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ESTIMATION OF ERROR OF CUBATURE FORMULA IN BESOV SPACE

E.S. Smailov, N.T. Tleukhanova

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Abstract. In this paper estimates of the error of a cubature formula in Besov classes are obtained. The method of research is essentially based on the choice of a Lizorkin system.

1 Introduction

Let F be a normed space embedded in the space of continuons functions $C[0,1]^n$, $c = \{c_1, ..., c_M\}, \quad t = \{t_1, ..., t_M\}$, $c_k \in \mathbb{R}$, $t_k \in [0,1)^n$, $k = 1, 2, ..., M$.

It is said that the quantity

$$\delta_M(F; c, t) = \sup_{\|f\|_F = 1} \left| \int_{[0,1]^n} f(y)dy - \sum_{k=1}^{M} c_k f(t_k) \right|,$$

is the error of the cubature formula

$$\sum_{k=1}^{M} c_k f(t_k)$$

in the class of functions F.

Note that there are many works dedicated to estimation of the error of a cubature formula and construction of cubature formulas, which are best possible in order, in various spaces with dominating mixed derivatives, namely in Sobolev spaces W_p^α (N.S. Bakhvalov [1], K.K. Frolov [2], V.N. Temlyakov [9]), in Korobov spaces E^α (N.M. Korobov [3], I.F. Sharygin [8], N.S. Bakhvalov [1]), in Nikol'skii spaces H_p^α (V.N. Temlyakov [9], [10]).

In this paper estimates of the error of a cubature formula in Besov classes $B_{p\theta}^\alpha[0,1]^2$ are obtained. The method of research is essentially related to the choice of a Lizorkin system.

Such systems are ortogonal and are constructed on the basis of the trigonometric system. They also possess the basic properties of the Haar systems, for example the property of being an unconditional basis in the space L_p. These systems were introduced by P.I. Lizorkin [4], and new theorems on multipliers of the Fourier series were obtained with the help of such systems.

2 Lizorkin block bases and some of their properties

Let $\{e^{2\pi ikx}\}_{n=-\infty}^{\infty}$ be the trigonometric system. The Lizorkin block system $E = \{E_n(x)\}_{n=0}^{\infty}$ is defined in the following way.

Let $n = 2^{m-1}+s$, where $m = 1, 2, ..., s = 0, ..., 2^{m-1}-1$, $E_0(x) = 1$, $E_1(x) = e^{2\pi ix}$,

$$E_n(x) = E^{(s)+}_{2^{m-1}-1}(x), \quad E_{-n}(x) = E^{(s)-}_{2^{m-1}-1}(x),$$

where

$$E^{(s)\pm}_{2^{m-1}-1}(x) = \frac{1}{2^{m-1}} \sum_{k=2^{m-1}}^{2^m-1} e^{\pm 2\pi ik(x-\frac{2s+1}{2^m})}.$$

We need some properties of the Lizorkin system.

Lemma 1. *Let $\nu \in \mathbb{Z}_+, 0 \leqslant s < 2^{\nu-1}, \quad m \in \mathbb{N}$. Then*

$$d^s_{\nu m} := \frac{1}{2^m} \sum_{r=0}^{2^m-1} E^{(s)\pm}_{2^{\nu-1}-1}\left(\frac{r}{2^m}\right) = \begin{cases} 0, & 1 \leqslant \nu \leqslant m, \\ 1, & \nu = 0, \\ \frac{1}{2^{\nu-1}} \sum\limits_{q=2^{\nu-m-1}}^{2^{\nu-m}-1} e^{\pm 2\pi i(2s+1)\frac{q}{2^{\nu-m}}}, & \nu > m. \end{cases}$$

Here $E^{(s)\pm}_{2^{\nu-1}}(x) := 1$ if $\nu = 0$.

Proof. Note that

$$d^s_{\nu m} = \frac{1}{2^m} \sum_{r=0}^{2^m-1} \frac{1}{2^{\nu-1}} \sum_{l=2^{\nu-1}}^{2^\nu-1} e^{\pm 2\pi il(\frac{r}{2^m}-\frac{2s+1}{2^\nu})} =$$

$$= \frac{1}{2^{m+\nu-1}} \sum_{l=2^{\nu-1}}^{2^\nu-1} e^{\mp 2\pi il\frac{2s+1}{2^\nu}} \sum_{r=0}^{2^m-1} e^{\pm 2\pi il\frac{r}{2^m}}.$$

It is well known that

$$\sum_{r=0}^{2^m-1} e^{\pm 2\pi il\frac{r}{2^m}} = \begin{cases} 0, & l \neq 2^m q, q \in \mathbb{Z}, \\ 2^m, & l = 2^m q. \end{cases}$$

Let $\nu \leqslant m$, then $l \leqslant 2^{m-1}$ and l is not an integer multiple of 2^m. Hence $d^s_{\nu m} = 0$.

