EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 1, Number 2 (2010), 5 – 16

INVERSE PROBLEM FOR AN OPERATOR PENCIL WITH NONSEPARATED BOUNDARY CONDITIONS

A.M. Akhtyamov, V.A. Sadovnichy, Ya.T. Sultanaev

Communicated by E.D. Nursultanov

Key words: inverse eigenvalue problem, nonseparated boundary conditions, quadratic pencil.

AMS Mathematics Subject Classification: 34A55, 34B05, 34B07.

Abstract. In this work an inverse problem of spectral analysis for a quadratic pencil of operators with general nonselfadjoint nonseparated boundary conditions is considered. Uniqueness and duality theorems are proved, an algoritm for solving the problem is presented. Apropriate examples and counterexample are given.

1 Introduction

The inverse Sturm-Liouville problem for the equation

$$-y'' + q(x)y = \lambda y = s^2 y$$

with separated or nonseparated boundary conditions has been thoroughly investigated (see [1] – [3], [9] – [14], [16]). Inverse problems for operator pencils with separated boundary conditions were considered, for example, in [4] – [6], [15], [17]. However, inverse problems for operator pencils with nonseparated boundary conditions and their numerical solutions were apparently not studied. In this paper, we try to fill this gap.

2 Setting up an inverse problem

Consider the following three boundary value problems.
Problem L:

$$y'' + (s^2 + i\,s\,q_1(x) + q(x))\,y = 0, \tag{1}$$

$$\begin{aligned} U_1(y) = y'(0) + (a_{11} + i\,s\,a_{12})\,y(0) \\ +(a_{13} + i\,s\,a_{14})\,y(\pi) = 0, \end{aligned} \tag{2}$$

$$\begin{aligned} U_1(y) = y'(\pi) + (a_{21} + i\,s\,a_{22})\,y(0) \\ +(a_{23} + i\,s\,a_{24})\,y(\pi) = 0, \end{aligned} \tag{3}$$

Problem $\mathbf{L}_1$:

$$
\begin{aligned}
&y'' + (\mu^2 + i\,\mu\, q_1(x) + q(x))\, y = 0,\\
&V_1(y) = y'(0) + (a_{11} + i\,\mu\, a_{12})\, y(0) = 0,\\
&V_2(y) = y'(\pi) + (a_{23} + i\,\mu\, a_{24})\, y(\pi) = 0,
\end{aligned}
$$

Problem $\mathbf{L}_2$:

$$
\begin{aligned}
&y'' + (\nu^2 + i\,\nu\, q_1(x) + q(x))\, y = 0,\\
&y(0) = 0,\\
&V_2(y) = y'(\pi) + (a_{23} + i\,s\, a_{24})\, y(\pi) = 0.
\end{aligned}
$$

Here $q \in L_1[0,\pi]$ and $q_1 \in W_1^1[0,\pi]$ are complex-valued functions and a_{ij} $(i = 1,2,\ j = 1,2,3,4)$ are complex numbers such that $a_{12} \neq \pm 1$, $a_{24} \neq \pm 1$. The conditions $a_{12} \neq \pm 1$, $a_{24} \neq 1$ exclude from consideration Redzhe type problems, which need a special consideration (see [5]).

The inverse problem is formulated as follows.

Problem. Given the eigenvalues $\{s_k\}$, $\{\mu_k\}$, and $\{\nu_k\}$ of problems L, L_1, L_2 respectively, find the coefficients of the pencil L, i.e., the coefficients $q(x)$, $q_1(x)$, a_{ij} $(i = 1,2,\ j = 1,2,3,4)$.

3 The duality and uniqueness theorems

Together with problem L, we consider problem L^-, that differs from L in the coefficients a_{13}, a_{14}, a_{21}, a_{22}:

Problem $\mathbf{L}^-$:

$$
y'' + (s^2 + i\,s\, q_1(x) + q(x))\, y = 0, \tag{4}
$$

$$
\begin{aligned}
U_1(y) = y'(0) + (a_{11} + i\,s\, a_{12})\, y(0)&\\
+(-a_{21} - i\,s\, a_{22})\, y(\pi) = 0,&
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
U_1(y) = y'(\pi) + (-a_{13} - i\,s\, a_{14})\, y(0)&\\
+(a_{23} + i\,s\, a_{24})\, y(\pi) = 0,&
\end{aligned} \tag{6}
$$

In what follows, problems of the types L, L^- but with different coefficients $\widetilde{q}, \widetilde{q}_1$ in the equation and with different parameters $\widetilde{a}_{ij}$ in the boundary forms is denoted by $\widetilde{L}$, $\widetilde{L}^-$ respectively. Throughout this paper, we assume that a symbol with a tilde in problem $\widetilde{L}$ denotes an object similar to that in problem L.

