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ON A CONTROL PROBLEM ASSOCIATED WITH THE HEAT TRANSFER PROCESS

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Abstract. Mathematical models of thermocontrol processes are considered. In the model under consideration, the temperature inside a domain is controlled by m convectors acting on the boundary. The control parameter is a vector-function, which components are equal to the magnitude of output of hot or cold air producing by each convector. The necessary and sufficient conditions for achieving the given projection of the temperature into some m-dimensional subspace are studied.

1 Introduction

Consider the heat equation

$$u_t(x,t) \;=\; \Delta u(x,t) \;-\; p(x)u(x,t), \quad p(x) \geq 0, \quad x \in \Omega, \quad t > 0, \tag{1}$$

with the boundary conditions

$$\frac{\partial u(x,t)}{\partial n} \;=\; \mu_k(t)a_k(x), \quad x \in \Gamma_k, \quad t > 0, \tag{2}$$

and

$$\frac{\partial u(x,t)}{\partial n} \;+\; h(x)u(x,t) \;=\; 0, \quad x \in \partial\Omega \setminus \Gamma, \quad t > 0, \tag{3}$$

and the initial condition

$$u(x,0) \;=\; 0. \tag{4}$$

Here $\Omega \subset \mathbb{R}^n$ is a domain with piecewise smooth boundary $\partial\Omega$, Γ_k are some disjoint subsets of $\partial\Omega$ (convectors, i.e. heaters or coolers) with piecewise smooth boundaries $\partial\Gamma_k$ and $\Gamma = \bigcup\limits_{k=1}^{m} \Gamma_k$.

We suppose that $h(x)$ (the thermal conductivity of the walls) and $a_k(x)$ (the power density of the k-th convector) are given piecewise smooth non-negative functions, which are not identically zeros, and the function $p(x)$ is sufficiently smooth in $\overline{\Omega} = \Omega \cup \partial\Omega$.

Boundary conditions (2) and (3) mean that each convector produces a hot or cold flow with magnitude of output given by a measurable real-valued function $\mu_k(t)$, and on the surface $\partial\Omega \setminus \Gamma$ a heat exchange takes place according to Newton's law (see, e.g. [11], Sec. III.4).

We extend the functions $h(x)$ and $a_k(x)$ to the whole boundary $\partial\Omega$ by setting $h(x) = 0$ for $x \in \Gamma$ and $a_k(x) = 0$ for $x \in \partial\Omega \setminus \Gamma_k$.

Introduce the vector-function $a : \partial\Omega \to \mathbb{R}^m$ by

$$a(x) \ = \ (a_1(x), a_2(x), ...a_m(x)), \quad x \in \partial\Omega, \tag{5}$$

and the vector-function $\mu : [0, +\infty) \to \mathbb{R}^m$ by

$$\mu(t) = (\mu_1(t), \mu_2(t), ..., \mu_m(t)), \quad t \geq 0. \tag{6}$$

With this notation we may rewrite the conditions (2) and (3) in the following form

$$\frac{\partial u(x,t)}{\partial n} + h(x)u(x,t) \ = \ \mu(t) \cdot a(x), \quad x \in \partial\Omega, \ t > 0, \tag{7}$$

where $u \cdot v$ denotes the scalar product of two vectors u and v in $\mathbb{R}^m$.

We say that the function $\mu : [0, +\infty) \to \mathbb{R}^m$ is an *admissible control* if all of the functions $\mu_j(t)$ are measurable and for $t \geq 0$ satisfy the following constraints

$$|\mu_j(t)| \ \leq 1, \quad j = 1, 2, ..., m, \quad t \geq 0. \tag{8}$$

Consider the following eigenvalue problem for the Laplace operator

$$-\Delta v(x) \ + \ p(x)v(x) \ = \ \lambda v(x), \quad x \in \Omega, \tag{9}$$

with the boundary condition

$$\frac{\partial v(x)}{\partial n} + h(x)v(x) = 0, \quad x \in \partial\Omega. \tag{10}$$

We define the generalized solution of problem (9)-(10) as the function $v(x)$ in the Sobolev space $W_2^1(\Omega)$, which satisfies the equality

$$\int_\Omega [\nabla v(x) \cdot \nabla \eta(x) + p(x)v(x)\eta(x)] \ dx \ + \int_{\partial\Omega} h(x)v(x)\eta(x)d\sigma(x) \ = \ = \ \lambda \int_\Omega v(x)\eta(x)dx, \tag{11}$$

for an arbitrary function $\eta \in W_2^1(\Omega)$ (see [9], Sec. III.6, formula (6.3)).

