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# HYPERSINGULARLY PERTURBED LOADS FOR A NONLINEAR TRACTION BOUNDARY VALUE PROBLEM. A FUNCTIONAL ANALYTIC APPROACH

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**Abstract.** Let $\Omega^i$ and $\Omega^o$ be two bounded open subsets of $\mathbb{R}^n$ containing 0. Let $G^i$ be a (nonlinear) map from $\partial\Omega^i \times \mathbb{R}^n$ to $\mathbb{R}^n$. Let $a^o$ be a map from $\partial\Omega^o$ to the set $M_n(\mathbb{R})$ of $n \times n$ matrices with real entries. Let $g$ be a function from $\partial\Omega^o$ to $\mathbb{R}^n$. Let $\gamma$ be a positive valued function defined on a right neighborhood of 0 in the real line. Let $T$ be a map from $]1-(2/n), +\infty[\times M_n(\mathbb{R})$ to $M_n(\mathbb{R})$. Then we consider the problem

$$\begin{cases} \mathrm{div}\,(T(\omega, Du)) = 0 & \text{in } \Omega^o \setminus \epsilon \mathrm{cl}\,\Omega^i\,, \\ -T(\omega, Du(x))\nu_{\epsilon\Omega^i}(x) = \frac{1}{\gamma(\epsilon)} G^i(x/\epsilon, \gamma(\epsilon)\epsilon^{-1}(\log\epsilon)^{-\delta_{2,n}}u(x)) & \forall x \in \epsilon\partial\Omega^i\,, \\ T(\omega, Du(x))\nu^o(x) = a^o(x)u(x) + g(x) & \forall x \in \partial\Omega^o\,, \end{cases}$$

where $\nu_{\epsilon\Omega^i}$ and $\nu^o$ denote the outward unit normal to $\epsilon\partial\Omega^i$ and $\partial\Omega^o$, respectively, and where $\epsilon > 0$ is a small parameter. Here $(\omega - 1)$ plays the role of ratio between the first and second Lamé constants, and $T(\omega, \cdot)$ denotes (a constant multiple of) the linearized Piola Kirchhoff stress tensor, and $\delta_{2,n}$ denotes the Kronecker symbol. Under the condition that $\gamma$ generates a very strong singularity, *i.e.*, the case in which $\lim_{\epsilon\to 0^+} \frac{\gamma(\epsilon)}{\epsilon^{n-1}}$ exists in $[0, +\infty[$, we prove that under suitable assumptions the above problem has a family of solutions $\{u(\epsilon, \cdot)\}_{\epsilon\in]0,\epsilon'[}$ for $\epsilon'$ sufficiently small and we analyze the behavior of such a family as $\epsilon$ is close to 0 by an approach which is alternative to those of asymptotic analysis.

## 1 Introduction

In [3], we have considered a linearly elastic homogeneous isotropic body with a small hole subject to a traction on the boundary of the internal hole, which depends singularly on the singular perturbation parameter $\epsilon$ which determines the size of the hole, and we have analyzed the cases which are 'microscopically weakly singular' and 'microscopically singular' in a sense which we illustrate below.

In this paper, we concentrate on the 'singular' and 'hypersingular' cases.

We assume that the constitutive relations of our body are expressed by means of the linearized tensor $T(\omega,\cdot)$ defined by

$$T(\omega, A) \equiv (\omega-1)(\operatorname{tr} A)I + (A + A^t) \qquad \forall A \in M_n(\mathbb{R}),$$

where $\omega \in ]1-(2/n), +\infty[$ is a parameter such that $(\omega-1)$ plays the role of ratio between the first and second Lamé constants, $M_n(\mathbb{R})$ denotes the set of $n\times n$ matrices with real entries, $\operatorname{tr} A$ and $A^t$ and $I$ denote the trace and the transpose matrix of the matrix $A$ and the identity matrix, respectively. We also note that the classical linearization of the (first) Piola Kirchhoff tensor equals the second Lamé constant times $T(\omega,\cdot)$.

Next we introduce a problem in the case our body has no hole. We assume that the body with no hole occupies an open bounded connected subset $\Omega^o$ of $\mathbb{R}^n$ of class $C^{m,\alpha}$ for some $m \in \mathbb{N}\setminus\{0\}$ and $\alpha \in ]0,1[$ and such that $0 \in \Omega^o$ and such that the exterior of $\Omega^o$ is also connected. Then we assign a function $a^o$ from $\partial\Omega^o$ to $M_n(\mathbb{R})$ of class $C^{m-1,\alpha}$, and a function $g$ from $\partial\Omega^o$ to $\mathbb{R}^n$ of class $C^{m-1,\alpha}$, and we consider the linear boundary value problem

$$\begin{cases} \operatorname{div}(T(\omega, Du)) = 0 & \text{in } \Omega^o, \\ T(\omega, Du(x))\nu^o(x) = a^o(x)u(x) + g(x) & \forall x \in \partial\Omega^o, \end{cases} \tag{1}$$

where $\nu^o$ denotes the outward unit normal to $\partial\Omega^o$. Here $a^o(x)u(x)+g(x)$ plays the role of the reciprocal of the second Lamé constant times a field of forces applied to the boundary of the body. We know that under certain nondegeneracy assumptions on the function $a^o$, the linear traction boundary value problem in (1) admits a unique solution $\tilde{u} \in C^{m,\alpha}(\operatorname{cl}\Omega^o, \mathbb{R}^n)$. Here $\operatorname{cl}\Omega^o$ denotes the closure of $\Omega^o$. Next we make a hole in the body $\Omega^o$. Namely, we consider another bounded open connected subset $\Omega^i$ of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $0 \in \Omega^i$ and such that the exterior of $\Omega^i$ is also connected, and we take $\epsilon_0 \in ]0,1[$ such that $\epsilon \operatorname{cl}\Omega^i \subseteq \Omega^o$ for $|\epsilon| < \epsilon_0$, and we consider the perforated domain

$$\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon \operatorname{cl}\Omega^i.$$

Obviously, $\partial\Omega(\epsilon) = (\epsilon\partial\Omega^i) \cup \partial\Omega^o$. Next we wish to define a boundary value problem in $\Omega(\epsilon)$. To do so, we assign a function $G^i$ from $\partial\Omega^i \times \mathbb{R}^n$ to $\mathbb{R}^n$ and a function $\gamma$ from $]0,\epsilon_0[$ to $]0,+\infty[$, and we consider the following nonlinear problem

$$\begin{cases} \operatorname{div}(T(\omega, Du)) = 0 & \text{in } \Omega^o \setminus \epsilon \operatorname{cl}\Omega^i, \\ -T(\omega, Du(x))\nu_{\epsilon\Omega^i}(x) = \frac{1}{\gamma(\epsilon)} G^i(x/\epsilon, u(x)) & \forall x \in \epsilon\partial\Omega^i, \\ T(\omega, Du(x))\nu^o(x) = a^o(x)u(x) + g(x) & \forall x \in \partial\Omega^o, \end{cases} \tag{2}$$

where $\nu_{\epsilon\Omega^i}$ denotes the outward unit normal to $\epsilon\partial\Omega^i$. Here the function $\frac{1}{\gamma(\epsilon)} G^i(x/\epsilon, u(x))$ of $x \in \epsilon\partial\Omega^i$ plays the role of the reciprocal of the second Lamé constant times a field of forces applied to the inner boundary of the body. Since we allow the function $\gamma$ to tend to 0 as $\epsilon$ tends to 0, the presence of the factor $1/\gamma(\epsilon)$ in the second equation of (2) determines a singularity of the problem as $\epsilon$ tends to 0. In [3], we have analyzed (2) under the following assumptions

$$\gamma_m \equiv \lim_{\epsilon\to 0}\gamma^{-1}(\epsilon)\epsilon(\log\epsilon)^{\delta_{2,n}} = 0\,, \qquad \gamma_m \equiv \lim_{\epsilon\to 0}\gamma^{-1}(\epsilon)\epsilon(\log\epsilon)^{\delta_{2,n}} \in \mathbb{R}\setminus\{0\}\,, \tag{3}$$

which we address to as the 'microscopically weakly singular' and 'microscopically singular' cases. Here $\delta_{2,n}$ denotes the Kronecker symbol defined by $\delta_{i,j} = 1$ if $i = j$, $\delta_{i,j} = 0$ if $i \neq j$ for all $i, j = 1, \ldots, n$. In this paper, we shall analyze the case in which the singularity is very strong, *i.e.*, the case in which

$$\gamma_M \equiv \lim_{\epsilon\to 0}\tfrac{\gamma(\epsilon)}{\epsilon^{n-1}} \in ]0, +\infty[\,, \tag{4}$$

$$\gamma_M \equiv \lim_{\epsilon\to 0}\tfrac{\gamma(\epsilon)}{\epsilon^{n-1}} = 0\,, \tag{5}$$

which we address to as the 'singular' and 'hypersingular' cases, respectively. Such a terminology is justified by the behavior of the families of solutions which we will consider as $\epsilon$ tends to zero. We note however that in cases (4) and (5), we consider a problem in which the right hand side of the second equation of (2) is replaced by

$$\frac{1}{\gamma(\epsilon)}G^i(x/\epsilon, \gamma(\epsilon)\epsilon^{-1}(\log\epsilon)^{-\delta_{2,n}}u(x))\,,$$

and thus we shall consider the boundary value problem

$$\begin{cases} \mathrm{div}\,(T(\omega, Du)) = 0 & \text{in } \Omega^o \setminus \epsilon\mathrm{cl}\,\Omega^i\,, \\ -T(\omega, Du(x))\nu_{\epsilon\Omega^i}(x) = \frac{1}{\gamma(\epsilon)}G^i(x/\epsilon, \gamma(\epsilon)\epsilon^{-1}(\log\epsilon)^{-\delta_{2,n}}u(x)) & \forall x \in \epsilon\partial\Omega^i\,, \\ T(\omega, Du(x))\nu^o(x) = a^o(x)u(x) + g(x) & \forall x \in \partial\Omega^o\,. \end{cases} \tag{6}$$

For a discussion on such a choice, see [4]. We note that in case $G^i(\cdot,\cdot)$ depends linearly upon the second variable, our problem becomes linear. Both under assumptions (4) and (5), we shall introduce a limiting boundary value problem (see (24)), and we shall show that under the assumption that such a problem admits a solution satisfying certain conditions, possibly shrinking $\epsilon_0$, problem (6) has a solution $u(\epsilon,\cdot) \in C^{m,\alpha}(\mathrm{cl}\,\Omega(\epsilon), \mathbb{R}^n)$ which is locally unique in a sense which we shall clarify. Whereas under condition (4), $u(\epsilon,\cdot)$ approaches a function related to the solution of the limiting problem (24), under the 'hypersingular' case (5), $u(\epsilon,\cdot)$ has a singular behavior as $\epsilon \to 0^+$, and we show that by taking the limit of

$$\frac{\gamma(\epsilon)}{\epsilon^{n-1}}u(\epsilon,\cdot)\,,$$

we obtain again a limiting function associated to the solution of the limiting problem. However, our main interest is focused on the description of the behavior of $u(\epsilon,\cdot)$ when $\epsilon$ is near 0, and not only on the limiting value. Actually, we pose the following three questions

(j) Let $x$ be a fixed point in $\mathrm{cl}\,\Omega^o \setminus \{0\}$. What can be said on the map $\epsilon \mapsto u(\epsilon, x)$ when $\epsilon$ is close to 0 and positive?

(jj) Let $x$ be a fixed point in $\mathbb{R}^n \setminus \Omega^i$. What can be said on the map $\epsilon \mapsto u(\epsilon, \epsilon x)$ when $\epsilon$ is close to 0 and positive?

(jjj) What can be said on the energy integral

$$\mathcal{E}(\omega, u(\epsilon, \cdot)) \equiv \frac{1}{2} \int_{\Omega(\epsilon)} \mathrm{tr}\Big( T(\omega, D_x u(\epsilon, x))(D_x u(\epsilon, x))^t \Big)\, dx \qquad (7)$$

when $\epsilon$ is close to 0 and positive?

Questions of this type have long been investigated for linear problems with the methods of Asymptotic Analysis and of the Calculus of the Variations. Here, we mention Dal Maso and Murat [5], Kozlov, Maz'ya and Movchan [8], Maz'ya, Nazarov and Plamenewskii [14], Ozawa [18], Ward and Keller [22]. We also mention the seminal paper of Ball [1] on nonlinear elastic cavitation. For more comments, see also [2].

Here instead, we wish to represent the maps of (j)–(jjj) in terms of real analytic maps and in terms of possibly singular at 0, but known functions of $\epsilon$ (such as $\epsilon^{-1}$, $\log\epsilon$, $1/\gamma(\epsilon)$, *etc.*.) Our main results in this sense are Theorems 4 and 5. Theorem 4, answers questions (j), (jj). In particular, Theorem 4 implies that for $x$ fixed as in (j), the function $\frac{\gamma(\epsilon)}{\epsilon^{n-1}} u(\epsilon, x)$ of the variable $\epsilon$ equals a real analytic map of three variables defined in a neighborhood of $(0, \gamma_M, 1 - \delta_{2,n})$ in $\mathbb{R}^3$ computed at $(\epsilon, \frac{\gamma(\epsilon)}{\epsilon^{n-1}}, (\log\epsilon)^{-\delta_{2,n}})$ for $\epsilon$ small enough and positive. Also, such a statement ensures that we can expand $\frac{\gamma(\epsilon)}{\epsilon^{n-1}} u(\epsilon, x)$ into a convergent power series of $\epsilon$, $(\frac{\gamma(\epsilon)}{\epsilon^{n-1}} - \gamma_M)$ for $n \geq 3$, and of $\epsilon$, $(\frac{\gamma(\epsilon)}{\epsilon} - \gamma_M)$, $(\log\epsilon)^{-1}$ for $n = 2$. Theorem 5 instead answers question (jjj).

The paper is organized as follows. Section 2 is a section of preliminaries. In Section 3, we transform our problem (6) into a problem for integral equations, and we identify the limiting problem (24), and we define our family of solutions $\{u(\epsilon, \cdot)\}_{\epsilon \in ]0, \epsilon'[}$ with $\epsilon' \in ]0, \epsilon_0[$. In Section 4, we prove Theorems 4 and 5. In Section 5, we prove the local uniqueness of our family of solutions. In Section 6, we present a sufficient condition in order that the (nonlinear) limiting boundary value problem (24) has a solution.