Theorem 1. *If $\{s_k\} = \{\widetilde{s}_k\}$, $\{\mu_k\} = \{\widetilde{\mu}_k\}$, $\{\nu_k\} = \{\widetilde{\nu}_k\}$, then either $L = \widetilde{L}$ or $L = \widetilde{L}^-$. Thus, given three spectra, the coefficients of (1)–(3) are dually determined: either $q(x) = \widetilde{q}(x)$, $q_1(x) = \widetilde{q}_1(x)$, $a_{ij} = \widetilde{a}_{ij}$, $i = 1,2$, $j = 1,2,3,4$, or $q(x) = \widetilde{q}(x)$, $q_1(x) = \widetilde{q}_1(x)$, $a_{11} = \widetilde{a}_{11}$, $a_{12} = \widetilde{a}_{12}$, $a_{13} = -\widetilde{a}_{21}$, $a_{14} = -\widetilde{a}_{22}$, $a_{21} = -\widetilde{a}_{13}$, $a_{22} = -\widetilde{a}_{14}$, $a_{23} = \widetilde{a}_{23}$, $a_{24} = \widetilde{a}_{24}$.*

Proof. Corollary 1 to Theorem 1 on the uniqueness of a solution to an inverse problem for L_1, and L_2 in [4] implies that $q(x) = \widetilde{q}(x)$, $a_{11} = \widetilde{a}_{11}$, $a_{12} = \widetilde{a}_{12}$, $a_{23} = \widetilde{a}_{23}$, $a_{24} = \widetilde{a}_{24}$.

The remaining equalities $a_{13} = \widetilde{a}_{13}$, $a_{14} = \widetilde{a}_{14}$, $a_{21} = \widetilde{a}_{21}$, $a_{22} = \widetilde{a}_{22}$ or $a_{13} = -\widetilde{a}_{21}$, $a_{14} = -\widetilde{a}_{22}$, $a_{21} = -\widetilde{a}_{13}$, $a_{22} = -\widetilde{a}_{14}$ are proved by using linear independence of certain functions in the expansion of the characteristic determinant $\Delta(s)$ of problem L, which is an entire function of the first order.

Let $\varphi(x,s)$ and $\psi(x,s)$ be solutions of (1) satisfying the conditions

$$\varphi(0,s) = \psi(\pi,s) = 1, \qquad V_1(\varphi) = V_2(\psi) = 0.$$

Then function

$$\Delta(s) = \begin{vmatrix} U_1(\varphi) & U_1(\psi) \\ U_2(\varphi) & U_2(\psi) \end{vmatrix}$$

is the characteristic function of problem (1)–(3).

We have

$$\Delta(s) = \begin{vmatrix} V_1(\varphi) + (a_{13} + i\,s\,a_{14})\,\varphi(\pi) & V_1(\psi) + (a_{13} + i\,s\,a_{14})\,\psi(\pi) \\ V_2(\varphi) + (a_{21} + i\,s\,a_{22})\,\varphi(0) & V_2(\psi) + (a_{21} + i\,s\,a_{22})\,\psi(0) \end{vmatrix}$$

$$= \begin{vmatrix} 0 + (a_{13} + i\,s\,a_{14})\,\varphi(\pi) & V_1(\psi) + (a_{13} + i\,s\,a_{14})\cdot 1 \\ V_2(\varphi) + (a_{21} + i\,s\,a_{22})\cdot 1 & 0 + (a_{21} + i\,s\,a_{22})\,\psi(0) \end{vmatrix}.$$

By the equality $V_2(\varphi) = -V_1(\psi)$ (see [4]) it follows that

$$\Delta(s) = (a_{13} + i\,s\,a_{14})\,(a_{21} + i\,s\,a_{22})\,\left[\varphi(\pi)\,\psi(0) - 1\right]$$

$$+(a_{13} + i\,s\,a_{14} - a_{21} - i\,s\,a_{22})\,V_1(\psi) + V_1^2(\psi)$$

$$= \left[-s^2\,a_{14}\,a_{22} + i\,s\,(a_{13}\,a_{22} + a_{14}\,a_{21}) + a_{13}\,a_{21}\right]\,f_1(s)$$

$$+\left[a_{13} - a_{21} + i\,s\,(a_{14} - a_{22})\right]\,f_2(s) + f_3(s),$$

where

$$f_1(s) = \varphi(\pi,s)\,\psi(0,s) - 1, \quad f_2(s) = V_1(\psi(x,s)), \quad f_3(s) = V_1^2(\psi(x,s)).$$