We consider this problem in real Hilbert space $L^2(\Omega)$ with the scalar product

$$(u, v) \ = \int_\Omega u(x)v(x)dx$$

and the norm $\|u\| = \sqrt{(u,u)}$.

It is well known that under the above assumptions there exists a sequence of positive eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ such that

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \leq ... , \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and the corresponding eigenfunctions $v_k(x)$ form an orthonormal basis $\{v_k\}_{k=1}^{\infty}$ in $L^2(\Omega)$ (see, e. g. [9], Sec. III.6) .

Set

$$H_m = \{u \in L_2(\Omega) : u = \sum_{j=1}^{m} \alpha_j v_j(x), \quad \alpha_j \in \mathbf{R}\}. \tag{12}$$

Let P_m be the orthogonal projector onto H_m, i. e.

$$P_m u(x) = \sum_{j=1}^{m} (u, v_j) v_j(x).$$

Note that the solution $u(x,t)$ of initial-boundary value problem (1)-(4) for every fixed $t \geq 0$ belongs to $L_2(\Omega)$ as a function of x, and therefore this function may be decomposed via the eigenfunctions:

$$u(x,t) = \sum_{k=1}^{\infty} c_k(t) v_k(x), \quad t \geq 0, \quad x \in \Omega.$$

We denote by $C_m[0,+\infty)$ the space of all vector-functions $f : [0,+\infty) \to \mathbb{R}^m$ such that all components of f are continuous on $[0,+\infty)$.

In the present work we consider the following problem.

Problem HC. *For a given vector-function $f \in C_m[0,\infty)$ a heat control problem HC consists in finding the admissible control μ ensuring that the solution $u(x,t)$ of initial-boundary value problem (1)-(4) exists, is unique and for all $t \geq 0$ satisfies the equalities*

$$\int_{\Omega} u(x,t)\, v_k(x) dx = f_k(t), \quad k = 1, 2, ..., m, \quad t \geq 0. \tag{13}$$

Note that the detailed information on the control problems for distributed parameter systems is given in the monographs [10] and [5]. More recent results concerned with the heat control problem for partial differential equations of parabolic type were established in [2]-[4] and [6].

For the solving the problem HC we need some spectral properties of the corresponding elliptic operator.

Consider the following boundary value problem for the Laplace equation

$$-\Delta w_k(x) + p(x) w_k(x) = 0, \quad x \in \Omega, \tag{14}$$

with the boundary condition

$$\frac{\partial w_k(x)}{\partial n} + h(x)w_k(x) = a_k(x), \quad x \in \partial\Omega. \tag{15}$$

The physical meaning of the function $w_k(x)$ is clear: this is the temperature at the point $x \in \Omega$ in the case where only k-th convector works and it produces heat or cold with maximal capacity (output).

Set

$$L_m \ = \ \{w \in L_2(\Omega) : w = \sum_{k=1}^{m} \beta_k w_k(x), \quad \beta_k \in \mathbf{R}\}. \tag{16}$$

Let Q_m be the orthogonal projector onto L_m.

Introduce the following characteristic of closeness of the subspaces L_m and H_m, which is known as the aperture of these subspaces (see [1], Chapter III, sec. 39), or the gap between these subspaces (see [7], Chapter IV, Sec. 2):

$$\theta(L_m, H_m) \ = \ \|P_m - Q_m\|_{L_2(\Omega)\to L_2(\Omega)}. \tag{17}$$

It is well known that $\theta(L_m, H_m) \le 1$, and the less is θ the closer are subspaces L_m and H_m.

We show that this characteristic is essential for describing the set of functions f for which the problem HC has a unique solution. First of all we may note that according to (4) the function f must satisfy the condition

$$f(0) \ = \ 0. \tag{18}$$

Furthermore, because of inertness of heat conduction, it is reasonable to assume that the function f does not change very quickly, i.e. the derivatives of all components are bounded.