## 2 Preliminaries and Notation

We denote the norm on a (real) normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the product space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x, y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ $\forall (x, y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. For standard definitions of Calculus in normed spaces, we refer to Prodi and Ambrosetti [20]. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. Throughout the paper,

$$n \text{ is an element of } \mathbb{N} \setminus \{0, 1\}\,.$$

The inverse function of an invertible function $f$ is denoted $f^{(-1)}$, as opposed to the reciprocal of a complex-valued function $g$, or the inverse of a matrix $A$, which are denoted $g^{-1}$ and $A^{-1}$, respectively. A dot '$\cdot$' denotes the inner product in $\mathbb{R}^n$, or the matrix product between matrices with real entries. Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\mathrm{cl}\,\mathbb{D}$ denotes the closure of $\mathbb{D}$ and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. For all $R > 0$, $x \in \mathbb{R}^n$, $x_j$ denotes the $j$-th coordinate of $x$, $|x|$ denotes the Euclidean modulus of $x$ in $\mathbb{R}^n$, and $\mathbb{B}_n(x, R)$ denotes the ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. Let $\Omega$ be an open subset of $\mathbb{R}^n$. The space of $m$ times continuously differentiable real-valued functions on $\Omega$ is denoted by $C^m(\Omega, \mathbb{R})$, or more simply by $C^m(\Omega)$. Let $f \in (C^m(\Omega))^n$. The $s$-th component of $f$ is denoted $f_s$, and $Df$ (or $\nabla f$) denotes the gradient matrix $\left(\frac{\partial f_s}{\partial x_l}\right)_{s,l=1,\dots,n}$. Let $\eta \equiv (\eta_1, \dots, \eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \cdots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \dots \partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions $f$ such that $f$ and its derivatives $D^\eta f$ of order $|\eta| \le m$ can be extended with continuity to $\mathrm{cl}\,\Omega$ is denoted $C^m(\mathrm{cl}\,\Omega)$. The subspace of $C^m(\mathrm{cl}\,\Omega)$ whose functions have $m$-th order derivatives that are Hölder continuous with exponent $\alpha \in ]0, 1]$ is denoted $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, (cf. *e.g.* Gilbarg and Trudinger [6].) The subspace of $C^m(\mathrm{cl}\,\Omega)$ of those functions $f$ such that $f_{|\mathrm{cl}\,(\Omega \cap \mathbb{B}_n(0,R))} \in C^{m,\alpha}(\mathrm{cl}\,(\Omega \cap \mathbb{B}_n(0, R)))$ for all $R \in ]0, +\infty[$ is denoted $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\Omega)$. Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{D})$ denotes $\{f \in (C^{m,\alpha}(\mathrm{cl}\,\Omega))^n : f(\mathrm{cl}\,\Omega) \subseteq \mathbb{D}\}$. Now let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. Then $C^m(\mathrm{cl}\,\Omega)$ endowed with the norm $\|f\|_{C^m(\mathrm{cl}\,\Omega)} \equiv \sum_{|\eta| \le m} \sup_{\mathrm{cl}\,\Omega} |D^\eta f|$ is a Banach space. If $f \in C^{0,\alpha}(\mathrm{cl}\,\Omega)$, then its Hölder constant $|f : \Omega|_\alpha$ is defined as $\sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} : x, y \in \mathrm{cl}\,\Omega, x \ne y\right\}$. The space $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, equipped with its usual norm $\|f\|_{C^{m,\alpha}(\mathrm{cl}\,\Omega)} = \|f\|_{C^m(\mathrm{cl}\,\Omega)} + \sum_{|\eta|=m} |D^\eta f : \Omega|_\alpha$, is well-known to be a Banach space. We say that a bounded open subset of $\mathbb{R}^n$ is of class $C^m$ or of class $C^{m,\alpha}$, if it is a manifold with boundary imbedded in $\mathbb{R}^n$ of class $C^m$ or $C^{m,\alpha}$, respectively (cf. *e.g.*, Gilbarg and Trudinger [6, §6.2].) For standard properties of the functions of class $C^{m,\alpha}$ both on a domain of $\mathbb{R}^n$ or on a manifold imbedded in $\mathbb{R}^n$ we refer to Gilbarg and Trudinger [6] (see also [12, §2, Lem. 3.1, 4.26, Thm. 4.28], Lanza and Rossi [13, §2].) We retain the standard notation of $L^p$ spaces and of corresponding norms. We note that throughout the paper 'analytic' means 'real analytic'. For the definition and properties of analytic operators, we refer to Prodi and Ambrosetti [20, p. 89].

We denote by $S_n$ the function of $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(\xi) \equiv \begin{cases} \frac{1}{s_n} \log|\xi| & \forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n = 2\,, \\ \frac{1}{(2-n)s_n} |\xi|^{2-n} & \forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n > 2\,, \end{cases}$$

where $s_n$ denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n$. $S_n$ is well-known to be the fundamental solution of the Laplace operator.

We denote by $\Gamma_n(\cdot, \cdot)$ the matrix valued function from $(\mathbb{R} \setminus \{-1\}) \times (\mathbb{R}^n \setminus \{0\})$ to $M_n(\mathbb{R})$ which takes a pair $(\omega, \xi)$ to the matrix $\Gamma_n(\omega, \xi)$ defined by

$$\Gamma^j_{n,i}(\omega, \xi) \equiv \frac{\omega + 2}{2(\omega + 1)} \delta_{i,j} S_n(\xi) - \frac{\omega}{2(\omega + 1)} \frac{1}{s_n} \frac{\xi_i \xi_j}{|\xi|^n}\,.$$

As is well known, $\Gamma_n(\omega, \xi)$ is the fundamental solution of the operator

$$L[\omega] \equiv \Delta + \omega \nabla \mathrm{div}\,.$$

We note that the classical operator of linearized homogeneous and isotropic elastostatics equals $L[\omega]$ times the second constant of Lamé, and that $L[\omega]u = \operatorname{div} T(\omega, Du)$ for all regular vector valued functions $u$, and that the classical fundamental solution of the operator of linearized homogeneous and isotropic elastostatics equals $\Gamma_n(\omega,\xi)$ times the reciprocal of the second constant of Lamé. We find also convenient to set

$$\Gamma_n^j(\cdot,\cdot) \equiv (\Gamma_{n,i}^j(\cdot,\cdot))_{i=1,\dots,n}\,,$$

which we think of as a column vector for all $j = 1,\dots,n$. Let $\alpha \in ]0,1[$. Let $\Omega$ be an open bounded connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. We shall denote by $\nu_\Omega$ the outward unit normal to $\partial\Omega$. We also set

$$\Omega^- \equiv \mathbb{R}^n \setminus \operatorname{cl}\Omega\,.$$

Let $\omega \in ]1-(2/n), +\infty[$. Then we set

$$\begin{aligned} v[\omega,\mu](x) &\equiv \int_{\partial\Omega} \Gamma_n(\omega, x-y)\mu(y)\,d\sigma_y\,, \\ w[\omega,\mu](x) &\equiv -\left(\int_{\partial\Omega} \mu^t(y) T(\omega, D_\xi \Gamma_n^i(\omega, x-y))\nu_\Omega(y)\;d\sigma_y\right)_{i=1,\dots,n}\,, \end{aligned}$$

for all $x \in \mathbb{R}^n$, and

$$v_*[\omega,\mu](x) \equiv \int_{\partial\Omega} \sum_{l=1}^n \mu_l(y) T(\omega, D_\xi \Gamma_n^l(\omega, x-y))\nu_\Omega(x)\,d\sigma_y \qquad \forall x \in \partial\Omega\,,$$

for all $\mu \equiv (\mu_j)_{j=1,\dots,n} \in L^p(\partial\Omega,\mathbb{R}^n)$ with $p>1$. As is well known, if $\mu \in C^{0,\alpha}(\partial\Omega,\mathbb{R}^n)$, then $v[\omega,\mu]$ is continuous in the whole of $\mathbb{R}^n$, and we set

$$v^+[\omega,\mu] \equiv v[\omega,\mu]_{|\operatorname{cl}\Omega} \qquad v^-[\omega,\mu] \equiv v[\omega,\mu]_{|\operatorname{cl}\Omega^-}\,.$$

Also if $\mu \in C^{0,\alpha}(\partial\Omega,\mathbb{R}^n)$, then $w[\omega,\mu]_{|\Omega}$ admits a unique continuous extension to $\operatorname{cl}\Omega$, which we denote by $w^+[\omega,\mu]$, and $w[\omega,\mu]_{|\Omega^-}$ admits a unique continuous extension to $\operatorname{cl}\Omega^-$, which we denote by $w^-[\omega,\mu]$.

We now shortly review some facts on the linear traction problem, which we need in the sequel. Let $a$ be a continuous map from $\partial\Omega$ to $M_n(\mathbb{R})$ satisfying the following assumptions.

$$\text{The determinant } \det a(\cdot) \text{ of } a(\cdot) \text{ does not vanish identically in } \partial\Omega\,, \tag{8}$$

$$\xi^t a(x)\xi \geq 0 \qquad \forall x \in \partial\Omega,\ \forall \xi \in \mathbb{R}^n\,. \tag{9}$$

For each $G \in C^0(\partial\Omega\times\mathbb{R}^n,\mathbb{R}^n)$, we denote by $F_G$ the (nonlinear) composition operator from $C^0(\partial\Omega,\mathbb{R}^n)$ to itself which maps $v \in C^0(\partial\Omega,\mathbb{R}^n)$ to the function $F_G[v]$ defined by

$$F_G[v](t) \equiv G(t, v(t)) \qquad \forall t \in \partial\Omega\,.$$

In the next proposition, we transform our nonlinear boundary value problem into a problem for integral equations (see [2, Prop. 2.3].)

**Proposition 1.** *Let $\alpha \in ]0,1[$, $\omega \in ]1-(2/n),+\infty[$. Let $m \in \mathbb{N} \setminus \{0\}$. Let $\Omega$ be an open bounded connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $G$ be a function of class $C^0(\partial\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ such that $F_G$ maps $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ to itself. Then the map from the set*

$$\left\{(d,\mu) \in \mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)_0 : -\frac{1}{2}\mu + v_*[\omega,\mu] = F_G[v[\omega,\mu]_{|\partial\Omega} + d]\right\}, \qquad (10)$$

*to the set*

$$\Big\{u \in C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{R}^n) : \\ \mathrm{div}\,(T(\omega, Du)) = 0 \text{ in } \Omega,\ T(\omega, Du)\nu_\Omega = F_G[u_{|\partial\Omega}] \text{ on } \partial\Omega\Big\},$$

*which takes $(d,\mu)$ to the function $v^+[\omega,\mu] + d$ is a bijection.*

## 3 Formulation of the problem in terms of integral equations, and existence of the solution $u(\epsilon,\cdot)$

We now provide a formulation of problem (6) in terms of integral equations. We shall consider the following assumptions for some $\alpha \in ]0,1[$ and for some natural $m \geq 1$.

*Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$.* (11)
*Let $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$ be connected. Let $0 \in \Omega$.*

Now let $\Omega^i$, $\Omega^o$ be as in (11). Then there exists

$$\epsilon_0 \in ]0,1[ \text{ such that } \epsilon \mathrm{cl}\,\Omega^i \subseteq \Omega^o,\ \forall \epsilon \in ]-\epsilon_0, \epsilon_0[\,. \qquad (12)$$

A simple topological argument shows that $\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon\mathrm{cl}\,\Omega^i$ is connected, and that $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega(\epsilon)$ has exactly the two connected components $\epsilon\Omega^i$ and $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega^o$, and that $\partial\Omega(\epsilon) = (\epsilon\partial\Omega^i) \cup \partial\Omega^o$, for all $\epsilon \in ]-\epsilon_0, \epsilon_0[\setminus\{0\}$. For brevity, we set

$$\nu^i \equiv \nu_{\Omega^i}, \qquad \nu^o \equiv \nu_{\Omega^o}.$$

Obviously,

$$\nu_{\Omega(\epsilon)}(x) = -\nu^i(x/\epsilon)\,\mathrm{sgn}(\epsilon) \qquad \forall x \in \epsilon\partial\Omega^i, \qquad (13)$$

$$\nu_{\Omega(\epsilon)}(x) = \nu^o(x) \qquad \forall x \in \partial\Omega^o, \qquad (14)$$

for all $\epsilon \in ]-\epsilon_0, \epsilon_0[\setminus\{0\}$, where $\mathrm{sgn}(\epsilon) = 1$ if $\epsilon > 0$, $\mathrm{sgn}(\epsilon) = -1$ if $\epsilon < 0$. Then we consider the following assumptions

$$G^i \in C^0(\partial\Omega^i \times \mathbb{R}^n, \mathbb{R}^n), \qquad (15)$$

$$F_{G^i} \text{ maps } C^{m-1,\alpha}(\partial\Omega^i, \mathbb{R}^n) \text{ to itself}, \qquad (16)$$

$$g \in C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n),\ a^o \in C^{m-1,\alpha}(\partial\Omega^o, M_n(\mathbb{R})), \qquad (17)$$

and we set

$$G^o(t,\xi) \equiv g(t) + a^o(t)\xi \qquad \forall (t,\xi) \in \partial\Omega^o \times \mathbb{R}^n\,. \tag{18}$$

If $G^i \in C^0(\partial\Omega^i \times \mathbb{R}^n, \mathbb{R}^n)$, $G^o \in C^0(\partial\Omega^o \times \mathbb{R}^n, \mathbb{R}^n)$, we denote by $G$ the function from $\partial\Omega(\epsilon) \times \mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$\begin{aligned} G(s,\xi) &\equiv G^o(s,\xi) \qquad \text{if } (s,\xi) \in \partial\Omega^o \times \mathbb{R}^n, \\ G(s,\xi) &\equiv G^i(s/\epsilon,\xi) \qquad \text{if } (s,\xi) \in \epsilon\partial\Omega^i \times \mathbb{R}^n\,. \end{aligned} \tag{19}$$

We now convert our boundary value problems (6) into a system of integral equations. We could exploit Proposition 1. However, we note that the corresponding representation formulas include integration on the $\epsilon$-dependent set $\partial\Omega(\epsilon)$. In order to get rid of such dependence, we introduce the following theorem in which we properly rescale the restriction of the unknown function to $\epsilon\partial\Omega^i$. We note that the transformation we operate (cf. (22)) differs considerably from that we have operated for the treatment of the nonlinear conditions on $\partial\Omega^o$ of [2], or for the microscopically weakly singular case of [3]. We find convenient to introduce the following notation. We set

$$X_{m,\alpha} \equiv C^{m-1,\alpha}(\partial\Omega^i, \mathbb{R}^n) \times C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n)\,,$$

and we introduce the map $M = (M_1, M_2, M_3)$ from $]-\epsilon_0, \epsilon_0[\times\mathbb{R}^2 \times \mathbb{R}^n \times X_{m,\alpha}$ to $\mathbb{R}^n \times X_{m,\alpha}$ defined by

$$\begin{aligned} &M_1[\epsilon,\epsilon_1,\epsilon_2,c,\eta,\rho] \equiv \int_{\partial\Omega^i} \eta\, d\sigma + \int_{\partial\Omega^o} \rho\, d\sigma\,, \\ &M_2[\epsilon,\epsilon_1,\epsilon_2,c,\eta,\rho](t) \\ &\quad \equiv \frac{1}{2}\eta(t) + v_*[\omega,\eta](t) + \epsilon^{n-1}\int_{\partial\Omega^o} \sum_{l=1}^n \rho_l(s) T(\omega, D_\xi \Gamma_n^l(\omega, \epsilon t - s))\nu^i(t)\; d\sigma_s \\ &\quad\quad + G^i\biggl(t, \epsilon_2 v[\omega,\eta](t) + \frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i}\eta\, d\sigma + \epsilon^{n-2}\epsilon_2 v[\omega,\rho](\epsilon t) + \epsilon^{n-2}\epsilon_2 c\biggr) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall t \in \partial\Omega^i\,, \\ &M_3[\epsilon,\epsilon_1,\epsilon_2,c,\eta,\rho](t) \\ &\quad \equiv -\frac{1}{2}\rho(t) + v_*[\omega,\rho](t) + \int_{\partial\Omega^i} \sum_{l=1}^n \eta_l(s) T(\omega, D_\xi\Gamma_n^l(\omega, t - \epsilon s))\nu^o(t)\; d\sigma_s \\ &\quad\quad - a^o(t)\biggl\{\int_{\partial\Omega^i} \Gamma_n(\omega, t-\epsilon s)\eta(s)\, d\sigma_s + v[\omega,\rho](t) + c\biggr\} - \epsilon_1 g(t) \quad \forall t \in \partial\Omega^o\,, \end{aligned} \tag{20}$$

for all $(\epsilon,\epsilon_1,\epsilon_2,c,\eta,\rho) \in ]-\epsilon_0,\epsilon_0[\times\mathbb{R}^2 \times \mathbb{R}^n \times X_{m,\alpha}$. As we shall see in the next two statements, the map $M$ will play an important role in the analysis of problem (6). In the first statement, we consider case $\epsilon > 0$. We clarify that at this stage, we do not claim existence neither for boundary value problem (6) nor for its equivalent counterpart (21) in terms of zeros of $M$.