Similarly, we find that the characteristic function of problem $\widetilde{L}$ has the form

$$\widetilde{\Delta}(s) = \left[-s^2\,\widetilde{a}_{14}\,\widetilde{a}_{22} + i\,s\,(\widetilde{a}_{13}\,\widetilde{a}_{22} + \widetilde{a}_{14}\,\widetilde{a}_{21}) + \widetilde{a}_{13}\,\widetilde{a}_{21}\right]\,f_1(s)$$

$$+\left[\widetilde{a}_{13} - \widetilde{a}_{21} + i\,s\,(\widetilde{a}_{14} - \widetilde{a}_{22})\right]\,f_2(s) + f_3(s),$$

As is shown in [4]

$$\varphi(x,s) = \frac{1 - a_{12}}{2}\,\exp(isx - Q(x))\,[1] + \frac{1 + a_{12}}{2}\,\exp(-isx + Q(x))\,[1],$$

$$\varphi'(x,s) = \frac{1 - a_{12}}{2}\,i\,s\,\exp(isx - Q(x))\,[1] - \frac{1 + a_{12}}{2}\,i\,s\,\exp(-isx + Q(x))\,[1],$$

$$\psi(x,s) = \frac{1+a_{24}}{2}\,\exp(is(\pi - x) - Q(\pi) + Q(x))\,[1]$$
$$+\frac{1-a_{24}}{2}\,\exp(-is(\pi - x) + Q(\pi) - Q(x))\,[1],$$
$$\psi'(x,s) = -\frac{1+a_{24}}{2}\,i\,s\,\exp(is(\pi - x) - Q(\pi) + Q(x))\,[1]$$
$$+\frac{1-a_{24}}{2}\,i\,s\,\exp(-is(\pi - x) + Q(\pi) - Q(x))\,[1],$$

for $|s| \to \infty$ uniformly in $x \in [0,\pi]$, where

$$Q(x) = \int_0^x q_1(t)\,dt, \qquad [1] = 1 + O\left(\frac{1}{s}\right).$$

By the above and the general theory of differential operators [8, pp. 1–27] it follows that $\Delta(s)$ is an entire function of s of the first order. Hence by the and Hadamard theorem ([7]) it follows that the function $\Delta(s)$ can be reconstructed from its zeros up to the factor $C\,e^{a\,s}$, where a and C are numbers and $C \neq 0$.

It is known (see [8, pp. 24–27]) that the zeros of the determinant $\Delta(s)$ are the eigenvalues of problem L and the multiplicity of the zero of function $\Delta(s)$ coincides with the algebraic multiplicity of the corresponding eigenvalue of problem L.

Since the eigenvalues of problems L and $\widetilde{L}$ are identical and have the same algebraic multiplicity, the function $\Delta(s)$ of problem L and the function $\widetilde{\Delta}(s)$ of problem $\widetilde{L}$ satisfy the identity

$$\Delta(s) \equiv C\,\widetilde{\Delta}(s)\,\exp(a\,s). \tag{7}$$

Let us show that $a = 0$ with the help of a reducio ad absurdum proof. If $a \neq 0$ then the functions

$$f_1(s),\ s\,f_1(s),\ s^2\,f_1(s),\ f_1(s)\,e^{as},\ s\,f_1(s)\,e^{as},\ s^2\,f_1(s)\,e^{as},$$

$$f_2(s),\ s\,f_2(s),\ f_2(s)\,e^{as},\ s\,f_2(s)\,e^{as},\ f_3(s),\ f_3(s)\,e^{as}$$

form a linearly independent system of functions.

Indeed,

$$f_1(s) = \varphi(\pi,s)\,\psi(0,s) - 1 =$$
$$\left(\frac{1-a_{12}}{2}\,\exp(is\pi - Q(\pi))\,[1] + \frac{1+a_{12}}{2}\,\exp(-is\pi + Q(\pi))\,[1]\right)$$
$$\times\left(\frac{1+a_{24}}{2}\,\exp(is\pi - Q(\pi))\,[1] + \frac{1-a_{24}}{2}\,\exp(-is\pi + Q(\pi))\,[1]\right) - 1$$
$$\left(\frac{1-a_{12}}{2}\,\frac{1-a_{24}}{2} + \frac{1+a_{12}}{2}\,\frac{1+a_{24}}{2}\right)\,[1] - 1$$
$$+\left(\frac{1-a_{12}}{2}\,\frac{1+a_{24}}{2}\,\exp(2is\pi - 2Q(\pi)) +\right.$$
$$\left.+\frac{1+a_{12}}{2}\,\frac{1-a_{24}}{2}\,\exp(-2is\pi + 2Q(\pi))\right)\,[1]$$