Consider the matrix

$$\widehat{E}(t) \ = \ \begin{vmatrix} e^{-\lambda_1 t} & 0 & 0 & \dots & 0 \\ 0 & e^{-\lambda_2 t} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & e^{-\lambda_m t} \end{vmatrix} \tag{19}$$

Every time when a vector from $\mathbb{R}^m$ is on the right-hand side of some matrix $m \times m$ we assume that this is a column-vector.

Let $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^m$ be two vectors with non-negative components: $a_j \ge 0,\ b_j \ge 0$. We say that vector a precedes vector b and denote this by $a \prec b$, if $a_j \le b_j$ for all $j = 1, 2, ..., m$.

For an arbitrary vector $b \in \mathbb{R}^m$ set

$$[b] \ = \ (|b_1|, |b_2|, ..., |b_m|).$$

We suppose that the function f has the derivative $f' \in C_m[0,\infty)$ and for some $b \in \mathbb{R}^m$ this derivatives satisfy the following condition

$$[f'(t)] \prec \widehat{E}(t)[b], \quad t \geq 0. \tag{20}$$

In the case in which the convectors are properly disposed conditions (18) and (20) are sufficient for solvability of the problem HC.

Theorem 1. *Let the condition*

$$\theta(L_m, H_m) < 1 \tag{21}$$

be satisfied. Then there exists an m-dimensional polyhedron $P \subset \mathbb{R}^m$ such that:

i) P is a convex neighborhood of the origin;

ii) for any $b \in P$ and for any continuously differentiable vector-function f, which satisfies conditions (18) and (20), the solution of the problem HC exists and is unique;

iii) for every $b \notin P$ there exists a function f which satisfies both conditions (18) and (20) but the solution of the problem HC does not exists.

The set P contains the ball with the center at the origin, hence, the following statement is valid.

Corollary 1. *Let the inequality (21) be satisfied. Then there exists $R > 0$ such that for any continuously differentiable vector-function f, which satisfies conditions (18) and the condition*

$$|f_j'(t)| \leq Re^{-\lambda_j t}, \tag{22}$$

the solution of the problem HC exists and is unique.

The next theorem shows that condition (21) is essential for solvability of the problem HC.

Theorem 2. *Let*

$$\theta(L_m, H_m) = 1. \tag{23}$$

Then for any $R > 0$ there exists a continuously differentiable vector-function f which satisfies conditions (18) and (22), but the solution of the problem HC does not exist.

To prove these theorems we show that the unknown function μ satisfies the following Volterra integral equation of the first kind

$$\int_0^t \widehat{K}(t-s)\mu(s)ds = f(t), \quad t \geq 0. \tag{24}$$

Here the kernel $\widehat{K}$ is a matrix-function defined by the Green function of the initial-bounded value problem for the heat conduction equation. Since this integral operator is compact and has no inverse bounded operator it is necessary to regularize equation

(24). With this purpose we differentiate this equation and consider the following Volterra integral equation of the second kind

$$\widehat{K}(0)\mu(t)+\int\limits_0^t \widehat{K}'(t-s)\mu(s)ds=f'(t). \tag{25}$$

In Section 2 we study the properties of the kernel $\widehat{K}$ and its derivative $\widehat{K}'$. In Section 3 we prove the solvability of equations (24) and (25) with constraint (8), and in Section 4 we prove Theorem 2.

2 Properties of the kernel of the main integral equation

To prove Theorem 1 we consider the Green function G , which we define by its spectral expansion:

$$G(x,y,t) \ = \ \sum_{j=1}^{\infty} e^{-\lambda_j t} v_j(x) v_j(y), \quad x\in\Omega\cup\partial\Omega, \quad y\in\Omega\cup\partial\Omega, \quad t>0. \tag{26}$$

The following statements are evident and some of them are well known (see, e.g., [8] and [12]).