**Theorem 1.** *Let $\alpha \in ]0,1[$, $\omega \in ]1-(2/n), +\infty[$, $m \in \mathbb{N}\setminus\{0\}$. Let $\Omega^i$, $\Omega^o$ be as in (11). Let $\epsilon_0$ be as in (12). Let $G^i$, $G^o$ be as in (15), (16), (17), (18). Let $\gamma(\cdot)$ be a* *map from* $]0, \epsilon_0[$ *to* $]0, +\infty[$. *Let* $\epsilon \in ]0, \epsilon_0[$. *The map* $u_\epsilon[\cdot, \cdot, \cdot]$ *from the set of solutions* $(c, \eta, \rho) \in \mathbb{R}^n \times X_{m,\alpha}$ *of the following system of integral equations*

$$M[\epsilon, \frac{\gamma(\epsilon)}{\epsilon^{n-1}}, (\log \epsilon)^{-\delta_{2,n}}, c, \eta, \rho] = 0 \tag{21}$$

*to the set of solutions* $u \in C^{m,\alpha}(\mathrm{cl}\,\Omega(\epsilon), \mathbb{R}^n)$ *of (6) which takes* $(c, \eta, \rho)$ *to* $v^+[\omega, \mu] + d$, *where*

$$\mu(x) \equiv \frac{\epsilon^{n-1}}{\gamma(\epsilon)}\rho(x) \quad \text{if } x \in \partial\Omega^o, \quad \mu(x) \equiv \frac{1}{\gamma(\epsilon)}\eta(x/\epsilon) \quad \text{if } x \in \epsilon\partial\Omega^i\,,$$
$$d \equiv \frac{\epsilon^{n-1}}{\gamma(\epsilon)} c\,, \tag{22}$$

*is a bijection.*

**Proof.** Let $\epsilon \in ]0, \epsilon_0[$. A simple computation based on the rule of change of variables in integrals over $\partial\Omega^i$ and on (13), (14) shows that $(c, \eta, \rho)$ solves (21) if and only if the pair $(c, \mu)$ belongs to the set in (10) with $\Omega = \Omega(\epsilon)$ and $G$ as in (19). Thus the statement follows by Proposition 1. □

Theorem 1 reduces the analysis of problem (6), which has been considered only for $\epsilon \in ]0, \epsilon_0[$ to that of equation $M = 0$. However equation $M = 0$, contrary to problem (6) makes sense also for $\epsilon = 0$. Then we state the following Theorem which analyses case $\epsilon = 0$. Once more, also in case $\epsilon = 0$, at this stage we do not claim existence neither for equation $M = 0$ nor for its equivalent counterpart in terms of boundary value problem (24).

**Theorem 2.** *Let the same assumptions of Theorem 1 hold. Let* $\gamma_M \in [0, +\infty[$. *Then the map* $(u^i[\cdot, \cdot, \cdot], u^o[\cdot, \cdot, \cdot])$ *from the set of solutions* $(c, \eta, \rho) \in \mathbb{R}^n \times X_{m,\alpha}$ *of the following system of integral equations*

$$M[0, \gamma_M, 1 - \delta_{2,n}, c, \eta, \rho] = 0 \tag{23}$$

*to the set of solutions* $(u^i, u^o) \in C^{m,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega^i, \mathbb{R}^n) \times C^{m,\alpha}(\mathrm{cl}\,\Omega^o, \mathbb{R}^n)$ *of the 'limiting boundary value problem'*

$$\begin{cases} \mathrm{div}\,(T(\omega, Du^i)) = 0 & \text{in } \mathbb{R}^n \setminus \mathrm{cl}\,\Omega^i, \\ -T(\omega, Du^i(t))\nu^i(t) & \\ \quad = G^i\Big(t, (1-\delta_{2,n})u^i(t) + \frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i} T(\omega, Du^i)\nu^i\, d\sigma\Big) & \forall t \in \partial\Omega^i, \\ \sup_{x\in\Omega^{i-}} |x|^{n-2+\delta_{2,n}}|u^{*,i}(x)| < \infty, & \\ \sup_{x\in\Omega^{i-}} |x|^{n-1+\delta_{2,n}}|Du^{*,i}(x)| < \infty, & \\ \mathrm{div}\,(T(\omega, Du^o)) = 0 & \text{in } \Omega^o, \\ T(\omega, Du^o(t))\nu^o(t) - a^o(t)u^o(t) & \\ \quad = -\sum_{l,j=1}^n \left(\int_{\partial\Omega^i} T_{lj}(\omega, Du^i)\nu^i_j\, d\sigma\right) T(\omega, D\Gamma^l_n(\omega, t))\nu^o(t) & \\ \qquad + a^o(t)\left\{\Gamma_n(\omega, t)\int_{\partial\Omega^i} T(\omega, Du^i)\nu^i\, d\sigma\right\} + \gamma_M g(t) & \forall t \in \partial\Omega^o, \end{cases} \tag{24}$$

*where* $\nu^i_j$ *denotes the* $j$*-th component of* $\nu^i$ *and*

$$u^*(x) = u^i(x) - \delta_{2,n}\Gamma(\omega, x)\int_{\partial\Omega^i} T(\omega, Du^i)\nu^i\, d\sigma \qquad \forall x \in \mathbb{R}^n \setminus \Omega^i\,,$$

*which takes the triple* $(c, \eta, \rho)$ *to*

$$\begin{aligned} u^i[c,\eta,\rho] &\equiv v^-[\omega,\eta] && \text{in } \mathbb{R}^n \setminus \Omega^i\,, \\ u^o[c,\eta,\rho] &\equiv v^+[\omega,\rho] + c && \text{on } \mathrm{cl}\,\Omega^o\,, \end{aligned} \tag{25}$$

*is a bijection.*

**Proof.** Let $(c, \eta, \rho)$ solve equation (23). Then by standard jump properties of the elastic layer potentials, and by the boundedness of the functions

$$|x|(\Gamma_2(\omega, x-y) - \Gamma_2(\omega, x))\,, \qquad |x|^2(D_x\Gamma_2(\omega, x-y) - D_x\Gamma_2(\omega, x))\,,$$

for all $(x, y) \in (\mathbb{R}^2 \setminus \mathrm{cl}\,\mathbb{B}_2(0,R)) \times \partial\Omega^i$ where $R > 0$ is such that $\mathrm{cl}\,\Omega^i \subseteq \mathbb{B}_n(0,R)$, and by the equality

$$\int_{\partial\Omega^i} T(\omega, Du^i)\nu^i\, d\sigma = \int_{\partial\Omega^i}\left\{\frac{1}{2}\eta + v_*[\omega,\eta]\right\} d\sigma = \int_{\partial\Omega^i} \eta\, d\sigma\,,$$

which follows by standard properties of elastic layer potentials (cf. *e.g.*, [2, (A.7)]), the functions $u^i$, $u^o$ defined in (25) solve problem (24) and have the required regularity (cf. *e.g.*, [2, Thm. A.2]).)

Next we prove that the map $(u^i[\cdot,\cdot,\cdot], u^o[\cdot,\cdot,\cdot])$ is injective. Thus we now assume that $(\hat{c}, \hat{\eta}, \hat{\rho})$, $(\breve{c}, \breve{\eta}, \breve{\rho}) \in \mathbb{R}^n \times X_{m,\alpha}$, and that the corresponding pairs $(u^i, u^o)$ defined in (25) coincide, and we show that $(\hat{c}, \hat{\eta}, \hat{\rho})$ equals $(\breve{c}, \breve{\eta}, \breve{\rho})$. To do so, we set $(\overline{c}, \overline{\eta}, \overline{\rho}) \equiv (\hat{c} - \breve{c}, \hat{\eta} - \breve{\eta}, \hat{\rho} - \breve{\rho})$ and we show that $(\overline{c}, \overline{\eta}, \overline{\rho}) = (0,0,0)$. Clearly,

$$v^-[\omega, \overline{\eta}] = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega^i, \qquad v^+[\omega, \overline{\rho}] + \overline{c} = 0 \quad \mathrm{cl}\,\Omega^o\,. \tag{26}$$

Then by standard jump properties of simple elastic layer potentials, we have

$$\frac{1}{2}\overline{\eta} + v_*[\omega, \overline{\eta}] = 0 \quad \text{on } \partial\Omega^i\,, \qquad -\frac{1}{2}\overline{\rho} + v_*[\omega, \overline{\rho}] = 0 \quad \text{on } \partial\Omega^o\,. \tag{27}$$

Since $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega^i$ is connected, a classical result in potential theory implies that $\overline{\eta} = 0$ (cf. *e.g.*, [2, Rmk. A.8].) Thus by the first component of equation (23) applied to $(\hat{c}, \hat{\eta}, \hat{\rho})$, $(\breve{c}, \breve{\eta}, \breve{\rho})$, we have $\int_{\partial\Omega^o} \overline{\rho}\, d\sigma = 0$. As known classically, the space of constant functions of the form $v^+[\omega, \overline{\rho}]_{|\partial\Omega^o}$ with $\overline{\rho}$ as in the second equation of (27) and such that $\int_{\partial\Omega^o} \overline{\rho}\, d\sigma = 0$ contains only the constant vector 0 (cf. *e.g.*, [2, Thm. A.5 (iv)].) Hence, the second equation in (26) implies that $\overline{c} = 0$. Then by the second equality of (26), and by the second equation of (27) and by equality $\int_{\partial\Omega^o} \overline{\rho}\, d\sigma = 0$, we deduce that $\overline{\rho} = 0$ (cf. *e.g.*, [2, Thm. A.5 (ii)].)

Now we show that if $(u^i, u^o)$ is as in the statement and satisfies the limiting problem (24), then there exists a solution $(c, \eta, \rho) \in \mathbb{R}^n \times X_{m,\alpha}$ of equation (23) such that $u^i = u^i[c, \eta, \rho]$, $u^o = u^o[c, \eta, \rho]$. By classical properties of elastic layer potentials, the equation

$$\frac{1}{2}\eta + v_*[\omega, \eta] = T(\omega, Du^i)\nu^i$$

has a unique solution $\eta \in C^{m-1,\alpha}(\partial\Omega^i, \mathbb{R}^n)$ (cf. *e.g.*, [2, Rmk. A.8].) Since

$$\int_{\partial\Omega^i} T(\omega, Dv^-[\omega, \eta])\nu^i \, d\sigma = \int_{\partial\Omega^i} \eta \, d\sigma = \int_{\partial\Omega^i} T(\omega, Du^i)\nu^i \, d\sigma \,,$$

both $u^i$ and $v^-[\omega, \eta]$ satisfy the system

$$\begin{cases} \operatorname{div}(T(\omega, Du)) = 0 & \text{in } \mathbb{R}^n \setminus \operatorname{cl}\Omega^i \,, \\ T(\omega, Du)\nu^i = T(\omega, Du^i)\nu^i & \text{on } \partial\Omega^i \,, \end{cases}$$

in the unknown $u$, together with the third and fourth condition of problem (24). Accordingly, $u^i = v^-[\omega, \eta]$ in $\mathbb{R}^n \setminus \Omega^i$ (cf. *e.g.*, [3, Thm. 2.3 (ii)].) Next we consider the problem

$$\begin{cases} \int_{\partial\Omega^o} \rho \, d\sigma = -\int_{\partial\Omega^i} \eta \, d\sigma \,, \\ -\frac{1}{2}\rho(t) + v_*[\omega, \rho](t) + (v[\omega, \rho](t) + c) \\ \qquad\qquad = T(\omega, Du^o(t))\nu^o(t) + u^o(t) \quad \forall t \in \partial\Omega^o \,. \end{cases} \tag{28}$$

Problem (28) in the unknown $(c, \rho)$ is associated to a linear traction boundary value problem. Since the matrix-valued function $a \equiv I$ on $\partial\Omega^o$ satisfies assumptions (8), (9), a classical result implies that problem (28) admits a unique solution $(c, \rho)$ in $\mathbb{R}^n \times X_{m,\alpha}$ (cf. *e.g.*, [2, Th. 2.2 (ii)].) Then both $u^o$ and $v^+[\omega, \rho] + c$ solve the boundary value problem

$$\begin{cases} \operatorname{div}(T(\omega, Du)) = 0 & \text{in } \Omega^o \,, \\ T(\omega, Du)\nu^o + u = T(\omega, Du^o)\nu^o + u^o & \text{on } \partial\Omega^o \,, \end{cases}$$

in the unknown $u$. Then a standard uniqueness argument implies that $u^o = v^+[\omega, \rho] + c$ (cf. *e.g.*, [2, Prop. 2.1].)

By equalities $u^i = v^-[\omega, \eta]$ in $\mathbb{R}^n \setminus \Omega^i$ and $u^o = v^+[\omega, \rho] + c$ in $\operatorname{cl}\Omega^o$, and by standard jump relations for the normal derivative of the elastic simple layer potential, and by the second and sixth equation in (24), we deduce that the second and third components of (23) hold. By (28), the first component of (23) holds. Hence, $(c, \eta, \rho)$ solves equation (23). □

Theorems 1, 2 reduce the analysis of problem (6) and of the boundary value problem (24) to that of the analysis of the set of zeros of $M$. We shall now show that, both in the singular and hypersingular case, if problem (24) has a solution $(\tilde{u}^i, \tilde{u}^o)$ satisfying certain nondegeneracy conditions, then for $\epsilon$ sufficiently small, problem (6) has a solution. We shall also see that such a solution is unique in a local sense which we clarify in section 5.