$$= \frac{a_{12}\,a_{24}-1}{2}\,[1] + \frac{1}{2}\,\cos\left(2\,s\pi + 2\,i\,Q(\pi)\right)\cdot(1-a_{12}\,a_{24})\,[1]$$
$$+\frac{i}{2}\,\sin\left(2\,s\pi + 2\,i\,Q(\pi))\right)\cdot(a_{24}-a_{12})\,[1],$$
$$f_2(s) = V_1(\psi(x,s)) = \gamma\,s\,\sin\left((s-w)\pi\right)\,[1],$$
$$f_3(s) = V_1^2(\psi(x,s)) = \gamma^2\,s^2\,\sin^2\left((s-w)\pi\right)\,[1],$$

where (see [4])

$$\gamma = \sqrt{(1-a_{12}^2)(1-a_{24}^2)},\quad w = \frac{1}{2\,i\,\pi}\left(\ln\frac{(1+a_{12})(1-a_{24})}{(1-a_{12})(1+a_{24})} + \int_0^\pi q_1(t)\,dt\right).$$

By (7) it follows that the coefficient γ^2 at $s^2\,\sin^2\left((s-w)\pi\right)\,[1]$ in the expansion of

$$\Delta(s) - \widetilde{\Delta}(s)\,\exp(as+b)$$

equals zero.

However the equality

$$\gamma^2 = (1-a_{12}^2)(1-a_{24}^2) = 0$$

contrdicts to the conditions $a_{12}\neq\pm1$, $a_{24}\neq\pm1$.

So $a=0$.

Relation (7), together with linear independence of the functions

$$f_1(s),\ s\,f_1(s),\ s^2\,f_1(s),\ f_2(s),\ s\,f_2(s),\ f_3(s)$$

implies that

$$a_{14}\,a_{22} = \widetilde{a}_{14}\,\widetilde{a}_{22}, \tag{8}$$
$$a_{13}\,a_{22} + a_{14}\,a_{21} = \widetilde{a}_{13}\,\widetilde{a}_{22} + \widetilde{a}_{14}\,\widetilde{a}_{21}, \tag{9}$$
$$a_{13}\,a_{21} = \widetilde{a}_{13}\,\widetilde{a}_{21}, \tag{10}$$
$$a_{13} - a_{21} = \widetilde{a}_{13} - \widetilde{a}_{21}, \tag{11}$$
$$a_{14} - a_{22} = \widetilde{a}_{14} - \widetilde{a}_{22}. \tag{12}$$

By (8) and (12) it follows that

$$a_{14} = \widetilde{a}_{14},\quad a_{22} = \widetilde{a}_{22}, \tag{13}$$

or

$$a_{14} = -\widetilde{a}_{22},\quad a_{22} = -\widetilde{a}_{14}. \tag{14}$$

By (10) and (11) it follows that

$$a_{13} = \widetilde{a}_{13},\quad a_{21} = \widetilde{a}_{21}, \tag{15}$$

or

$$a_{13} = -\widetilde{a}_{21},\quad a_{21} = -\widetilde{a}_{13}. \tag{16}$$

Thus, the following four cases are possible:

(i) the equations (13), (15),

(ii) the equations (14), (16),

(iii) the equations (14), (15),

(iv) the equations (13), (16).

Let us show that cases (iii) and (iv) are particular cases of (i) and (ii).

In case (iii) equation (9) can be reduced to the form

$$(a_{13} + a_{21})(a_{14} + a_{22}) = 0.$$

Therefore $a_{13} + a_{21} = 0 \;\; a_{14} + a_{22} = 0$. If $a_{13} + a_{21} = 0$, then

$$a_{13} = -a_{21} = -\widetilde{a}_{21}, \quad a_{21} = -a_{13} = -\widetilde{a}_{13},$$

i.e. we have case (ii).

If $a_{14} + a_{22} = 0$, then

$$a_{14} = -a_{22} = \widetilde{a}_{14}, \quad a_{22} = -a_{14} = \widetilde{a}_{22},$$

i.e. we have case (i).

Hence, case (iii) is particular case of (i) and (ii).

In case (iv) equation (9) can be reduced to the form

$$(a_{13} + a_{21})(a_{14} + a_{22}) = 0.$$

Therefore $a_{13} + a_{21} = 0 \;\; a_{14} + a_{22} = 0$. If $a_{13} + a_{21} = 0$, then

$$a_{13} = -a_{21} = \widetilde{a}_{13}, \quad a_{21} = -a_{13} = \widetilde{a}_{21},$$

i.e. we have case (i).