We suppose that $h(x)\geq 0$ for all $x\in\partial\Omega$ and $h\not\equiv 0$. Then the Green function is non-negative:

$$G(x,y,t) \ \geq \ 0, \quad (x,y)\in\overline{\Omega}\times\overline{\Omega}, \ t>0,$$

and solution of initial-boundary value problem (1)+(4)+(7) may be represented by the Green function:

$$u(x,t)=\int\limits_0^t ds\int\limits_{\partial\Omega} G(x,y,t-s)\ \mu(s)\cdot a(y)\,d\sigma(y), \tag{27}$$

where vectors a and μ are defined by equalities (5) and (6).

By the definition of the scalar product we may write

$$u(x,t)=\sum_{k=1}^{m}\int\limits_0^t \mu_k(s)ds\int\limits_{\partial\Omega} G(x,y,t-s)a_k(y)\,d\sigma(y) \tag{28}$$

By condition (13), for $t>0$ the following equation

$$\int\limits_{\Omega} u(x,t)v_j(x)\,dx \ =$$

$$= \ \int\limits_{\Omega} v_j(x)\,dx\sum_{k=1}^{m}\int\limits_0^t \mu_k(s)ds\int\limits_{\partial\Omega} G(x,y,t-s)a_k(y)\,d\sigma(y) \ = \ f_j(t) \tag{29}$$

must be satisfied.

It follows by (26) that

$$\int_{\Omega} G(x,y,t)v_j(x)dx \;=\; v_j(y)e^{-\lambda_j t}, \quad y \in \Omega \cup \partial\Omega, \quad t > 0.$$

Then equation (29) takes the following form

$$\sum_{k=1}^{m} \int_0^t e^{\lambda_j(s-t)}\mu_k(s)ds \int_{\partial\Omega} v_j(y)a_k(y)d\sigma(y) = f_j(t), \quad t > 0. \tag{30}$$

Set

$$a_{jk} \;=\; \int_{\partial\Omega} v_j(y)a_k(y)d\sigma(y). \tag{31}$$

Now we introduce the kernel of the main integral operator as a matrix $\widehat{K}(t) = ||K_{jk}(t)||$ with the elements

$$K_{jk}(t) = \; a_{jk}\, e^{-\lambda_j t}. \tag{32}$$

Hence we may rewrite equation (30) as follows

$$\sum_{k=1}^{m} \int_0^t K_{jk}(t-s)\mu_k(s)ds = f_j(t), \quad t > 0. \tag{33}$$

It is clear that the matrices $\widehat{A} = ||a_{jk}||$ and $\widehat{K}$ satisfy the equality

$$\widehat{K}(t) \;=\; \widehat{E}(t)\widehat{A},$$

where $\widehat{E}$ is the matrix defined by (19).

Introduce the following integral operator with matrix kernel:

$$K\mu(t) \;=\; \int_0^t \widehat{K}(t-s)\mu(s)ds, \quad t > 0. \tag{34}$$

Equation (33) takes the form of the following Volterra integral equation of the first kind

$$\int_0^t \widehat{E}(t-s)\widehat{A}\mu(s)ds = f(t), \quad t > 0, \tag{35}$$

which is the main equation of the considered problem HC. Since the integral operator in the left hand side is compact and quasinilpotent, we need to study not only the kernel (34) of this operator but also its derivatives.

After differentiating equality (35) we get the equation (since $\widehat{E}(0) = I$)

$$\widehat{A}\mu(t) \ + \int\limits_0^t \widehat{E}'(t-s)\widehat{A}\mu(s)ds = f'(t), \quad t>0. \tag{36}$$

It is clear that in the case in which $f' \in C_m[0,+\infty)$ and the function f satisfies condition (18) equations (35) and (36) are equivalent.

Set

$$\omega(t) \ = \ \widehat{A}\mu(t).$$

Then

$$\omega(t) \ + \int\limits_0^t \widehat{E}'(t-s)\omega(s)ds = f'(t), \quad t>0, \tag{37}$$

and this is a Volterra integral equation of the second kind.

It is well known that equation (37) has unique solution for every $f' \in C_m[0,+\infty)$. The problem is to find the solution ω for which $\omega(t) \ = \ \widehat{A}\mu(t)$ and $\mu(t)$ satisfies restrictions (8).