**Theorem 3.** *Let* $\alpha \in ]0,1[$*,* $\omega \in ]1-(2/n),+\infty[$*,* $m \in \mathbb{N}\setminus\{0\}$*. Let* $\Omega^i$*,* $\Omega^o$ *be as in (11). Let* $\epsilon_0$ *be as in (12). Let* $\gamma(\cdot)$ *be a map from* $]0,\epsilon_0[$ *to* $]0,+\infty[$*. Let* $\lim_{\epsilon\to 0}\frac{\gamma(\epsilon)}{\epsilon^{n-1}} \equiv \gamma_M$ *exist in* $[0,+\infty[$*. Let (15)–(18) hold. Let*

$$F_{G^i} \text{ be real analytic in } C^{m-1,\alpha}(\partial\Omega^i,\mathbb{R}^n)\,. \tag{29}$$

*Let* $-a^o$ *satisfy conditions (8), (9) on* $\partial\Omega^o$*. Assume that the limiting boundary value problem (24) admits a solution* $(\tilde{u}^i,\tilde{u}^o) \in C^{m,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega^i,\mathbb{R}^n)\times C^{m,\alpha}(\mathrm{cl}\,\Omega^o,\mathbb{R}^n)$*. Let* $\mathcal{G}^i$ *be the function from* $\partial\Omega^i$ *to* $M_n(\mathbb{R})$ *defined by*

$$\mathcal{G}^i(t) \equiv -D_\xi G^i\left(t,(1-\delta_{2,n})\tilde{u}^i(t)+\frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i} T(\omega,D\tilde{u}^i)\nu^i\,d\sigma\right)\ \forall t\in\partial\Omega^i\,. \tag{30}$$

*If* $n=2$*, we assume that the matrix* $I-\frac{\omega+2}{4\pi(\omega+1)}\int_{\partial\Omega^i}\mathcal{G}^i\,d\sigma$ *is invertible.*

*If* $n\geq 3$*, we assume that* $\mathcal{G}^i(\cdot)$ *satisfies conditions (8), (9) on* $\partial\Omega^i$*.*

*Let* $(\tilde{c},\tilde{\eta},\tilde{\rho}) \in \mathbb{R}^n\times X_{m,\alpha}$ *be the unique solution of the system of integral equations* $M[0,\gamma_M,1-\delta_{2,n},\tilde{c},\tilde{\eta},\tilde{\rho}]=0$ *such that*

$$\tilde{u}^i = u^i[\tilde{c},\tilde{\eta},\tilde{\rho}] \qquad \tilde{u}^o = u^o[\tilde{c},\tilde{\eta},\tilde{\rho}]\,,$$

*(see Theorem 1.) Then there exist* $\epsilon' \in ]0,\epsilon_0[$*, and an open neighborhood* $\mathcal{U}_{\gamma_M}$ *of the pair* $(\gamma_M,1-\delta_{2,n})$ *in* $\mathbb{R}^2$*, and an open neighborhood* $\mathcal{V}$ *of* $(\tilde{c},\tilde{\eta},\tilde{\rho})$ *in* $\mathbb{R}^n\times X_{m,\alpha}$*, and a real analytic operator* $(C,E,R)$ *from* $]-\epsilon',\epsilon'[\times\mathcal{U}_{\gamma_M}$ *to* $\mathcal{V}$ *such that*

$$\left(\frac{\gamma(\epsilon)}{\epsilon^{n-1}},(\log\epsilon)^{-\delta_{2,n}}\right)\in\mathcal{U}_{\gamma_M} \qquad \forall\epsilon\in]0,\epsilon'[\,, \tag{31}$$

*and such that the set of zeros of* $M$ *in* $]-\epsilon',\epsilon'[\times\mathcal{U}_{\gamma_M}\times\mathcal{V}$ *coincides with the graph of* $(C,E,R)$*. In particular*

$$(C[0,\gamma_M,1-\delta_{2,n}],E[0,\gamma_M,1-\delta_{2,n}],R[0,\gamma_M,1-\delta_{2,n}]) = (\tilde{c},\tilde{\eta},\tilde{\rho})\,.$$

**Proof.** We plan to apply the Implicit Function Theorem to equation

$$M[\epsilon,\epsilon_1,\epsilon_2,c,\eta,\rho]=0\,,$$

around $(0,\gamma_M,1-\delta_{2,n},\tilde{c},\tilde{\eta},\tilde{\rho})$. By assumption (29), and by standard properties of the elastic potentials (cf. *e.g.*, [2, Thm. A.2]) and by known properties of (nonsingular) integral operators (cf. *e.g.*, [11, Thm. 6.2]), we conclude that the map $M$ is real analytic. By definition of $(\tilde{c},\tilde{\eta},\tilde{\rho})$, we have $M[0,\gamma_M,1-\delta_{2,n},\tilde{c},\tilde{\eta},\tilde{\rho}]=0$. By standard Calculus in Banach space (see also [11, Prop. 6.3]), the differential of $M$ at the point

$(0, \gamma_M, 1-\delta_{2,n}, \tilde{c}, \tilde{\eta}, \tilde{\rho})$ with respect to the variable $(c, \eta, \rho)$ is delivered by the formula

$$\partial_{(c,\eta,\rho)} M_1[0, \gamma_M, 1-\delta_{2,n}, \tilde{c}, \tilde{\eta}, \tilde{\rho}](\overline{c}, \overline{\eta}, \overline{\rho}) = \int_{\partial\Omega^i} \overline{\eta}\, d\sigma + \int_{\partial\Omega^o} \overline{\rho}\, d\sigma\,,$$

$$\partial_{(c,\eta,\rho)} M_2[0, \gamma_M, 1-\delta_{2,n}, \tilde{c}, \tilde{\eta}, \tilde{\rho}](\overline{c}, \overline{\eta}, \overline{\rho}) = \frac{1}{2}\overline{\eta} + v_*[\omega, \overline{\eta}]$$
$$-\mathcal{G}^i \cdot \left\{(1-\delta_{2,n}) v[\omega, \overline{\eta}] + \frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i} \overline{\eta}\, d\sigma\right\} \quad \text{on } \partial\Omega^i\,,$$

$$\partial_{(c,\eta,\rho)} M_3[0, \gamma_M, 1-\delta_{2,n}, \tilde{c}, \tilde{\eta}, \tilde{\rho}](\overline{c}, \overline{\eta}, \overline{\rho})(t) = -\frac{1}{2}\overline{\rho}(t) + v_*[\omega, \overline{\rho}](t)$$
$$+\sum_{l=1}^{n}\left(\int_{\partial\Omega^i} \overline{\eta}_l\, d\sigma\right) T(\omega, D\Gamma_n^l(\omega,t))\nu^o(t)$$
$$-a^o(t) \cdot \left\{\Gamma_n(\omega,t)\int_{\partial\Omega^i} \overline{\eta}\, d\sigma + v[\omega, \overline{\rho}](t) + \overline{c}\right\} \qquad \forall t \in \partial\Omega^o\,,$$

for all $(\overline{c}, \overline{\eta}, \overline{\rho}) \in \mathbb{R}^n \times X_{m,\alpha}$. We now prove that $\partial_{(c,\eta,\rho)} M[0, \gamma_M, 1-\delta_{2,n}, \tilde{c}, \tilde{\eta}, \tilde{\rho}]$ is a linear homeomorphism of $\mathbb{R}^n \times X_{m,\alpha}$ onto itself. As a first step, we shall show that $\partial_{(c,\eta,\rho)} M[0, \gamma_M, 1-\delta_{2,n}, \tilde{c}, \tilde{\eta}, \tilde{\rho}]$ is a Fredholm operator of index 0. First we note that the map of $\mathbb{R}^n \times X_{m,\alpha}$ to itself which takes $(\overline{c}, \overline{\eta}, \overline{\rho})$ to

$$\Biggl(\int_{\partial\Omega^i} \overline{\eta}\, d\sigma + \int_{\partial\Omega^o} \overline{\rho}\, d\sigma, -\mathcal{G}^i \cdot \left\{\frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i} \overline{\eta}\, d\sigma\right\},$$
$$\sum_{l=1}^{n}\left(\int_{\partial\Omega^i} \overline{\eta}_l\, d\sigma\right) T(\omega, D\Gamma_n^l(\omega,t))\nu^o(t) - a^o(t) \cdot \left\{\Gamma_n(\omega,t)\int_{\partial\Omega^i} \overline{\eta}\, d\sigma + \overline{c}\right\}\Biggr)$$

is compact as a linear map with finite dimensional image. Then we note that the operator from $\mathbb{R}^n \times X_{m,\alpha}$ to itself which takes a triple $(\overline{c}, \overline{\eta}, \overline{\rho})$ to the triple $(0, (1-\delta_{2,n})v[\omega, \overline{\eta}], v[\omega, \overline{\rho}])$ is compact. Indeed, such a map has a range contained in $\mathbb{R}^n \times X_{m+1,\alpha}$, which is compactly imbedded in $\mathbb{R}^n \times X_{m,\alpha}$ (cf. *e.g.*, [2, Thm. A.2].) Then the map from $\mathbb{R}^n \times X_{m,\alpha}$ to itself which takes $(\overline{c}, \overline{\eta}, \overline{\rho})$ to $(0, -(1-\delta_{2,n})\mathcal{G}^i \cdot v[\omega, \overline{\eta}], -a^o \cdot v[\omega, \overline{\rho}])$ is compact. Hence, $\partial_{(c,\eta,\rho)} M[0, \gamma_M, 1-\delta_{2,n}, \tilde{c}, \tilde{\eta}, \tilde{\rho}]$ is a compact perturbation of the operator $\Lambda$ from $\mathbb{R}^n \times X_{m,\alpha}$ to itself defined by

$$\Lambda[\overline{c}, \overline{\eta}, \overline{\rho}] \equiv \left(0, \frac{1}{2}\overline{\eta} + v_*[\omega, \overline{\eta}], -\frac{1}{2}\overline{\rho} + v_*[\omega, \overline{\rho}]\right)\,.$$

Since a compact perturbation of a Fredholm operator of index 0 is a Fredholm operator of index 0, it suffices to show that $\Lambda$ is a Fredholm operator of index 0. Since $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega^i$ is connected, a classical result of potential theory implies that the second component of $\Lambda$ induces a homeomorphism from $C^{m-1,\alpha}(\partial\Omega^i, \mathbb{R}^n)$ onto itself (cf. *e.g.*, [2, Rmk. A.8].) By classical results, the third component of $\Lambda$ is Fredholm of index 0 in $C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n)$ (cf. *e.g.*, [2, Thm. A.9].) Hence, we immediately deduce that $\Lambda$ is Fredholm of index 0. Now that we have proved that $\partial_{(c,\eta,\rho)} M[0, \gamma_M, 1-\delta_{2,n}, \tilde{c}, \tilde{\eta}, \tilde{\rho}]$ is Fredholm of index 0, it suffices to show that its kernel is trivial. Thus we now assume that $(\overline{c}, \overline{\eta}, \overline{\rho}) \in \mathbb{R}^n \times X_{m,\alpha}$ and that

$$\partial_{(c,\eta,\rho)} M[0, \gamma_M, 1-\delta_{2,n}, \tilde{c}, \tilde{\eta}, \tilde{\rho}](\overline{c}, \overline{\eta}, \overline{\rho}) = 0\,, \tag{32}$$

and we prove that $(\overline{c}, \overline{\eta}, \overline{\rho}) = 0$.

If $n \geq 3$, we consider the second equation

$$\frac{1}{2}\overline{\eta} + v_*[\omega, \overline{\eta}] - \mathcal{G}^i \cdot v[\omega, \overline{\eta}] = 0\,,$$

of (32). Since $\mathcal{G}^i$ satisfies (8), (9), a classical result on the integral equations corresponding to an exterior linear traction boundary value problem implies that $\overline{\eta} = 0$ (cf. *e.g.*, [3, Thm. 2.2 (v)]).

If $n = 2$, we consider the second equation

$$\frac{1}{2}\overline{\eta} + v_*[\omega, \overline{\eta}] - \frac{1}{4\pi}\frac{\omega+2}{\omega+1}\mathcal{G}^i \int_{\partial\Omega^i} \overline{\eta}\, d\sigma = 0 \qquad \text{on } \partial\Omega^i\,, \tag{33}$$

of (32). By equality $\int_{\partial\Omega^i} v_*[\omega, \overline{\eta}]\, d\sigma = \frac{1}{2}\int_{\partial\Omega^i} \overline{\eta}\, d\sigma$, which follows by classical properties of elastic layer potentials, we obtain that

$$\left(I - \frac{\omega+2}{4\pi(\omega+1)}\int_{\partial\Omega^i} \mathcal{G}^i\, d\sigma\right)\left(\int_{\partial\Omega^i} \overline{\eta}\, d\sigma\right) = 0\,,$$

(cf. *e.g.*, [2, (A.7)].) Then by our assumption in case $n = 2$, we have $\int_{\partial\Omega^i} \overline{\eta}\, d\sigma = 0$. Next we go back to equality (33) and we obtain that $\frac{1}{2}\overline{\eta} + v_*[\omega, \overline{\eta}] = 0$, which as above implies that $\overline{\eta} = 0$. Thus both in cases $n \geq 3$ and $n = 2$, we have $\overline{\eta} = 0$. Hence, equality (32) implies that

$$\begin{cases} \int_{\partial\Omega^o} \overline{\rho}\, d\sigma = 0 \\ -\frac{1}{2}\overline{\rho} + v_*[\omega, \overline{\rho}] - a^o \cdot \{v[\omega, \overline{\rho}] + \overline{c}\} = 0 \quad \text{on } \partial\Omega^o\,, \end{cases}$$

which is an integral equation corresponding to a linear traction boundary value problem. Since $-a^o$ satisfies (8), (9), we can prove classically that $\overline{\rho} = 0$ and $\overline{c} = 0$ (cf. *e.g.*, [2, Thm. 2.2 (ii)]). Then we can invoke the Implicit Function Theorem and deduce the existence of $(C, E, R)$ as in the statement. □

In order to simplify the notation of (31), we introduce the function $\Xi_{M,n}$ from $]0, \epsilon'[$ to $\mathcal{U}_{\gamma_M}$ by setting

$$\Xi_{M,n}[\epsilon] \equiv \left(\frac{\gamma(\epsilon)}{\epsilon^{n-1}}, (\log\epsilon)^{-\delta_{2,n}}\right) \qquad \forall \epsilon \in ]0, \epsilon'[\,. \tag{34}$$

Theorem 3 enables us to introduce our family of solutions.

**Definition 1.** *Let the assumptions of Theorem 3 hold. Then we set*

$$u(\epsilon, t) \equiv u_\epsilon[C[\epsilon, \Xi_{M,n}[\epsilon]], E[\epsilon, \Xi_{M,n}[\epsilon]], R[\epsilon, \Xi_{M,n}[\epsilon]]](t) \qquad \forall t \in \operatorname{cl}\Omega(\epsilon)\,,$$

*for all* $\epsilon \in ]0, \epsilon'[$.

## 4 A functional analytic representation Theorem for the family $\{u(\epsilon,\cdot)\}_{\epsilon\in]0,\epsilon'[}$ and for its energy integral

**Theorem 4.** *Let the assumptions of Theorem 3 hold. Then the following statements hold.*

*(i) Let $\tilde{\Omega}$ be a bounded open subset of $\Omega^o\setminus\{0\}$ such that $0\notin \mathrm{cl}\,\tilde{\Omega}$. Then there exist $\epsilon_{\tilde{\Omega}}\in]0,\epsilon'[$ and a real analytic operator $U_{\tilde{\Omega}}$ from $]-\epsilon_{\tilde{\Omega}},\epsilon_{\tilde{\Omega}}[\times\mathcal{U}_{\gamma_M}$ to $C^{m,\alpha}(\mathrm{cl}\,\tilde{\Omega},\mathbb{R}^n)$ such that $\tilde{\Omega}\subseteq\Omega(\epsilon)$ for all $\epsilon\in]-\epsilon_{\tilde{\Omega}},\epsilon_{\tilde{\Omega}}[$ and such that*

$$u(\epsilon,t)_{|\mathrm{cl}\,\tilde{\Omega}}=\frac{\epsilon^{n-1}}{\gamma(\epsilon)}U_{\tilde{\Omega}}[\epsilon,\Xi_{M,n}[\epsilon]](t)\qquad\forall t\in\mathrm{cl}\,\tilde{\Omega}\,,$$

*for all $\epsilon\in]0,\epsilon_{\tilde{\Omega}}[$. Moreover,*

$$\lim_{\epsilon\to0}\frac{\gamma(\epsilon)}{\epsilon^{n-1}}u(\epsilon,t)=\Gamma_n(\omega,t)\int_{\partial\Omega^i}T(\omega,D\tilde{u}^i)\nu^i\,d\sigma+\tilde{u}^o(t)\qquad\forall t\in\mathrm{cl}\,\tilde{\Omega}\,,$$

*where $(\tilde{u}^i,\tilde{u}^o)$ is as in Theorem 3.*

*(ii) Let $U^{r,1}$ be the real analytic map from $]-\epsilon',\epsilon'[\times\mathcal{U}_{\gamma_M}$ to $\mathbb{R}^n$ defined by*

$$U^{r,1}[\epsilon,\epsilon_1,\epsilon_2]\equiv\frac{\omega+2}{4\pi(\omega+1)}\int_{\partial\Omega^i}E[\epsilon,\epsilon_1,\epsilon_2]\,d\sigma$$