If $a_{14} + a_{22} = 0$, then

$$a_{14} = -a_{22} = -\widetilde{a}_{22}, \quad a_{22} = -a_{14} = -\widetilde{a}_{14},$$

i.e. we have case (ii).

Hence, case (iv) is particular case of (i) and (ii).

Thus, only two cases (i) and (ii) are possible. □

As special cases of Theorem 1, we can obtain various uniqueness theorems. Below are examples.

Theorem 2. *If* $\{s_k\} = \{\widetilde{s}_k\}$, $\{\mu_k\} = \{\widetilde{\mu}_k\}$, $\{\nu_k\} = \{\widetilde{\nu}_k\}$, $a_{13} = \widetilde{a}_{13}$ *and* $a_{14} = \widetilde{a}_{14}$, *then* $L = \widetilde{L}$. *Thus, given three spectra, the coefficients of pencil (1)–(3) are uniquely determined if* a_{13} *and* a_{14} *are known.*

Proof. Theorem 1 implies that $q(x) = \widetilde{q}(x)$, $a_{11} = \widetilde{a}_{11}$, $a_{12} = \widetilde{a}_{12}$, $a_{23} = \widetilde{a}_{23}$, $a_{24} = \widetilde{a}_{24}$ and one of the cases

(i) the equations (13), (15),

(ii) the equations (14), (16)

is possible.

By the assumptions of the theorem $a_{13} = \widetilde{a}_{13}$, $a_{14} = \widetilde{a}_{14}$, so we have only one case (i). □

Theorem 3. *If* $\{s_k\} = \{\widetilde{s}_k\}$, $\{\mu_k\} = \{\widetilde{\mu}_k\}$, $\{\nu_k\} = \{\widetilde{\nu}_k\}$, $a_{14}, \widetilde{a}_{14}, a_{22}, \widetilde{a}_{22} \in \mathbb{R}$ *and* $\operatorname{sign} a_{14} = \operatorname{sign} \widetilde{a}_{14}$, $\operatorname{sign} a_{22} = \operatorname{sign} \widetilde{a}_{22}$, *then* $L = \widetilde{L}$. *Thus, given three spectra, the coefficients of pencil (1)–(3) are uniquely determined if* $\operatorname{sign} a_{14}$ *and* $\operatorname{sign} a_{22}$ *are known.*

Proof. Theorem 1 implies that $q(x) = \widetilde{q}(x)$, $a_{11} = \widetilde{a}_{11}$, $a_{12} = \widetilde{a}_{12}$, $a_{23} = \widetilde{a}_{23}$, $a_{24} = \widetilde{a}_{24}$, and one of the cases:

(i) the equations (13), (15),

(ii) the equations (14), (16)

is possible.

If signs (plus or minus) of coefficients a_{14} and a_{22} are known, then only one case of cases (i) or (ii) is possible. □

Theorem 4. *If* $\{s_k\} = \{\widetilde{s}_k\}$, $\{\mu_k\} = \{\widetilde{\mu}_k\}$, $\{\nu_k\} = \{\widetilde{\nu}_k\}$, $a_{21} = \widetilde{a}_{21}$, $a_{22} = \widetilde{a}_{22}$ *then* $L = \widetilde{L}$. *Thus, given three spectra, the coefficients of pencil (1)–(3) are uniquely determined if* a_{21} *and* a_{22} *are known.*

Proof. Theorem 1 implies that $q(x) = \widetilde{q}(x)$, $a_{11} = \widetilde{a}_{11}$, $a_{12} = \widetilde{a}_{12}$, $a_{23} = \widetilde{a}_{23}$, $a_{24} = \widetilde{a}_{24}$, and one of the cases

(i) the equations (13), (15),

(ii) the equations (14), (16)

is possible.

By the assumptions the theorem $a_{21} = \widetilde{a}_{21}$, $a_{22} = \widetilde{a}_{22}$, so we have only one case (i). □

The question arises how, given the spectra, to find the two solutions discribedin Theorem 1. Below, we propose a method for constructing these two solutions and prove another (stronger) duality theorem. It differs in that it uses only five eigenvalues instead of the entire spectrum of problem L.

4 An algorithm for solving the inverse problem

The coefficients $q(x)$, $q_1(x)$, a_{11}, a_{12}, a_{23}, and a_{24} can be found from the spectra of problems L_1 and L_2 with the help of the method described in [4].

It remains to show how to find a_{13}, a_{14}, a_{21}, and a_{22}.