We note that vector equation (37) splits up into m scalar equations

$$\omega_j(t) \ - \ \lambda_j \int\limits_0^t e^{-\lambda_j(t-s)}\omega_j(s)ds \ = \ f_j'(t), \quad t>0. \tag{38}$$

Set

$$S_j v(t) \ = \ \lambda_j \int\limits_0^t e^{-\lambda_j(t-s)} v(s)ds. \tag{39}$$

Then we may rewrite equation (38) as follows:

$$(I - S_j)\omega_j(t) \ = \ f_j'(t). \tag{40}$$

We need the following two properties of integral operator (39).

Proposition 1. *Let v be a measurable function on the half-line $[0,+\infty)$, bounded on the segment $[0,T]$ for any $T>0$. If*

$$(I - S_j)v(t) \geq 0$$

then

$$v(t) \geq 0$$

almost everywhere on $[0,+\infty)$.

Proof. Follows by the positiveness of the eigenvalues. Indeed, since $\lambda_j > 0$ the kernel of integral operator (39) is a positive function. Hence, if we set $g(t) = (I - S_j)v(t)$, then

$$v(t) \ = \ \sum_{k=0}^{\infty} S_j^k g(t) \geq 0. \qquad \square$$

Proposition 2. *Let v be a measurable function on the half-line $[0,+\infty)$, bounded on the segment $[0,T]$ for any $T>0$. If for some $M>0$ the inequality*

$$|(I-S_j)v(t)| \ \le \ Me^{-\lambda_j t}$$

is satisfied, then

$$|v(t)| \ \le \ M$$

almost everywhere on $[0,+\infty)$.

Proof. Set $v_\pm(t)=M\pm v(t)$. Then

$$(I-S_j)v_\pm(t) \ = \ (I-S_j)M \ \pm \ (I-S_j)v(t).$$

Note that

$$(I-S_j)M \ = \ M - M\lambda_j\frac{1-e^{-\lambda_j t}}{\lambda_j} \ = \ Me^{-\lambda_j t}.$$

Hence

$$(I-S_j)v_\pm(t) \ = \ Me^{-\lambda_j t} \ \pm \ (I-S_j)v(t) \ \ge \ 0,$$

and, according to Proposition 1, $v_\pm(t)\ge 0$. □

3 Existence and uniqueness of the solution of the problem HC

First we study the properties of the matrix $A=||a_{jk}||$ defined by (31).

Proposition 3. *The elements of the matrix $\widehat{A}$ have the following form:*

$$a_{jk} \ = \ \lambda_j\int\limits_\Omega v_j(x)\,w_k(x)\,dx. \tag{41}$$

Here λ_j and v_j are the eigenvalues and eigenfunctions of the problem (9)-(10) and w_k are the solutions of boundary value problem (14)-(15).

Proof. Indeed, it follows immediately by the Green formula

$$\int\limits_\Omega [v_j(x)\Delta w_k(x) - w_k(x)\Delta v_j(x)]\,dx \ = $$

$$= \int\limits_{\partial\Omega}\left[v_j(x)\frac{\partial w_k(x)}{\partial n} - w_k(x)\frac{\partial v_j(x)}{\partial n}\right]d\sigma(x)$$

that

$$\int\limits_{\partial\Omega} v_j(x)\,a_k(x)\,d\sigma(x) \ = \ \lambda_j\int\limits_\Omega v_j(x)\,w_k(x)\,dx,$$

and taking into account definition (31) we get required equality (41). □

Proposition 4. *Let the condition (21) be satisfied. Then* $\det \widehat{A} \neq 0$.

Proof. It follows from condition (21) that the vector $u \in H_m$, which is orthogonal to L_m, must be equal to 0 (see [1], Sec. 34).

Assume that there exists c_j such that

$$\sum_{j=1}^{m} c_j\, a_{jk} \;=\; 0. \tag{42}$$

According to (41),

$$\sum_{j=1}^{m} c_j\, a_{jk} \;=\; \int_{\Omega} \left[\sum_{j=1}^{m} c_j \lambda_j v_j(x)\right] w_k(x)\, dx. \tag{43}$$