*for all $(\epsilon,\epsilon_1,\epsilon_2)\in]-\epsilon',\epsilon'[\times\mathcal{U}_{\gamma_M}$. Let $\tilde{\Omega}$ be a bounded open subset of $\mathbb{R}^n\setminus\mathrm{cl}\,\Omega^i$. Then there exist $\epsilon_{\tilde{\Omega},r}\in]0,\epsilon'[$, and two real analytic functions $U^{r,j}_{\tilde{\Omega}}[\cdot,\cdot,\cdot]$ for $j=2$, $3$ from $]-\epsilon_{\tilde{\Omega},r},\epsilon_{\tilde{\Omega},r}[\times\mathcal{U}_{\gamma_M}$ to $C^{m,\alpha}(\mathrm{cl}\,\tilde{\Omega},\mathbb{R}^n)$ such that*

$$\tilde{\Omega}\subseteq\frac{1}{\epsilon}\Omega(\epsilon)\qquad\forall\epsilon\in]-\epsilon_{\tilde{\Omega},r},\epsilon_{\tilde{\Omega},r}[\setminus\{0\}\tag{35}$$

*and such that*

$$u(\epsilon,\epsilon t)=\frac{\epsilon}{\gamma(\epsilon)}\Big\{\delta_{2,n}U^{r,1}[\epsilon,\Xi_{M,n}[\epsilon]]\log\epsilon+U^{r,2}_{\tilde{\Omega}}[\epsilon,\Xi_{M,n}[\epsilon]](t)+U^{r,3}_{\tilde{\Omega}}[\epsilon,\Xi_{M,n}[\epsilon]](t)\epsilon^{n-2}\Big\}\qquad\forall t\in\mathrm{cl}\,\tilde{\Omega}\,,\tag{36}$$

*for all $\epsilon\in]0,\epsilon_{\tilde{\Omega},r}[$. Moreover,*

$$\begin{aligned}U^{r,1}[0,\gamma_M,1-\delta_{2,n}]&=\frac{\omega+2}{4\pi(\omega+1)}\int_{\partial\Omega^i}T(\omega,D\tilde{u}^i)\nu^i\,d\sigma\,,\\ U^{r,2}_{\tilde{\Omega}}[0,\gamma_M,1-\delta_{2,n}]&=\tilde{u}^i_{|\mathrm{cl}\,\tilde{\Omega}}\,,\\ U^{r,3}_{\tilde{\Omega}}[0,\gamma_M,1-\delta_{2,n}]&=\tilde{u}^o(0)\,,\end{aligned}\tag{37}$$

*where $(\tilde{u}^i,\tilde{u}^o)$ is as in Theorem 3 and*

$$\lim_{\epsilon\to0}\frac{\gamma(\epsilon)}{\epsilon}(\log\epsilon)^{-\delta_{2,n}}u(\epsilon,\epsilon\,\cdot\,)=\frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i}T(\omega,D\tilde{u}^i)\nu^i\,d\sigma+(1-\delta_{2,n})\tilde{u}^i_{|\mathrm{cl}\,\tilde{\Omega}}(\cdot)\,,$$

*in $C^{m,\alpha}(\mathrm{cl}\,\tilde{\Omega},\mathbb{R}^n)$.*

**Proof.** We first consider statement (i). Let $\epsilon^*_{\tilde{\Omega}} \in ]0, \epsilon'[$ be such that $\tilde{\Omega} \subseteq \Omega(\epsilon)$ for all $\epsilon \in [-\epsilon^*_{\tilde{\Omega}}, \epsilon^*_{\tilde{\Omega}}]$. Let $\epsilon_{\tilde{\Omega}} \in ]0, \epsilon^*_{\tilde{\Omega}}[$ be such that $\epsilon \mathrm{cl}\, \Omega^i \subseteq \epsilon^*_{\tilde{\Omega}} \Omega^i$ for all $\epsilon \in [-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}]$. By definition of $u(\epsilon, \cdot)$, we have

$$\begin{aligned} u(\epsilon, t) \equiv \frac{\epsilon^{n-1}}{\gamma(\epsilon)} \int_{\partial\Omega^i} \Gamma_n(\omega, t - \epsilon s) E[\epsilon, \Xi_{M,n}[\epsilon]](s)\, d\sigma_s \\ + \frac{\epsilon^{n-1}}{\gamma(\epsilon)} \int_{\partial\Omega^o} \Gamma_n(\omega, t - s) R[\epsilon, \Xi_{M,n}[\epsilon]](s)\, d\sigma_s + \frac{\epsilon^{n-1}}{\gamma(\epsilon)} C[\epsilon, \Xi_{M,n}[\epsilon]]\,, \end{aligned}$$

for all $t \in \mathrm{cl}\, \Omega(\epsilon^*_{\tilde{\Omega}})$ and for all $\epsilon \in ]0, \epsilon_{\tilde{\Omega}}[$. Thus it is natural to define

$$\begin{aligned} U_{\Omega(\epsilon^*_{\tilde{\Omega}})}[\epsilon, \epsilon_1, \epsilon_2](t) \equiv \int_{\partial\Omega^i} \Gamma_n(\omega, t - \epsilon s) E[\epsilon, \epsilon_1, \epsilon_2](s)\, d\sigma_s \\ + \int_{\partial\Omega^i} \Gamma_n(\omega, t - s) R[\epsilon, \epsilon_1, \epsilon_2](s)\, d\sigma_s + C[\epsilon, \epsilon_1, \epsilon_2] \qquad \forall t \in \mathrm{cl}\, \Omega(\epsilon^*_{\tilde{\Omega}})\,, \end{aligned} \tag{38}$$

for all $(\epsilon, \epsilon_1, \epsilon_2) \in ]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[ \times \mathcal{U}_{\gamma_M}$. Thus we are now reduced to show that the right hand side of (38) defines a real analytic operator from $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[ \times \mathcal{U}_{\gamma_M}$ to $C^{m,\alpha}(\mathrm{cl}\, \Omega(\epsilon^*_{\tilde{\Omega}}), \mathbb{R}^n)$. Indeed, $\tilde{\Omega} \subseteq \Omega(\epsilon^*_{\tilde{\Omega}})$ and thus we can take $U_{\tilde{\Omega}}$ equal to $U_{\Omega(\epsilon^*_{\tilde{\Omega}})}$ composed with the restriction operator from $\mathrm{cl}\, \Omega(\epsilon^*_{\tilde{\Omega}})$ to $\mathrm{cl}\, \tilde{\Omega}$. Since $\mathrm{cl}\, \Omega(\epsilon^*_{\tilde{\Omega}}) \subseteq \mathrm{cl}\, \Omega^o$, known regularity properties of the elastic layer potentials (cf. *e.g.*, [2, Thm. A.2]), and the real analyticity of $R$ imply that the map from $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[ \times \mathcal{U}_{\gamma_M}$ to $C^{m,\alpha}(\mathrm{cl}\, \Omega^o, \mathbb{R}^n)$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to $v^+[\omega, R[\epsilon, \epsilon_1, \epsilon_2]]$ is real analytic. By standard properties of integral operators with real analytic kernel and with no singularity (see also [11, Prop. 6.1]), the map from $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[ \times L^1(\partial\Omega^i, \mathbb{R}^n)$ to $C^{m+1}(\mathrm{cl}\, \Omega(\epsilon^*_{\tilde{\Omega}}), \mathbb{R}^n)$ which takes $(\epsilon, f)$ to the function $\int_{\partial\Omega^i} \Gamma_n(\omega, t - \epsilon s) f(s)\, d\sigma_s$ of the variable $t \in \mathrm{cl}\, \Omega(\epsilon^*_{\tilde{\Omega}})$ is real analytic. Since $E$ is real analytic from $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[ \times \mathcal{U}_{\gamma_M}$ to $C^{m-1,\alpha}(\partial\Omega^i, \mathbb{R}^n)$ and since $C^{m-1,\alpha}(\partial\Omega^i, \mathbb{R}^n)$ is continuously imbedded into the space $L^1(\partial\Omega^i, \mathbb{R}^n)$ and $C^{m+1}(\mathrm{cl}\, \Omega(\epsilon^*_{\tilde{\Omega}}), \mathbb{R}^n)$ is continuously imbedded into $C^{m,\alpha}(\mathrm{cl}\, \Omega(\epsilon^*_{\tilde{\Omega}}), \mathbb{R}^n)$, we conclude that the function from $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[ \times \mathcal{U}_{\gamma_M}$ to $C^{m,\alpha}(\mathrm{cl}\, \Omega(\epsilon^*_{\tilde{\Omega}}), \mathbb{R}^n)$ which takes a triple $(\epsilon, \epsilon_1, \epsilon_2)$ to the function

$$\int_{\partial\Omega^i} \Gamma_n(\omega, t - \epsilon s) E[\epsilon, \epsilon_1, \epsilon_2](s)\, d\sigma_s$$

of the variable $t \in \mathrm{cl}\, \Omega(\epsilon^*_{\tilde{\Omega}})$ is real analytic. Also, Theorem 3 and definition (38) imply that

$$\begin{aligned} U_{\Omega(\epsilon^*_{\tilde{\Omega}})}[0, \gamma_M, 1 - \delta_{2,n}](t) &= \Gamma_n(\omega, t) \int_{\partial\Omega^i} E[0, \gamma_M, 1 - \delta_{2,n}](s)\, d\sigma_s \\ &\qquad + v[\omega, R[0, \gamma_M, 1 - \delta_{2,n}]](t) + C[0, \gamma_M, 1 - \delta_{2,n}] \\ &= \Gamma_n(\omega, t) \int_{\partial\Omega^i} T(\omega, D\tilde{u}^i(y)) \nu^i(y)\, d\sigma_y + \tilde{u}^o(t) \quad \forall t \in \mathrm{cl}\, \Omega(\epsilon^*_{\tilde{\Omega}})\,, \end{aligned}$$

where $(\tilde{u}^i, \tilde{u}^o)$ is the solution of the limiting problem (24) of Theorem 3. Here $v[\omega, R[0, \gamma_M, 1 - \delta_{2,n}]]$ denotes the simple elastic layer potential associated to $R[0, \gamma_M, 1 - \delta_{2,n}]$.

We now prove statement (ii). Let $\epsilon^*_{\tilde{\Omega},r} \in ]0,\epsilon'[$ be such that $\mathrm{cl}\,\tilde{\Omega} \subseteq \frac{1}{\epsilon}\Omega^o$ for all $\epsilon \in [-\epsilon^*_{\tilde{\Omega},r}, \epsilon^*_{\tilde{\Omega},r}] \setminus \{0\}$. Since the restriction map from $C^{m,\alpha}\left(\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\mathrm{cl}\,\Omega(\epsilon^*_{\tilde{\Omega},r}), \mathbb{R}^n\right)$ to $C^{m,\alpha}(\mathrm{cl}\,\tilde{\Omega}, \mathbb{R}^n)$ is linear and continuous, it clearly suffices to construct first $U^{r,j}_{\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\Omega(\epsilon^*_{\tilde{\Omega},r})}$ and then to define $U^{r,j}_{\tilde{\Omega}}$ for $j = 2, 3$ to be the composition of $U^{r,j}_{\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\Omega(\epsilon^*_{\tilde{\Omega},r})}$ with the restriction operator from $\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\mathrm{cl}\,\Omega(\epsilon^*_{\tilde{\Omega},r})$ to $\mathrm{cl}\,\tilde{\Omega}$ for $\epsilon$ ranging in a possibly smaller interval. Let $\tilde{R} > 0$ be such that $\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\Omega^o \subseteq \mathbb{B}_n(0, \tilde{R}/2)$. Let $\epsilon_{\tilde{\Omega},r} \in ]0, \epsilon^*_{\tilde{\Omega},r}[$ be such that $\mathrm{cl}\,\mathbb{B}_n(0, \tilde{R}\epsilon_{\tilde{\Omega},r}) \subseteq \Omega^o$. Clearly,

$$\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\Omega^o \subseteq \mathbb{B}_n(0, \tilde{R}/2) \subseteq \mathbb{B}_n(0, \tilde{R}) \subseteq \mathrm{cl}\,\mathbb{B}_n(0, \tilde{R}) \subseteq \frac{1}{\epsilon}\Omega^o$$

for all $\epsilon \in [-\epsilon_{\tilde{\Omega},r}, \epsilon_{\tilde{\Omega},r}] \setminus \{0\}$. By definition of $u(\epsilon, \cdot)$, we have

$$\begin{aligned} u(\epsilon, \epsilon t) = {} & \delta_{2,n}\frac{\epsilon^{n-1}\log\epsilon}{\gamma(\epsilon)}\frac{\omega+2}{4\pi(\omega+1)}\int_{\partial\Omega^i} E[\epsilon, \Xi_{M,n}[\epsilon]]\,d\sigma \\ & + \frac{\epsilon}{\gamma(\epsilon)}v[\omega, E[\epsilon, \Xi_{M,n}[\epsilon]]](t) + \frac{\epsilon^{n-1}}{\gamma(\epsilon)}\left(v[\omega, R[\epsilon, \Xi_{M,n}[\epsilon]]](\epsilon t) + C[\epsilon, \Xi_{M,n}[\epsilon]]\right) \end{aligned}$$

for all $t \in \frac{1}{\epsilon^*_{\tilde{\Omega},r}}\mathrm{cl}\,\Omega(\epsilon^*_{\tilde{\Omega},r})$ and for all $\epsilon \in ]-\epsilon_{\tilde{\Omega},r}, \epsilon_{\tilde{\Omega},r}[$. Then it is natural to set

$$\begin{aligned} U^{r,2}_{\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\Omega(\epsilon^*_{\tilde{\Omega},r})}[\epsilon, \epsilon_1, \epsilon_2](t) &\equiv v[\omega, E[\epsilon, \epsilon_1, \epsilon_2]](t)\,, \\ U^{r,3}_{\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\Omega(\epsilon^*_{\tilde{\Omega},r})}[\epsilon, \epsilon_1, \epsilon_2](t) &\equiv v[\omega, R[\epsilon, \epsilon_1, \epsilon_2]](\epsilon t) + C[\epsilon, \epsilon_1, \epsilon_2]\,, \end{aligned}$$

for all $t \in \frac{1}{\epsilon^*_{\tilde{\Omega},r}}\mathrm{cl}\,\Omega(\epsilon^*_{\tilde{\Omega},r})$ and for all $(\epsilon, \epsilon_1, \epsilon_2) \in ]-\epsilon_{\tilde{\Omega},r}, \epsilon_{\tilde{\Omega},r}[\times\mathcal{U}_{\gamma_M}$. By the real analyticity of $E$ and by known properties of the elastic simple layer potential (cf. *e.g.*, [2, Thm. A.2]), we deduce that the map $U^{r,2}_{\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\Omega(\epsilon^*_{\tilde{\Omega},r})}$ is real analytic from $]-\epsilon_{\tilde{\Omega},r}, \epsilon_{\tilde{\Omega},r}[\times\mathcal{U}_{\gamma_M}$ to $C^{m,\alpha}(\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\mathrm{cl}\,\Omega(\epsilon^*_{\tilde{\Omega},r}), \mathbb{R}^n)$. We now prove that $U^{r,3}_{\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\Omega(\epsilon^*_{\tilde{\Omega},r})}$ is real analytic. By standard properties of elastic single layer potentials and by the real analyticity of $R[\cdot,\cdot,\cdot]$, the map which takes $(\epsilon, \epsilon_1, \epsilon_2)$ in $]-\epsilon_{\tilde{\Omega},r}, \epsilon_{\tilde{\Omega},r}[\times\mathcal{U}_{\gamma_M}$ to $v[\omega, R[\epsilon, \epsilon_1, \epsilon_2]]$ in the space

$$C^0_{L[\omega]}(\mathrm{cl}\,\Omega^o, \mathbb{R}^n) \equiv \{u \in C^0(\mathrm{cl}\,\Omega^o, \mathbb{R}^n) \cap C^2(\Omega^o, \mathbb{R}^n) : \, L[\omega](u) = 0\}$$

endowed with the sup-norm is real analytic. By an analyticity result on the composition operator which is a variant of a result due to Preciso [19] (see [3, Prop. 6.2 of the Appendix]), the map from $]-\epsilon_{\tilde{\Omega},r}, \epsilon_{\tilde{\Omega},r}[\times\mathcal{U}_{\gamma_M}$ to the space $C^0_{L[\omega]}(\mathrm{cl}\,\mathbb{B}_n(0, \tilde{R}), \mathbb{R}^n)$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to the map $v^+[\omega, R[\epsilon, \epsilon_1, \epsilon_2]](\epsilon t)$ of the variable $t \in \mathrm{cl}\,\mathbb{B}_n(0, \tilde{R})$ is real analytic. By classical interior estimates for the solutions of equation $L[\omega]u = 0$, one can easily see that the restriction operator from $C^0_{L[\omega]}(\mathrm{cl}\,\mathbb{B}_n(0, \tilde{R}), \mathbb{R}^n)$ to