Let s_k be the eigenvalues of problem (1)–(3). Then they are zeros of the characteristic determinant $\Delta(s)$, i.e., satisfy the equalities

$$\begin{aligned} &[-s_k^2\, b_1 + i\, s_k\, b_2 + b_3]\ f_1(s_k) \\ &+ [b_4 + i\, s_k\, b_5]\ f_2(s_k) + f_3(s_k) = 0, \end{aligned} \tag{17}$$

where $f_1(s)$, $f_2(s)$, $f_3(s)$ are certain functions associated with linearly independent solutions to equation (1), and

$$\begin{aligned} &b_1 = a_{14}a_{22},\ b_2 = a_{13}a_{22} + a_{14}a_{21},\ b_3 = a_{13}a_{21}, \\ &b_4 = a_{13} - a_{21},\ b_5 = a_{14} - a_{22}. \end{aligned} \tag{18}$$

If the eigenvalues s_k, $k = 1, 2, 3, 4, 5$ of problem (1)–(3) are such that the determinant of the matrix

$$\| - s_k^2 f_1(s_k) \;\; is_k f_1(s_k) \;\; f_1(s_k) \;\; f_2(s_k) \;\; is_k f_2(s_k)\|$$

of system (17) is nonzero, then the system of algebraic equations with the unknowns b_1, b_2, b_3, b_4, and b_5 has a unique solution. The unknown coefficients a_{13}, a_{14}, a_{21}, and a_{22} are found using b_1, b_2, b_3, b_4, and b_5 from the system of nonlinear algebraic equations (18). This system has two solutions.

The above results in the following.

Theorem 5. *If the eigenvalues s_k ($k = 1, 2, 3, 4, 5$) of problem (1)–(3) are such that the determinant of the matrix*

$$\| - s_k^2 f_1(s_k) \;\; is_k f_1(s_k) \;\; f_1(s_k) \;\; f_2(s_k) \;\; is_k f_2(s_k)\|$$

of system (17) is nonzero, then the coefficients $q(x)$, $q_1(x)$, a_{11}, a_{12}, a_{23}, and a_{24} are uniquely determined and the coefficients a_{13}, a_{14}, a_{21}, and a_{22} are dually determined by these five eigenvalues and the spectra of problems L_1 and L_2.

5 Examples and a counterexample

Example 1. Suppose that the eigenvalues μ_k and ν_k of problems L_1, L_2 respectively, coincide with the roots of the equation

$$-2\cos\mu_k\pi + \left(\frac{3}{\mu_k} + \mu_k\right)\sin\mu_k\pi = 0,$$

$$\cos\nu_k\pi + 3\,\frac{\sin\nu_k\pi}{\nu_k} = 0,$$

and let five eigenvalues of problem L be

$$s_1 = 0.9056531 + 0.0614968\,i, \quad s_2 = 2.0410923 + 0.0617103\,i,$$

$$s_3 = 2.9137034 - 0.0504283\,i, \quad s_4 = 4.0270545 + 0.0804198\,i,$$

$$s_5 = 4.9425939 - 0.0746285\,i.$$

The coefficients $q(x)$, $q_1(x)$, a_{11}, a_{12}, a_{23}, and a_{24} are determined by the eigenvalues μ_k and ν_k of problems L_1 and L_2 by applying the method of [4]: $q(x) = q_1(x) = a_{12} = a_{24} = 0$, $a_{11} = 1$, and $a_{23} = 3$. To find the coefficients a_{13}, a_{14}, a_{21}, and a_{22} system (17) is solved by the Cramer rule. As a result, we have $b_1 = 8$, $b_2 = 10$, $b_3 = 3$, $b_4 = 2$ and $b_5 = 2$ with the accuracy 10^{-3}. Substituting these b_i in system (18) and solving the latter produces two solutions

(i) $a_{13} = 3$, $a_{14} = 4$, $a_{21} = 1$, $a_{22} = 2$;

(ii) $a_{13} = -1$, $a_{14} = -2$, $a_{21} = -3$, $a_{22} = -4$.

Thus, the indicated spectra can be possessed by the two eigenvalue problems:

Problem L.

$$y'' + s^2\, y = 0, \quad y'(0) + y(0) + (3 + 4is)\, y(\pi) = 0,$$
$$y'(\pi) + (1 + 2is)\, y(0) + 3\, y(\pi) = 0.$$

Problem L$^-$.

$$y'' + s^2\, y = 0, \quad y'(0) + y(0) + (-1 - 2is)\, y(\pi) = 0,$$
$$y'(\pi) + (-3 - 4is)\, y(0) + 3\, y(\pi) = 0.$$

If, in addition to the spectra of L, L_1, and L_2, we know that $a_{13} = 3$ and $a_{14} = 4$, we obtain one solution (rather than two), namely, that of problem L (Theorem 2).