Set

$$\phi(x) \;=\; \sum_{j=1}^{m} c_j v_j(x).$$

It is clear that $\phi \in H_m$ and $(-\Delta + p)\phi \in H_m$. It follows by (42) and (43) that

$$\int_{\Omega} (-\Delta + p)\phi(x)\, w_k(x)\, dx \;=\; 0.$$

Hence $(-\Delta + p)\phi(x)$ is orthogonal to L_m and, since $(-\Delta + p)\phi \in H_m$, due to assumption (21),

$$(-\Delta + p)\phi(x) \;\equiv\; 0.$$

Because of orthogonality $\lambda_j c_j = 0$ for all $j = 1, 2, ..., m$, and since $\lambda_j > 0$ we may state that all $c_j = 0$. This means that the rows of the matrix $\widehat{A}$ are linearly independent. Hence $\det \widehat{A} \neq 0$. □

Corollary 2. *Let condition (21) be satisfied. Then there exists the inverse matrix* $\widehat{A}^{-1}$.

Proof of Theorem 1. Set

$$Q \;=\; \{u \in \mathbb{R}^m \,:\, |u_j| \leq 1, \quad j = 1, 2, ..., m\}. \tag{44}$$

Denote the image of this cube under transform $\widehat{A}$ by $\widetilde{Q}$:

$$\widetilde{Q} \;=\; \{v \in \mathbb{R}^m \,:\, v = \widehat{A}u, \quad u \in Q\}. \tag{45}$$

The set $\widetilde{Q}$ is a convex polyhedron of dimension m and, according to Proposition 4, this polyhedron contains a ball centered at the origin.

We introduce the set

$$P \;=\; \{b \in \widetilde{Q} \;:\; [a] \prec [b] \;\Rightarrow\; a \in \widetilde{Q}\}.$$

It is clear that P is a convex polyhedron and a neighborhood of the origin.

1) Let $b \in P$. If the function f satisfies condition (20) then, according to Propositions 1 and 2, the solution ω of the equation (37) satisfies the condition

$$[\omega(t)] \prec [b].$$

Hence, due to the definition of the set P we may state that $\omega(t) \in \widetilde{Q}$ for all $t \geq 0$. Consequently, the vector $\mu(t) = \widehat{A}^{-1}\omega(t) \in Q$, or

$$|\mu_j(t)| \leq 1, \quad j = 1, 2, ..., m.$$

This means that μ is an admissible control.

2) Assume now that $b \notin P$. Then we may find a vector b^* such that $[b^*] \prec [b]$ and $b^* \notin \widetilde{Q}$.

Indeed, if $b \notin \widetilde{Q}$ then we just set $b^* = b$. In the case in which $b \in \widetilde{Q}$ the existence of such a vector b^* follows by the definition of the set P.

Set

$$f_j(t) = \frac{b_j^*}{\lambda_j}\left(1 - e^{-\lambda_j t}\right).$$

The corresponding vector-function f satisfies conditions (18) and (20). Indeed,

$$f_j'(t) = b_j^* e^{-\lambda_j t}$$

and

$$[f'(t)] = \widehat{E}(t)[b^*] \prec \widehat{E}(t)[b].$$

Note that the solution ω of equation (37) is continuous and $\omega(0) = f'(0) = b^* \notin \widetilde{Q}$. Since $\widetilde{Q}$ is closed, there exists $T > 0$ such that $\omega(t) \notin \widetilde{Q}$ for $0 \leq t \leq T$. Then $\mu(t) = \widehat{A}^{-1}\omega(t) \notin Q$ for $0 \leq t \leq T$. Hence, there is no admissible control and the problem HC has no solution. □

4 Non-existence of the solution

Proposition 5. *Let condition (23) be satisfied. Then* $\det \widehat{A} = 0$.

Proof. First we show that the next statement follows from [1]: *if*

$$\theta(L_m, H_m) = 1,$$

then there exists a vector $v \in L_m$ such that $v \perp H_m$, or there exists a vector $u \in H_m$ such that $u \perp L_m$.