$C^{m,\alpha}(\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\mathrm{cl}\,\Omega(\epsilon^*_{\tilde{\Omega},r}),\mathbb{R}^n)$ is real analytic (cf. *e.g.*, [3, Theorem 6.1 of the Appendix].) Then the map which takes a triple $(\epsilon,\epsilon_1,\epsilon_2)$ to the function $v^+[\omega,R[\epsilon,\epsilon_1,\epsilon_2]](\epsilon t)$ of $t\in\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\mathrm{cl}\,\Omega(\epsilon^*_{\tilde{\Omega},r})$ is real analytic from $]-\epsilon_{\tilde{\Omega},r},\epsilon_{\tilde{\Omega},r}[\times\mathcal{U}_{\gamma_M}$ to $C^{m,\alpha}(\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\mathrm{cl}\,\Omega(\epsilon^*_{\tilde{\Omega},r}),\mathbb{R}^n)$. Then by the real analyticity of $C$, we deduce that $U^{r,3}_{\frac{1}{\epsilon^*_{\tilde{\Omega},r}}\Omega(\epsilon^*_{\tilde{\Omega},r})}$ is real analytic. By Theorems 2 and 3 and by equality

$$\int_{\partial\Omega^i}T(\omega,D\tilde{u}^i)\nu^i\,d\sigma=\int_{\partial\Omega^i}\left\{\frac{1}{2}\tilde{\eta}+v_*[\omega,\tilde{\eta}]\right\}d\sigma=\int_{\partial\Omega^i}\tilde{\eta}\,d\sigma\,,$$

we are ready to deduce the validity of the equalities in (37). □

We now consider the energy integral of the family $\{u(\epsilon,\cdot)\}_{\epsilon\in]0,\epsilon'[}$, and we prove the following.

**Theorem 5.** *Let the assumptions of Theorem 3 hold. Then there exist* $\tilde{\epsilon}\in]0,\epsilon'[$ *and a real analytic operator* $\mathcal{F}[\cdot,\cdot,\cdot]$ *from* $]-\tilde{\epsilon},\tilde{\epsilon}[\times\mathcal{U}_{\gamma_M}$ *to* $\mathbb{R}$ *such that*

$$\frac{1}{2}\int_{\Omega(\epsilon)}\mathrm{tr}\Big(T(\omega,D_xu(\epsilon,x))(D_xu(\epsilon,x))^t\Big)\,dx=\frac{\epsilon^n}{\gamma^2(\epsilon)}(\log\epsilon)^{\delta_{2,n}}\mathcal{F}[\epsilon,\Xi_{M,n}[\epsilon]]\,,\qquad(39)$$

*for all* $\epsilon\in]0,\tilde{\epsilon}[$ *(cf. (34).) Moreover,*

$$\begin{aligned}\mathcal{F}[0,\gamma_M,1-\delta_{2,n}]=&-\delta_{2,n}\frac{\omega+2}{8\pi(\omega+1)}\left|\int_{\partial\Omega^i}T(\omega,D\tilde{u}^i)\nu^i\,d\sigma\right|^2\\&+(1-\delta_{2,n})\frac{1}{2}\int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega^i}\mathrm{tr}\big(T(\omega,D\tilde{u}^i)(D\tilde{u}^i)^t\big)\,dx\,,\end{aligned}\qquad(40)$$

*where* $(\tilde{u}^i,\tilde{u}^o)$ *is a solution of the limiting boundary value problem (24) satisfying the assumptions of Theorem 3.*

**Proof.** By the Divergence Theorem, we have

$$\begin{aligned}&\int_{\Omega(\epsilon)}\mathrm{tr}\left(T(\omega,D_xu(\epsilon,x))(D_xu(\epsilon,x))^t\right)dx\\&=-\int_{\epsilon\partial\Omega^i}u^t(\epsilon,s)T(\omega,D_su(\epsilon,s))\nu_{\epsilon\Omega^i}(s)\,d\sigma_s\\&\quad+\int_{\partial\Omega^o}u^t(\epsilon,s)T(\omega,D_su(\epsilon,s))\nu_{\Omega^o}(s)\,d\sigma_s\\&=\frac{1}{\gamma(\epsilon)}\int_{\epsilon\partial\Omega^i}u^t(\epsilon,s)G^i\left(s/\epsilon,\frac{\gamma(\epsilon)}{\epsilon(\log\epsilon)^{\delta_{2,n}}}u(\epsilon,s)\right)d\sigma_s\\&\quad+\int_{\partial\Omega^o}u^t(\epsilon,s)\{a^o(s)\cdot u(\epsilon,s)+g(s)\}\,d\sigma_s\\&=\frac{\epsilon^{n-1}}{\gamma(\epsilon)}\int_{\partial\Omega^i}u^t(\epsilon,\epsilon s)G^i\left(s,\frac{\gamma(\epsilon)}{\epsilon(\log\epsilon)^{\delta_{2,n}}}u(\epsilon,\epsilon s)\right)d\sigma_s\\&\quad+\int_{\partial\Omega^o}u^t(\epsilon,s)a^o(s)u(\epsilon,s)\,d\sigma_s+\int_{\partial\Omega^o}u^t(\epsilon,s)g(s)\,d\sigma_s\end{aligned}$$

for all $\epsilon \in ]0,\epsilon'[$. Hence it suffices to take $\tilde{\epsilon} \equiv \min\left\{\epsilon_{\Omega(\epsilon')}, \epsilon_{\frac{1}{\epsilon'}\Omega(\epsilon'),r}\right\}$ and to set

$$\begin{aligned}\mathcal{F}[\epsilon,\epsilon_1,\epsilon_2] \equiv \delta_{2,n}\frac{1}{2}\int_{\partial\Omega^i} &(U^{r,1}[\epsilon,\epsilon_1,\epsilon_2])^t\hat{G}^i[\epsilon,\epsilon_1,\epsilon_2]\,d\sigma\\ +\epsilon_2\frac{1}{2}\int_{\partial\Omega^i} &(U^{r,2}_{\frac{1}{\epsilon'}\Omega(\epsilon')}[\epsilon,\epsilon_1,\epsilon_2])^t\hat{G}^i[\epsilon,\epsilon_1,\epsilon_2]\,d\sigma\\ +\epsilon^{n-2}\epsilon_2\frac{1}{2}\int_{\partial\Omega^i} &(U^{r,3}_{\frac{1}{\epsilon'}\Omega(\epsilon')}[\epsilon,\epsilon_1,\epsilon_2])^t\hat{G}^i[\epsilon,\epsilon_1,\epsilon_2]\,d\sigma\\ +\epsilon^{n-2}\epsilon_2\frac{1}{2}\int_{\partial\Omega^o} &(U_{\Omega(\epsilon')}[\epsilon,\epsilon_1,\epsilon_2])^t\cdot a^o\cdot U_{\Omega(\epsilon')}[\epsilon,\epsilon_1,\epsilon_2]\,d\sigma\\ +\epsilon^{n-2}\epsilon_1\epsilon_2\frac{1}{2}\int_{\partial\Omega^o} &(U_{\Omega(\epsilon')}[\epsilon,\epsilon_1,\epsilon_2])^t g\,d\sigma\,,\end{aligned}$$

where $\hat{G}^i$ is defined by

$$\begin{aligned}\hat{G}^i[\epsilon,\epsilon_1,\epsilon_2](t) \equiv G^i\Bigg(&t,\delta_{2,n}U^{r,1}[\epsilon,\epsilon_1,\epsilon_2](t)\\ &+\epsilon_2 U^{r,2}_{\frac{1}{\epsilon'}\Omega(\epsilon')}[\epsilon,\epsilon_1,\epsilon_2](t)+\epsilon^{n-2}\epsilon_2 U^{r,3}_{\frac{1}{\epsilon'}\Omega(\epsilon')}[\epsilon,\epsilon_1,\epsilon_2](t)\Bigg)\quad \forall t\in\partial\Omega^i\,,\end{aligned}$$

for all $(\epsilon,\epsilon_1,\epsilon_2) \in ]-\tilde{\epsilon},\tilde{\epsilon}[\times\mathcal{U}_{\gamma_M}$. We note that Theorems 3, 4 and assumption (29) ensure that $\hat{G}^i[\epsilon,\epsilon_1,\epsilon_2]$ depends real analytically upon $(\epsilon,\epsilon_1,\epsilon_2)$.

Then by Theorem 4, we easily compute that

$$\begin{aligned}&\mathcal{F}[0,\gamma_M,1-\delta_{2,n}] \\ &\quad=\delta_{2,n}\frac{\omega+2}{8\pi(\omega+1)}\left(\int_{\partial\Omega^i}T(\omega,D\tilde{u}^i)\nu^i\,d\sigma\right)^t\int_{\partial\Omega^i}\hat{G}^i[0,\gamma_M,1-\delta_{2,n}]\,d\sigma\\ &\quad\quad+(1-\delta_{2,n})\frac{1}{2}\int_{\partial\Omega^i}(\tilde{u}^i)^t\hat{G}^i[0,\gamma_M,1-\delta_{2,n}]\,d\sigma\,.\end{aligned}\tag{41}$$

By the second equation of the limiting boundary value problem (24) with $u^i=\tilde{u}^i$, $u^o=\tilde{u}^o$ and by Theorem 4, we deduce that

$$\hat{G}^i[0,\gamma_M,1-\delta_{2,n}] = -T(\omega,D\tilde{u}^i)\nu^i\,.\tag{42}$$

Now let $R>0$ be such that $\mathrm{cl}\,\Omega^i\subseteq\mathbb{B}_n(0,R)$. By applying the Divergence Theorem to $\tilde{u}^i$ on $\mathbb{B}_n(0,R)\setminus\mathrm{cl}\,\Omega^i$, we obtain that

$$\begin{aligned}&\int_{\mathbb{B}_n(0,R)\setminus\mathrm{cl}\,\Omega^i}\mathrm{tr}\Big(T(\omega,D\tilde{u}^i)(D\tilde{u}^i)^t\Big)\,dx\\ &\qquad=-\int_{\partial\Omega^i}(\tilde{u}^i)^tT(\omega,D\tilde{u}^i)\nu^i\,d\sigma+\int_{\partial\mathbb{B}_n(0,R)}(\tilde{u}^i)^tT(\omega,D\tilde{u}^i)\nu_{\mathbb{B}_n(0,R)}\,d\sigma\,.\end{aligned}$$

Now by taking the limit as $R$ tends to infinity and by exploiting the third and fourth inequalities of (24), we obtain that

$$\int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega^i}\mathrm{tr}\Big(T(\omega,D\tilde{u}^i)(D\tilde{u}^i)^t\Big)\,dx=-\int_{\partial\Omega^i}(\tilde{u}^i)^tT(\omega,D\tilde{u}^i)\nu^i\,d\sigma\qquad\text{if } n\geq 3\,.\tag{43}$$

By equalities (41)–(43), we deduce immediately the validity of (40). □

## 5 Local uniqueness for the family $\{u(\epsilon,\cdot)\}_{\epsilon\in]0,\epsilon'[}$

We now show by means of the following theorem, that the family $\{u(\epsilon,\cdot)\}_{\epsilon\in]0,\epsilon'[}$ is locally essentially unique.

**Theorem 6.** *Let the assumptions of Theorem 3 hold. If $\{\varepsilon_j\}_{j\in\mathbb{N}}$ is a sequence of $]0,\epsilon_0[$ converging to 0 and if $\{u_j\}_{j\in\mathbb{N}}$ is a sequence of functions such that*

$$
\begin{aligned}
&u_j \in C^{m,\alpha}(\mathrm{cl}\,\Omega(\varepsilon_j),\mathbb{R}^n)\,,\\
&u_j \text{ solves (6) for } \epsilon=\varepsilon_j\,,\\
&\lim_{j\to\infty}\gamma(\varepsilon_j)\varepsilon_j^{-1}(\log\varepsilon_j)^{-\delta_{2,n}}u_j(\varepsilon_j\,\cdot\,)_{|\partial\Omega^i}\\
&\quad=(1-\delta_{2,n})\tilde{u}^i(\cdot)+\tfrac{\delta_{2,n}}{4\pi}\tfrac{\omega+2}{\omega+1}\int_{\partial\Omega^i}T(\omega,D\tilde{u}^i)\nu^i\,d\sigma\quad\text{in } C^{m-1,\alpha}(\partial\Omega^i,\mathbb{R}^n)\,,
\end{aligned}
\tag{44}
$$

*then there exists $j_0\in\mathbb{N}$ such that $u_j(\cdot)=u(\varepsilon_j,\cdot)$ for all $j\geq j_0$.*

**Proof.** Since $u_j$ solves (6), Theorem 1 ensures that there exist $(c_j,\eta_j,\rho_j)$ and $(\tilde{c},\tilde{\eta},\tilde{\rho})$ in $\mathbb{R}^n\times X_{m,\alpha}$ such that

$$
M[\varepsilon_j,\Xi_{M,n}[\varepsilon_j],c_j,\eta_j,\rho_j]=0\,,\qquad M[0,\gamma_M,1-\delta_{2,n},\tilde{c},\tilde{\eta},\tilde{\rho}]=0\,,
$$

and that

$$
u_j=v^+[\omega,\mu_j]+d_j\,,\qquad \tilde{u}^i=v^-[\omega,\tilde{\eta}]\,,\qquad \tilde{u}^o=v^+[\omega,\tilde{\rho}]+\tilde{c},
$$

where

$$
\begin{aligned}
&\mu_j(y)=\frac{\varepsilon_j^{n-1}}{\gamma(\varepsilon_j)}\rho_j(y)\quad\text{if } y\in\partial\Omega^o\,,\quad \mu_j(y)=\frac{1}{\gamma(\varepsilon_j)}\eta_j(y/\varepsilon_j)\quad\text{if } y\in\varepsilon_j\partial\Omega^i\,,\\
&d_j=\frac{\varepsilon_j^{n-1}}{\gamma(\varepsilon_j)}c_j\,.
\end{aligned}
$$

