{const} > 0$ *such that for every convex-valued bounded lower semicontinuous mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and for every vector $y \in \mathbb{R}^m$ there exists $x = (x_1, ..., x_n) \in \mathbb{R}^n$ such that $y \in P(x) + \Phi(x)$ and* (4.2) *holds.*

Proof. Applying the Michael continuous selection theorem we obtain a continuous mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ such that $\varphi(x) \in \Phi(x)$ for every $x \in \mathbb{R}^n$. Applying Theorem 4.1 to the mapping P, the perturbation φ and a vector $y \in \mathbb{R}^m$ since $\|\varphi\| \leq \|\Phi\|$, we obtain that the desired x exists. □

Here, we consider convex-valued bounded lower semicontinuous perturbations. Another types of perturbations can be considered using different technique based on various fixed point theorems for set-valued mappings (see, for example, [3, 5]).

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