If two solutions coincide, then we also obtain one solution. This case is considered in example 2.

Example 2. Suppose that the eigenvalues μ_k and ν_k of problems L_1, L_2 respectively, coincide with the roots of the equation

$$-2\, \cos \mu_k\, \pi + \left(\frac{3}{\mu_k} + \mu_k \right)\, \sin \mu_k\, \pi = 0,$$

$$\cos \nu_k\, \pi + 3\, \frac{\sin \nu_k\, \pi}{\nu_k} = 0,$$

and let five eigenvalues of problem L be

$$s_1 = 1.155913754, \quad s_2 = 2.163308594, \quad s_3 = 3.144507346,$$

$$s_4 = 4.124456887, \quad s_5 = 5.107505070.$$

The coefficients $q(x)$, $q_1(x)$, a_{11}, a_{12}, a_{23}, and a_{24} are determined by eigenvalues μ_k and ν_k of problems L_1 and L_2 by applying the method of [4]:

$$q(x) = q_1(x) = a_{12} = a_{24} = 0, \quad a_{11} = 1, \quad a_{23} = 3.$$

To find the coefficients a_{13}, a_{14}, a_{21}, and a_{22} system (17) is solved by the Cramer rule. As a result, we have

$$b_1 = 0, \quad b_2 = 0, \quad b_3 = 0, \quad b_4 = 0, \quad b_5 = 0,$$

with the accuracy 10^{-5}. Substituting these b_i in system (18) and solving the latter produces one solution: $a_{13} = 0, \quad a_{14} = 0, \quad a_{21} = 0, \quad a_{22} = 0.$

Example 3 (Counterexample). In Theorem 3, the condition that the determinant of the matrix of system (17) is nonzero cannot be omitted. A relevant example is as follows. Suppose that the eigenvalues μ_k and ν_k of problems L_1, L_2 respectively, coincide with the roots of the equation

$$-2\, \cos \mu_k\, \pi + \left(\frac{3}{\mu_k} + \mu_k \right)\, \sin \mu_k\, \pi = 0,$$

$$\cos\nu_k\,\pi + 3\,\frac{\sin\nu_k\,\pi}{\nu_k} = 0,$$

and let five eigenvalues of L be as follows up to 10^{-2}:

$$s_1 = 100, \quad s_2 = 200, \quad s_3 = 300,$$

$$s_4 = 400, \quad s_5 = 500.$$

The coefficients $q(x)$, $q_1(x)$, a_{11}, a_{12}, a_{23}, and a_{24} are determined by the eigenvalues μ_k and ν_k of problems L_1 and L_2 by applying the method of [4]: $q(x) = q_1(x) = a_{12} = a_{24} = 0$, $a_{11} = 1$, and $a_{23} = 3$. To find a_{13}, a_{14}, a_{21}, and a_{22}, we solve system (17):

$$4 - 2\,b_4 - 200\,i\,b_5 = 0, \quad 4 - 2\,b_4 - 400\,i\,b_5 = 0, \quad 4 - 2\,b_4 - 600\,i\,b_5 = 0,$$

$$4 - 2\,b_4 - 800\,i\,b_5 = 0, \quad 4 - 2\,b_4 - 1000\,i\,b_5 = 0.$$

As a result, we obtain not a unique solution but an infinite set of solutions: $b_1 = C_1$, $b_2 = C_2$, $b_3 = C_3$, $b_4 = 2$, and $b_5 = 0$.

Thus, it may happen that the indicated spectra are possessed not only by the two eigenvalue problems

Problem L.

$$y'' + s^2\,y = 0, \quad y'(0) + y(0) + (3 + 4is)\,y(\pi) = 0,$$
$$y'(\pi) + (1 + 2is)\,y(0) + 3\,y(\pi) = 0,$$

Problem L⁻.

$$y'' + s^2\,y = 0, \quad y'(0) + y(0) + (-1 - 2is)\,y(\pi) = 0,$$

$$y'(\pi) + (-3 - 4is)\,y(0) + 3\,y(\pi) = 0,$$

but by an infinite set of eigenvalue problems.

The reason for this is that the five values s_k were not the first five eigenvalues but the eigenvalues that are far away from zero. In this case, s_k are asymptotically close to the numbers k. As a result, $f_1(s_k)$, $s_k\,f_1(s_k)$, and $s_k^2\,f_1(s_k)$ are close to zero. Therefore, the determinant of system (17) becomes close to zero.

6 Conclusion

The results explained in the previous sections show that the eigenvalue boundary problem (1)–(3) for operator pencils with nonseparated boundary conditions can be reconstructed using three spectra by the numerical methods.