We now rewrite equation $M[\epsilon,\epsilon_1,\epsilon_2,c,\eta,\rho]=0$ in the following form

$$
\begin{aligned}
&M_1[\epsilon,\epsilon_1,\epsilon_2,c,\eta,\rho]=0\,,\\
&\frac{1}{2}\eta(t)+v_*[\omega,\eta](t)+\epsilon^{n-1}\int_{\partial\Omega^o}\sum_{l=1}^n\rho_l(s)T(\omega,D_\xi\Gamma_n^l(\omega,\epsilon t-s))\nu^i(t)\,d\sigma_s\\
&\quad-\mathcal{G}^i(t)\cdot\left\{\epsilon_2 v[\omega,\eta](t)+\frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i}\eta\,d\sigma+\epsilon^{n-2}\epsilon_2 v[\omega,\rho](\epsilon t)+\epsilon^{n-2}\epsilon_2 c\right\}\\
&\quad=-G^i\left(t,\epsilon_2 v[\omega,\eta](t)+\frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i}\eta\,d\sigma+\epsilon^{n-2}\epsilon_2 v[\omega,\rho](\epsilon t)+\epsilon^{n-2}\epsilon_2 c\right)\\
&\qquad-\mathcal{G}^i(t)\cdot\left\{\epsilon_2 v[\omega,\eta](t)+\frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i}\eta\,d\sigma+\epsilon^{n-2}\epsilon_2 v[\omega,\rho](\epsilon t)+\epsilon^{n-2}\epsilon_2 c\right\}\\
&\hspace{20em}\forall t\in\partial\Omega^i,\\
&-\frac{1}{2}\rho(t)+v_*[\omega,\rho](t)+\int_{\partial\Omega^i}\sum_{l=1}^n\eta_l(s)T(\omega,D_\xi\Gamma_n^l(\omega,t-\epsilon s))\nu^o(t)\,d\sigma_s\\
&\quad-a^o(t)\left\{\int_{\partial\Omega^i}\Gamma_n(\omega,t-\epsilon s)\eta(s)\,d\sigma_s+v[\omega,\rho](t)+c\right\}=\epsilon_1 g(t)\qquad\forall t\in\partial\Omega^o.
\end{aligned}
\tag{45}
$$

Next we denote by $N[\epsilon,\epsilon_1,\epsilon_2,c,\eta,\rho] \equiv (N_l[\epsilon,\epsilon_1,\epsilon_2,c,\eta,\rho])_{l=1,2,3}$ the function of the variable $(\epsilon,\epsilon_1,\epsilon_2,c,\eta,\rho)$ in $]-\epsilon_0,\epsilon_0[\times\mathbb{R}^{n+2}\times X_{m,\alpha}$ to $\mathbb{R}^n\times X_{m,\alpha}$ defined by $N_1\equiv M_1$ and such that $N_2$ and $N_3$ equal the left hand side of the second and the third equation in (45), respectively. Thus equation (45) can be rewritten as

$$\begin{aligned}
&N_1[\epsilon,\epsilon_1,\epsilon_2,c,\eta,\rho]=0 \qquad (46)\\
&N_2[\epsilon,\epsilon_1,\epsilon_2,c,\eta,\rho](t)\\
&\quad = -G^i\Big(t,\epsilon_2 v[\omega,\eta](t)+\frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i}\eta\,d\sigma+\epsilon^{n-2}\epsilon_2 v[\omega,\rho](\epsilon t)+\epsilon^{n-2}\epsilon_2 c\Big)\\
&\qquad -\mathcal{G}^i(t)\cdot\Big\{\epsilon_2 v[\omega,\eta](t)+\frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i}\eta\,d\sigma+\epsilon^{n-2}\epsilon_2 v[\omega,\rho](\epsilon t)+\epsilon^{n-2}\epsilon_2 c\Big\}\\
&\qquad\qquad \forall t\in\partial\Omega^i,\\
&N_3[\epsilon,\epsilon_1,\epsilon_2,c,\eta,\rho]=\epsilon_1 g \qquad \text{on } \partial\Omega^o.
\end{aligned}$$

By our assumption of analyticity of $F_{G^i}$, we can easily verify that the Fréchet differential of $F_{G^i}$ at a point $u\in C^{m-1,\alpha}(\partial\Omega^i,\mathbb{R}^n)$ is necessarily delivered by the formula

$$dF_{G^i}[u](v)=\sum_{l=1}^n F_{\partial_{\xi_l}G^i}[u]v_l \qquad \forall v\in C^{m-1,\alpha}(\partial\Omega^i,\mathbb{R}^n)\,,$$

and that $F_{\partial_{\xi_l}G^i}[u]\in C^{m-1,\alpha}(\partial\Omega^i,\mathbb{R}^n)$ for all $l\in\{1,\dots,n\}$ (see [11, Prop. 6.3], where the scalar case has been worked out, but the proof is the same for vector valued functions.) Hence, $\mathcal{G}^i$ must be an element of $C^{m-1,\alpha}(\partial\Omega^i,M_n(\mathbb{R}))$ (cf. (30).) Then by standard properties of integral operators with a real analytic kernel and with no singularity (see [11, Thm. 6.2]), and by standard properties of elastic layer potentials (cf. *e.g.*, [2, Thm. A.2]), the map $N$ is real analytic. Next, we note that $N[\epsilon,\epsilon_1,\epsilon_2,\cdot,\cdot,\cdot]$ is linear for all fixed $(\epsilon,\epsilon_1,\epsilon_2)\in]-\epsilon_0,\epsilon_0[\times\mathbb{R}^2$. Accordingly, the map from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}^2$ to $\mathcal{L}(\mathbb{R}^n\times X_{m,\alpha},\mathbb{R}^n\times X_{m,\alpha})$ which takes $(\epsilon,\epsilon_1,\epsilon_2)$ to $N[\epsilon,\epsilon_1,\epsilon_2,\cdot,\cdot,\cdot]$ is real analytic. Here $\mathcal{L}(\mathbb{R}^n\times X_{m,\alpha},\mathbb{R}^n\times X_{m,\alpha})$ denotes the space of linear and continuous operators from $\mathbb{R}^n\times X_{m,\alpha}$ to itself. We also note that

$$N[0,\gamma_M,1-\delta_{2,n},\cdot,\cdot,\cdot]=\partial_{(c,\eta,\rho)}M[0,\gamma_M,1-\delta_{2,n},\tilde c,\tilde\eta,\tilde\rho](\cdot,\cdot,\cdot)\,,$$

and thus that $N[0,\gamma_M,1-\delta_{2,n},\cdot,\cdot,\cdot]$ is a linear homeomorphism (see the proof of Theorem 3.) Since the set of linear homeomorphisms is open in the set of linear and continuous operators, and since the map which takes a linear invertible operator to its inverse is real analytic (cf. *e.g.*, Hille and Phillips [7, Thms. 4.3.2 and 4.3.4]), there exists an open neighborhood $\mathcal{W}$ of $(0,\gamma_M,1-\delta_{2,n})$ in $]-\epsilon_0,\epsilon_0[\times\mathbb{R}^2$ such that the map $(\epsilon,\epsilon_1,\epsilon_2)\mapsto N[\epsilon,\epsilon_1,\epsilon_2,\cdot,\cdot,\cdot]^{(-1)}$ is real analytic from $\mathcal{W}$ to $\mathcal{L}(\mathbb{R}^n\times X_{m,\alpha},\mathbb{R}^n\times X_{m,\alpha})$. Clearly, there exists $j_1\in\mathbb{N}$ such that $(\varepsilon_j,\Xi_{M,n}[\varepsilon_j])\in\mathcal{W}$ for all $j\geq j_1$. Since $M[\varepsilon_j,\Xi_{M,n}[\varepsilon_j],c_j,\eta_j,\rho_j]=0$, the invertibility of $N[\varepsilon_j,\Xi_{M,n}[\varepsilon_j],\cdot,\cdot,\cdot]$ and equality (46) guarantee that

$$\begin{aligned}
(c_j,\eta_j,\rho_j)=N\left[\varepsilon_j,\Xi_{M,n}[\varepsilon_j],\cdot,\cdot,\cdot\right]^{(-1)}\Big(&0,-F_{G^i}[\gamma(\varepsilon_j)\varepsilon_j^{-1}(\log\varepsilon_j)^{-\delta_{2,n}}u_j(\varepsilon_j\,\cdot)|_{\partial\Omega^i}]\\
&-\mathcal{G}^i\cdot\left(\gamma(\varepsilon_j)\varepsilon_j^{-1}(\log\varepsilon_j)^{-\delta_{2,n}}u_j(\varepsilon_j\,\cdot)|_{\partial\Omega^i}\right)\,,\,\frac{\gamma(\varepsilon_j)}{\varepsilon_j^{n-1}}g\Big)
\end{aligned}$$

for all $j \geq j_1$. By assumption (29), $F_{G^i}[\cdot]$ is continuous in $C^{m-1,\alpha}(\partial\Omega^i, \mathbb{R}^n)$. Then, by the third assumption in (44), we have

$$\begin{aligned} \lim_{j\to\infty} & -F_{G^i}[\gamma(\varepsilon_j)\varepsilon_j^{-1}(\log\varepsilon_j)^{-\delta_{2,n}}u_j(\varepsilon_j\,\cdot)_{|\partial\Omega^i}] \\ & -\mathcal{G}^i \cdot \left(\gamma(\varepsilon_j)\varepsilon_j^{-1}(\log\varepsilon_j)^{-\delta_{2,n}}u_j(\varepsilon_j\,\cdot)_{|\partial\Omega^i}\right) \\ & = -F_{G^i}\left[(1-\delta_{2,n})\tilde{u}^i(\cdot) + \frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i} T(\omega, D\tilde{u}^i)\nu^i\,d\sigma\right] \\ & -\mathcal{G}^i \cdot \left\{(1-\delta_{2,n})\tilde{u}^i(\cdot) + \frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i} T(\omega, D\tilde{u}^i)\nu^i\,d\sigma\right\} \end{aligned} \tag{47}$$

in $C^{m-1,\alpha}(\partial\Omega^i, \mathbb{R}^n)$. The analyticity of $(\epsilon, \epsilon_1, \epsilon_2) \mapsto N[\epsilon, \epsilon_1, \epsilon_2, \cdot, \cdot, \cdot]^{(-1)}$ guarantees that

$$\lim_{j\to\infty} N\left[\varepsilon_j, \Xi_{M,n}[\varepsilon_j], \cdot, \cdot, \cdot\right]^{(-1)} = N[0, \gamma_M, 1-\delta_{2,n}, \cdot, \cdot, \cdot]^{(-1)}\,, \tag{48}$$

in $\mathcal{L}(\mathbb{R}^n \times X_{m,\alpha}, \mathbb{R}^n \times X_{m,\alpha})$. Since the evaluation map from $\mathcal{L}(\mathbb{R}^n \times X_{m,\alpha}, \mathbb{R}^n \times X_{m,\alpha}) \times (\mathbb{R}^n \times X_{m,\alpha})$ to $\mathbb{R}^n \times X_{m,\alpha}$, which takes a pair $(A, v)$ to $A[v]$ is bilinear and continuous, the limiting relations of (47) and (48) imply that

$$\begin{aligned} \lim_{j\to\infty} & (c_j, \eta_j, \rho_j) \\ = & \lim_{j\to\infty} N\left[\varepsilon_j, \Xi_{M,n}[\varepsilon_j], \cdot, \cdot, \cdot\right]^{(-1)} \Big(0, -F_{G^i}[\gamma(\varepsilon_j)\varepsilon_j^{-1}(\log\varepsilon_j)^{-\delta_{2,n}}u_j(\varepsilon_j\,\cdot)_{|\partial\Omega^i}] \\ & \qquad -\mathcal{G}^i \cdot \left(\gamma(\varepsilon_j)\varepsilon_j^{-1}(\log\varepsilon_j)^{-\delta_{2,n}}u_j(\varepsilon_j\,\cdot)_{|\partial\Omega^i}\right),\ \frac{\gamma(\varepsilon_j)}{\varepsilon_j^{n-1}}g\Big) \\ = & N[0, \gamma_M, 1-\delta_{2,n}, \cdot, \cdot, \cdot]^{(-1)} \\ & \Big(0, -F_{G^i}\left[(1-\delta_{2,n})\tilde{u}^i(\cdot) + \frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i} T(\omega, D\tilde{u}^i)\nu^i\,d\sigma\right] \\ & \quad -\mathcal{G}^i \cdot \left\{(1-\delta_{2,n})\tilde{u}^i(\cdot) + \frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i} T(\omega, D\tilde{u}^i)\nu^i\,d\sigma\right\}, \gamma_M g\Big) \end{aligned} \tag{49}$$

in $\mathbb{R}^n \times X_{m,\alpha}$. Since $M[0, \gamma_M, 1-\delta_{2,n}, \tilde{c}, \tilde{\eta}, \tilde{\rho}] = 0$, the right hand side of (49) equals $(\tilde{c}, \tilde{\eta}, \tilde{\rho})$. Hence,

$$\lim_{j\to\infty} (\varepsilon_j, \Xi_{M,n}[\varepsilon_j], c_j, \eta_j, \rho_j) = (0, \gamma_M, 1-\delta_{2,n}, \tilde{c}, \tilde{\eta}, \tilde{\rho})$$

in $\mathbb{R}^{n+3} \times X_{m,\alpha}$. Thus Theorem 3 implies that there exists $j_0 \in \mathbb{N}$ such that

$$c_j = C\left[\varepsilon_j, \Xi_{M,n}[\varepsilon_j]\right]\,, \qquad \eta_j = E\left[\varepsilon_j, \Xi_{M,n}[\varepsilon_j]\right]\,, \qquad \rho_j = R\left[\varepsilon_j, \Xi_{M,n}[\varepsilon_j]\right]\,,$$

for all $j \geq j_0$. Accordingly, $u_j(\cdot) = u(\varepsilon_j, \cdot)$ for $j \geq j_0$ (see Definition 1). □

## 6 A sufficient condition for the existence of solutions of the limiting boundary value problem

In Theorem 3, we have assumed that the boundary value problem (24) admits at least a solution satisfying certain conditions. Here, we present a sufficient condition on the data to ensure existence of such solutions.

By Theorem 2, it suffices to show that the integral equation in (23) has a solution $(\tilde{c}, \tilde{\eta}, \tilde{\rho}) \in \mathbb{R}^n \times X_{m,\alpha}$ such that the assumptions on the matrix valued function $\mathcal{G}^i$ of Theorem 3 with $\tilde{u}^i = v^-[\omega, \tilde{\eta}]$ are satisfied.

We collect in the following Theorem, some basic facts of classical potential theory.

**Theorem 7.** *Let $\alpha \in ]0,1[$, $\omega \in ]1-(2/n), +\infty[$, $p \in ]1, +\infty[$. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following statements hold.*

*(i) The operator from $L^p(\partial\Omega, \mathbb{R}^n)$ to itself which takes $\eta$ to $\frac{1}{2}\eta + v_*[\omega, \eta]$ is Fredholm of index 0.*

*(ii) If $\eta \in L^p(\partial\Omega, \mathbb{R}^n)$ and $\frac{1}{2}\eta + v_*[\omega, \eta] \in C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$, then $\eta$ belongs to $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$.*

*(iii) If $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$ is connected, then the operator $\frac{1}{2}I + v_*[\omega, \cdot]$ is a linear homeomorphism from $L^p(\partial\Omega, \mathbb{R}^n)$ onto itself and from $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ onto itself.*

*(iv) If $p > (n-1)/(1-\alpha)$, then the operator $v[\omega, \cdot]|_{\partial\Omega}$ is linear and continuous from the space $L^p(\partial\Omega, \mathbb{R}^n)$ to the space $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$.*

**Proof.** For proof of statement (i) in case $n = 2$, we refer to the book of Muskhelishvili [16, Ch. 19] (see also Kupradze [9, Ch. VIII, §§5–6].) For a proof in case $n \geq 3$, we refer to the book of Mikhlin and Prössdorf [15, Ch. XIV, §6], who actually worked out the proof for the case $n = 3$. However, the proof is the same for $n \geq 3$.