Indeed, we may use the formula (see [1], Chapter III, Section 39, formula (2))

$$\theta(L_m, H_m) = \max\left\{\sup_{v \in L_m, \|v\|=1} \|(I - P_m)v\|, \sup_{u \in H_m, \|u\|=1} \|(I - Q_m)u\|\right\}.$$

In the case in which

$$\sup_{v\in L_m, \|v\|=1} \|(I-P_m)v\| \;=\; 1,$$

we may state that there exists $v \in L_m$ such that $\|v\| = 1$ and

$$\|(I-P_m)v\| \;=\; 1.$$

Hence

$$1 \;=\; \|v\|^2 \;=\; \|P_m v\|^2 \;+\; \|(I-P_m)v\|^2 \;=\; \|P_m v\|^2 \;+\; 1$$

and, consequently, $\|P_m v\| = 0$ or $v \perp H_m$.

Analogously, if

$$\sup_{u\in H_m, \|u\|=1} \|(I-Q_m)u\| \;=\; 1,$$

then there exists $u \in H_m$ such that $\|u\| = 1$ and

$$\|(I-Q_m)u\| \;=\; 1.$$

Hence

$$1 \;=\; \|u\|^2 \;=\; \|Q_m u\|^2 \;+\; \|(I-Q_m)u\|^2 \;=\; \|Q_m u\|^2 \;+\; 1$$

and, consequently, $\|Q_m u\| = 0$ or $u \perp L_m$.

Assume that there exists $u \in H_m$ such that $u \perp L_m$. Set

$$u(x) \;=\; \sum_{j=1}^{m} \alpha_j v_j(x), \quad \text{where} \quad \sum_{j=1}^{m} \alpha_j^2 = 1.$$

Then for any $k = 1, 2, ..., m$

$$0 \;=\; (u, w_k) \;=\; \sum_{j=1}^{m} \int_{\Omega} \alpha_j v_j(x)\, w_k(x)\, dx \;=\; \sum_{j=1}^{m} \frac{\alpha_j}{\lambda_j} a_{jk}.$$

Hence the rows of the matrix $\widehat{A}$ are linearly dependent and therefore $\det \widehat{A} = 0$.

Analogously, if there exists $v \in L_m$ such that $v \perp H_m$, and

$$v(x) \;=\; \sum_{k=1}^{m} \beta_k w_k(x), \quad \text{where} \quad \sum_{k=1}^{m} \beta_k^2 = 1,$$

then for any $j = 1, 2, ..., m$

$$0 = (v, v_j) = \int_{\Omega} \left[\sum_{k=1}^{m} \beta_k w_k(x)\right] v_j(x)\, dx \;=\; \frac{1}{\lambda_j} \sum_{k=1}^{m} \beta_k a_{jk}.$$

Hence, the columns of the matrix $\widehat{A}$ are linearly dependent and therefore $\det \widehat{A} = 0$.

$\square$

Corollary 3. *Let condition (23) be satisfied. Then* $\det \widehat{A}^* = 0$ *where* $\widehat{A}^*$ *is the transpose of matrix* $\widehat{A}$.

Corollary 4. *Let condition (23) be satisfied. Then there exists a vector* $\widetilde{\mu} \in \mathbb{R}^m$ *such that* $|\widetilde{\mu}| = 1$ *and the following equality is valid:*

$$\widehat{A}^*\widetilde{\mu} \;=\; 0.$$

Proof of Theorem 2. Consider, for $\lambda \geq \lambda_m$ and for sufficiently small $\varepsilon > 0$, the following function

$$f(t) \;=\; \varepsilon\widetilde{\mu}\left(1 - e^{-\lambda t}\right), \quad t \geq 0.$$

It is clear that $f(0) = 0$ and

$$f'(t) \;=\; \varepsilon\lambda\widetilde{\mu}e^{-\lambda t}, \quad t \geq 0.$$

If equation (36) has a solution $\mu(t)$, then obviously the vector $\widehat{A}\mu(t)$ is continuous and, putting $t = 0$, we get

$$\widehat{A}\mu(0) \;=\; f'(0) \;=\; \varepsilon\lambda\widetilde{\mu}.$$

Hence

$$\widehat{A}\mu(0) \cdot \widetilde{\mu} \;=\; \varepsilon\lambda\widetilde{\mu} \cdot \widetilde{\mu} \;=\; \varepsilon\lambda > 0.$$

But on the other hand

$$\widehat{A}\mu(0) \cdot \widetilde{\mu} \;=\; \mu(0) \cdot \widehat{A}^*\widetilde{\mu} \;=\; 0.$$

This contradiction proves the theorem. □

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