Acknowledgement. The research was supported by the Russian Foundation for Basic Research (grant 09-01-00440-a) and the Academy of Sciences of the Republic Bashkortostan (grants 08-01-97008-p_Povolzh'e_a and 08-01-97026-p_Povolzh'e_a).

References

[1] V.A. Ambarzumijan, *Uber eine Frage der Eigenwerttheorie*. Zeitshrift fur Physik, 53 (1929), 690 – 695.

[2] P.A. Binding, H. Volkmer, *Inverse Oscillation Theory for Sturm-Liouville Problems with Nonseparated Boundary Conditions*, Inverse Probl., 23, no. 1 (2007), 343 – 356.

[3] G. Borg, *Eine Umkehrung der Sturm Liouvilleschen Eigenwertanfgabe. Bestimmung der Differentialgleichung durch die Eigenwarte*. Acta Math., 78:1 (1946), 1 – 96.

[4] S.A. Buterin, V.A. Yurko, *The Inverse Spectral Problem for Pencils of Differential Operators on a Finite Interval*. Vestn. Bashkir. Univ., 4 (2006), 8 – 12 (in Russian).

[5] G.M. Gubreev, V.N. Pivovarchik, *Spectral Analysis of Redzhe Problem with Parameters*. Funkts. Anal. Ego Prilozh., 1 (1997), 70 – 74 (in Russian).

[6] I.M. Guseinov, I.M. Nabiev, *The Inverse Spectral Problem for Pencils of Differential Operators*. Sb. Math., 198, no. 11 (2007), 1579 – 1598 (in Russian).

[7] B.Ya. Levin, *Distribution of Zeros of Entire Functions*. Amer. Math Soc., Providence, 1980.

[8] M.A. Naimark, *Linear Differential Operators*. Ungar, New York, 1967.

[9] O.A. Plaksina, *Inverse Problems of Spectral Analysis for Sturm-Liouville Operators with Nonseparated Boundary Conditions, I*. Math. Ussr. Sb., 59, no. 1 (1988), 1 – 23 (in Russian).

[10] O.A. Plaksina, *Inverse Problems of Spectral Analysis for Sturm-Liouville Operators with Nonseparated Boundary Conditions, II*. Math. Ussr. Sb., 64, no. 1 (1989), 141 – 160 (in Russian).

[11] V.A. Sadovnichii, Ya.T. Sultanaev, A.M. Akhtyamov, *Analogues of Borg's Uniqueness Theorem in the Case of Nonseparated Boundary Conditions*, Doklady Mathematics, 60, no. 1 (1999), 115 – 117 (in Russian).

[12] V.A. Sadovnichii, Ya.T. Sultanaev, A.M. Akhtyamov, *Well-Posedness of the Inverse Sturm-Liouville Problem with Undecomposable Boundary Conditions*. Doklady Mathematical Sciences, 69, no. 2 (2004), 105 – 107 (in Russian).

[13] V.A. Sadovnichii, Ya.T. Sultanaev, A.M. Akhtyamov, *Inverse Sturm-Liouville Problems with Generalized Periodic Boundary Conditions*. Differential Equations, 45, no. 4 (2009), 526 – 538 (in Russian).

[14] V.A. Sadovnichii, Ya.T. Sultanaev, A.M. Akhtyamov, *Inverse Sturm-Liouville Problems with Nonseparated Boundary Conditions*. MSU, Moscow, 2009 (in Russian).

[15] Van der Mee, V.N. Pivovarchik, *A Sturm-Liouville Inverse Spectral Problem with Boundary Conditions Depending on the Spectral Parameter*. Funkts. Anal. Prilozh. Ego Prilozh., 36, no. 4 (2002), 74 – 77 (in Russian).

[16] V.A. Yurko, *Inverse Spectral Problems for Linear Differential Operators and their Applications*. Gordon and Breach, New York, 2000.

[17] V.A. Yurko, *An Inverse Problem for Differential Operator Pencils*. Sb. Math., 191, no. 10 (2000), 1561 – 1586 (in Russian).

Azamat Akhtyamov
Institute of Mechanics
Ufa Scientific Centre
71 Oktyabrya Pr,
450054 Ufa, Russia
E-mail: AkhtyamovAM@mail.ru

Victor Sadovnichy
M. V. Lomonosov Moscow State University
Leninskie gory
119992 Moscow, Russia
E-mail: rector@rector.msu.su

Yavdat Sultanaev
Bashkir State University
32 Zaki Validi St,
450074 Ufa, Russia
E-mail: SultanaevYT@bsu.bashedu.ru, yavdat@bgua.bashkiria.su

Received: 05.06.2010