For a proof of statement (ii), we refer for example to Ševčenko [21, p. 929 of Engl. transl.] and to Mikhlin and Prössdorf [15, Ch. XIII, Thm. 7.1]. Statement (iii) follows by statements (i), (ii) and by the known form of the kernel of $\frac{1}{2}I + v_*[\omega, \cdot]$ (see also [2, Rmk. A.8].)

We now consider statement (iv). Let $\beta \in ]0, 1-\alpha[$. By exploiting the definition of $\Gamma_n(\omega, \xi)$, we note that there exists a positive constant $C > 0$ such that

$$\begin{aligned} |\Gamma_n(\omega, x-y)| &\leq \frac{C}{|x-y|^{n-2+\delta_{2,n}\beta}}, \\ |\Gamma_n(\omega, x-y) - \Gamma_n(\omega, x'-y)| &\leq \frac{C|x-x'|^\alpha}{\inf\{|x-y|, |x'-y|\}^{n-2+\alpha+\delta_{2,n}\beta}} \end{aligned}$$

for all $x, x', y \in \partial\Omega$ with $x \neq y$ and $x' \neq y$. Then the proof of statement (iv) can be deduced by the regularizing properties of the integral operators with a kernel of class $G_1(n-2+\delta_{2,n}\beta, \alpha)$ with $\beta \in ]0, 1-\alpha-p^{-1}[$ (cf., *e.g.*, Kupradze *et al.* [10, Ch. IV Theorem 2.6]) and by a standard argument based on the existence of a partition of unity subordinated to an atlas of $\partial\Omega$. □

**Lemma 1.** *Let $\alpha \in ]0,1[$, $p \in ]\frac{n-1}{1-\alpha}, +\infty[$, $\omega \in ]1-(2/n), +\infty[$, $m \in \mathbb{N} \setminus \{0\}$. Let $\Omega$ be as in (11). Let $B \in C^{m-1,\alpha}(\partial\Omega^i, M_n(\mathbb{R}))$ be such that*

$$\begin{aligned} \det\left(I - \frac{1}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i} B\, d\sigma\right) &\neq 0 \qquad \text{if } n = 2, \\ B \text{ satisfies (8) and (9) on } \partial\Omega \qquad &\text{if } n \geq 3. \end{aligned} \tag{50}$$

*Let $L_B$, $Q$ be the linear operators in $L^p(\partial\Omega, \mathbb{R}^n)$ defined by*

$$
\begin{aligned}
Q[\eta] &\equiv (1-\delta_{2,n})v[\omega,\eta] + \frac{\delta_{2,n}}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega^i} \eta\, d\sigma \qquad (51)\\
L_B[\eta] &\equiv \frac{1}{2}\eta + v_*[\omega,\eta] - B\cdot Q[\eta]
\end{aligned}
$$

*for all $\eta \in L^p(\partial\Omega, \mathbb{R}^n)$. Then the following statements hold*

*(i) The operator $L_B$ induces an isomorphism from $L^p(\partial\Omega, \mathbb{R}^n)$ onto itself.*

*(ii) The operator $L_B$ induces an isomorphism from $C^{j-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ onto itself for all $j \in \{1, \ldots, m\}$.*

**Proof.** We first consider statement (i). By Theorem 7, $L_B$ is a compact perturbation of a Fredholm operator of index 0 in $L^p(\partial\Omega, \mathbb{R}^n)$. Thus it suffices to show that $L_B$ is injective.

We first consider case $n \geq 3$. Let $\eta \in L^p(\partial\Omega, \mathbb{R}^n)$, $L_B[\eta] = 0$. By Theorem 7 (iv) and by our assumptions on $B$, we have $Q[\eta] \in C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ and thus $\frac{1}{2}\eta + v_*[\omega,\eta] = B\cdot Q[\eta] \in C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$. Then by Theorem 7 (ii), we have $\eta \in C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$. Since $v^-[\omega,\eta]$ solves the classical homogeneous exterior linear traction boundary value problem associated to equation $L_B[\eta] = 0$ and $B$ satisfies (8) and (9), we have $\eta = 0$ and thus $L_B$ is injective (cf. *e.g.*, [3, Th. 2.2 (v)].)

We now consider case $n = 2$. By integrating equality $L_B[\eta] = 0$ on $\partial\Omega$, we obtain that

$$
0 = \int_{\partial\Omega} L_B[\eta]\, d\sigma = \int_{\partial\Omega} \eta\, d\sigma - \frac{1}{4\pi}\frac{\omega+2}{\omega+1}\int_{\partial\Omega} B\, d\sigma \int_{\partial\Omega} \eta\, d\sigma\, .
$$

Hence, condition (50) for $n = 2$ implies that $\int_{\partial\Omega} \eta\, d\sigma = 0$ and thus $L_B[\eta] = \frac{1}{2}\eta + v_*[\omega,\eta] = 0$ and accordingly $\eta = 0$ (cf. *e.g.*, Theorem 7 (iii).)

To prove statement (ii), we note that $\frac{1}{2}I + v_*[\omega,\cdot]$ is a Fredholm operator of index 0 in $C^{j-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ (cf. *e.g.*, [2, Thm. A.9]) and that $Q[\cdot]$ is linear and continuous from $C^{j-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ to $C^{j,\alpha}(\partial\Omega, \mathbb{R}^n)$, which is compactly imbedded in $C^{j-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ (cf. *e.g.*, [2, Thm. A.2].) Then $L_B$ is a Fredholm operator of index 0 in $C^{j-1,\alpha}(\partial\Omega, \mathbb{R}^n)$. Since $L_B$ has been proved to be injective in $L^p(\partial\Omega, \mathbb{R}^n)$, $L_B$ is also injective on $C^{j-1,\alpha}(\partial\Omega, \mathbb{R}^n)$, and thus statement (ii) holds. □

**Theorem 8.** *Let $\alpha \in ]0,1[$, $\omega \in ]1-(2/n), +\infty[$, $m \in \mathbb{N}\setminus\{0\}$. Let $\Omega$ be as in (11). Let $v_0 \in \mathbb{R}^n$. Let $G \in C^0(\partial\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ satisfy the following conditions.*

*$F_G$ is continuous from $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ to itself and maps bounded sets of $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ to bounded sets of $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$,* (52)

*$F_G$ maps $C^{j-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ to itself for all $j \in \{1, \ldots, m\}$.* (53)

*If there exist $\delta \in ]0,1[$ and $B \in C^{m-1,\alpha}(\partial\Omega, M_n(\mathbb{R}))$ such that (50) holds and such that*

$$
C_{G,B} \equiv \sup_{(t,\xi)\in\partial\Omega\times\mathbb{R}^n} \frac{|G(t,\xi) + B(t)\xi|}{(1+|\xi|)^\delta} < \infty\, .
$$

*Then the integral equation*

$$\frac{1}{2}\eta + v_*[\omega, \eta] + F_G[Q[\eta] + \upsilon_0] = 0 \qquad \text{on } \partial\Omega\,, \tag{54}$$

*has at least a solution* $\eta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ *(see (51).)*

**Proof.** By Lemma 1 (ii), equation (54) can be rewritten in the form

$$\eta = L_B^{(-1)}\left[F_{\tilde{G}}\left[Q[\eta] + \upsilon_0\right] + B\upsilon_0\right] \tag{55}$$

where

$$\tilde{G}(t,\xi) \equiv -G(t,\xi) - B(t)\xi \qquad \forall (t,\xi) \in \partial\Omega \times \mathbb{R}^n\,.$$

We now show that equation (55) admits at least a solution in $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$. By classical results on single layer potentials, $Q[\cdot]$ is linear and continuous from $C^{j-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ to $C^{j,\alpha}(\partial\Omega, \mathbb{R}^n)$, which is compactly imbedded into the space $C^{j-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ for all $j \in \{1, \dots, m\}$ (cf. *e.g.*, [2, Thm. A.2].) Since we have assumed that $F_G$ is continuous in $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ and that $B$ belongs to $C^{m-1,\alpha}(\partial\Omega, M_n(\mathbb{R}))$, we can easily see that the map $L_B^{(-1)}\left[F_{\tilde{G}}\left[Q[\cdot] + \upsilon_0\right] + B\upsilon_0\right]$ is continuous and maps bounded sets into sets with a compact closure in $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ (see also Lemma 1 (ii).) Then we now turn to prove the existence for equation (55) by exploiting the Leray-Schauder degree on a ball centered at the origin and with sufficiently large radius in the Banach space $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$. To do so, we need to show that there exists $R > 0$ such that

$$(\lambda, \eta) \in [0,1] \times C^{0,\alpha}(\partial\Omega, \mathbb{R}^n),\ \eta - \lambda L_B^{(-1)}\left[F_{\tilde{G}}\left[Q[\eta] + \upsilon_0\right] + B\upsilon_0\right] = 0 \tag{56}$$

$$\Rightarrow\ \|\eta\|_{C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)} < R\,.$$

We first fix $p \in ]\frac{n-1}{1-\alpha}, +\infty[$. By Lemma 1 (i), $L_B^{(-1)}$ is linear and continuous in $L^p(\partial\Omega, \mathbb{R}^n)$. Then all solutions $\eta \in C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ of the equation in (56) must satisfy the inequality

$$\begin{aligned}
\|\eta\|_{L^p(\partial\Omega,\mathbb{R}^n)} &\leq \|L_B^{(-1)}\|_{\mathcal{L}(L^p(\partial\Omega,\mathbb{R}^n), L^p(\partial\Omega,\mathbb{R}^n))} \|F_{\tilde{G}}\left[Q[\eta] + \upsilon_0\right] + B\upsilon_0\|_{L^p(\partial\Omega,\mathbb{R}^n)} \\
&\leq \|L_B^{(-1)}\|_{\mathcal{L}(L^p(\partial\Omega,\mathbb{R}^n), L^p(\partial\Omega,\mathbb{R}^n))} (\mathrm{meas}(\partial\Omega))^{1/p} \\
&\quad \cdot \left[ C_{G,B}(1 + \|Q\|_{\mathcal{L}(L^p(\partial\Omega,\mathbb{R}^n), C^{0,\alpha}(\partial\Omega,\mathbb{R}^n))} \|\eta\|_{L^p(\partial\Omega,\mathbb{R}^n)})^\delta + \|B\upsilon_0\|_{C^{0,\alpha}(\partial\Omega,\mathbb{R}^n)} \right],
\end{aligned}$$

an inequality which implies that there exists $R_1 > 0$ such that $\|\eta\|_{L^p(\partial\Omega,\mathbb{R}^n)} \leq R_1$ for all possible solutions $\eta \in C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ of the equation in (56). Then we have

$$\|Q[\eta] + \upsilon_0\|_{C^{0,\alpha}(\partial\Omega,\mathbb{R}^n)} \leq \|Q\|_{\mathcal{L}(L^p(\partial\Omega,\mathbb{R}^n), C^{0,\alpha}(\partial\Omega,\mathbb{R}^n))} R_1 + \|\upsilon_0\|_{C^{0,\alpha}(\partial\Omega,\mathbb{R}^n)}$$

for all possible solutions $\eta \in C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ of the equation in (56) (see also Theorem 7 (iv).) Then assumption (52) on $F_G$ and the linearity and continuity of $L_B^{(-1)}$ in $C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ imply that there exists $R > 0$ such that $\|\eta\|_{C^{0,\alpha}(\partial\Omega,\mathbb{R}^n)} < R$ for all possible solutions $\eta \in C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ of the equation in (56). By homotopy invariance of the Leray-Schauder Degree of $I - \lambda L_B^{(-1)} [F_{\tilde{G}} [Q[\cdot] + \upsilon_0] + B\upsilon_0]$ in the ball centered at 0 and radius $R$ as $\lambda \in [0, 1]$, we deduce that equation (54) has at least a solution $\eta \in C^{0,\alpha}(\partial\Omega, \mathbb{R}^n)$ with $\|\eta\|_{C^{0,\alpha}(\partial\Omega,\mathbb{R}^n)} < R$.

Next we show by finite induction that $\eta \in C^{j-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ for all $j \in \{1, \dots, m\}$. We already know that such a membership holds for $j = 1$. We now assume that such a membership holds for a fixed $j \in \{1, \dots, m-1\}$ and we show that it holds also for $j + 1$. By classical results on elastic layer potentials (cf. *e.g.*, [2, Thm. A.2]), we have $Q[\eta] \in C^{j,\alpha}(\partial\Omega, \mathbb{R}^n)$. Since $F_{\tilde{G}}$ and $L_B^{(-1)}$ map $C^{j,\alpha}(\partial\Omega, \mathbb{R}^n)$ to itself and $B\upsilon_0$ belongs to $C^{m-1,\alpha}(\partial\Omega, \mathbb{R}^n)$ which is contained in $C^{j,\alpha}(\partial\Omega, \mathbb{R}^n)$, we deduce that $L_B^{(-1)} [F_{\tilde{G}} [Q[\eta] + \upsilon_0] + B\upsilon_0] \in C^{j,\alpha}(\partial\Omega, \mathbb{R}^n)$. Hence, equation (55) implies that $\eta \in C^{j,\alpha}(\partial\Omega, \mathbb{R}^n)$ and the proof is complete. □

We note that assumptions (52) an (53) certainly hold if $G$ is smooth enough (see also end of [11, App. B].)

**Corollary 1.** *Let the assumptions of Theorem 1 hold. Let $\gamma_M$ be in $[0, +\infty[$. Let $-a^o$ satisfy conditions (8), (9). Let the function $G^i$ satisfy the assumptions of Theorem 8 on $G$ with $\Omega = \Omega^i$. Then equation (23) admits a solution $(\tilde{c}, \tilde{\eta}, \tilde{\rho}) \in \mathbb{R}^n \times X_{m,\alpha}$.*

*If $n = 2$ and if we further assume that*

$$\det \left( I + \frac{1}{4\pi} \frac{\omega + 2}{\omega + 1} \int_{\partial\Omega^i} D_\xi G^i(t, \tilde{\xi})\, d\sigma_t \right) \neq 0 \qquad \forall \tilde{\xi} \in \mathbb{R}^n ,$$

*then the matrix $\mathcal{G}^i$ of (30) with $\tilde{u}^i = v^-[\omega, \tilde{\eta}]$ satisfies the assumptions of Theorem 3.*

*If $n \geq 3$ and if we further assume that*

$$\zeta D_\xi G^i(t, \tilde{\xi}) \zeta^t \leq 0 \qquad \forall (t, \tilde{\xi}, \zeta) \in \partial\Omega^i \times \mathbb{R}^n \times \mathbb{R}^n ,$$
$$\text{there exists } t \in \partial\Omega^i \text{ such that } \det D_\xi G^i(t, \tilde{\xi}) \neq 0 \ \forall \tilde{\xi} \in \mathbb{R}^n ,$$

*then the matrix $\mathcal{G}^i$ of (30) with $\tilde{u}^i = v^-[\omega, \tilde{\eta}]$ satisfies the assumptions of Theorem 3.*

**Proof.** By Theorem 8 with $G = G^i$, $\Omega = \Omega^i$, $\upsilon_0 = 0$, the second component of equation (23) admits a solution $\tilde{\eta} \in C^{m-1,\alpha}(\partial\Omega^i, \mathbb{R}^n)$. Then by classical results on linear integral equations associated to interior linear traction boundary value problems, the first and third components of equation (23) admit a unique solution $(\tilde{c}, \tilde{\rho}) \in \mathbb{R}^n \times C^{m-1,\alpha}(\partial\Omega^o, \mathbb{R}^n)$ (cf. *e.g.*, [2, Thm. 2.2 (ii)].) Finally, the last part of the statement is obvious. □

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