Let $\nu > m$, then $2^{\nu-1} \leqslant l < 2^\nu$. In this interval the of numbers of l, which are integer multiples of 2^m, is equal to $2^{\nu-1-m}$.

Then we have

$$d^s_{\nu m} = \frac{2^m}{2^{m+\nu-1}} \sum_{l=2^{\nu-m-1}}^{2^{\nu-m}-1} e^{\pm 2\pi i2^m q\frac{2s+1}{2^\nu}} = \frac{1}{2^{\nu-1}} \sum_{q=2^{\nu-m-1}}^{2^{\nu-m}-1} e^{\pm 2\pi i(2s+1)\frac{q}{2^{\nu-m}}}.$$

□

Corollary 1. *Let* $\nu \in \mathbb{Z}_+, 0 \leqslant s < 2^{\nu-1}, \quad m \in \mathbb{N}$. *Then for numbers* $d^s_{\nu m}$ *the following estimation hold:*

$$|d^s_{\nu m}| \leqslant \begin{cases} 0, & 1 \leqslant \nu \leqslant m, \\ 1, & \nu = 0, \\ \frac{1}{2^m}, & \nu > m. \end{cases}$$

Let $1 < p < \infty, 0 < \theta \leqslant \infty, r > 0$. We consider the spaces $B^r_{p\theta}([0,1]^2$ ([5]), which are the sets of all trigonometric series (which are not necessarily convergent)

$$f(x,y) = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} a_{k_1 k_2} e^{2\pi i (k_1 x + k_2 y)}, \tag{1}$$

for which

$$\|f\|_{B^r_{p\theta}([0,1]^2)} = \left(\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} 2^{r(k_1+k_2)\theta} \|\delta_{k_1 k_2}(f)\|^{\theta}_{L_p} \right)^{1/\theta} < \infty, \text{ if } 0 < \theta < \infty$$

and

$$\|f\|_{B^r_{p\infty}([0,1]^2)} = \sup_{k_i \in \mathbb{N}} 2^{r(k_1+k_2)} \|\delta_{k_1 k_2}(f)\|_{L_p} < \infty, \text{ if } \theta = \infty,$$

where

$$\delta_{k_1 k_2}(f) = \sum_{[2^{k_1-1}] \leqslant |m_1| < 2^{k_1}} \sum_{[2^{k_2-1}] \leqslant |m_2| < 2^{k_2}} a_{m_1 m_2} e^{2\pi i (m_1 x + m_2 y)}.$$

Let $f \in C[0,1]^n$, $m \in \mathbb{N}^2$, $k_1 = 2^{m_1} + s_1, 0 \leqslant s_1 < 2^{m_1} - 1$, $k_2 = 2^{m_2} + s_2, 0 \leqslant s_2 < 2^{m_2} - 1$, and

$$\Gamma(k_1,k_2) = 2^{(m_1-2)+(m_2-2)} \int_0^1 \int_0^1 f(x,y) \overline{E_{k_1 k_2}(x,y)} dxdy,$$

where $E_{k_1 k_2}(x,y) = E_{k_1}(x) E_{k_2}(y)$.

Theorem 1. *The series (1) belongs to the space* $B^r_{p\theta}([0,1]^2)$, *where* $r > 0$, $1 < p < \infty, 1 \leqslant \theta \leqslant \infty$ *if and only if the quantities* $\Gamma(k_1,k_2)$ *satisfy the condition*

$$\beta^r_{p,\theta}(f) =$$

$$= \left(\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} 2^{(k_1+k_2)(r-\frac{1}{p})\theta} \left(\sum_{[2^{k_1-1}] \leqslant |m_1| < 2^{k_1}} \sum_{[2^{k_2-1}] \leqslant |m_2| < 2^{k_2}} |\Gamma(m_1,m_2)|^p \right)^{\frac{\theta}{p}} \right)^{\frac{1}{\theta}} < \infty$$

if $1 \leqslant \theta < \infty$ *and*

$$\beta^r_{p,\infty}(f) = \sup_{\substack{k_2 \geqslant 0 \\ k_1 \geqslant 0}} 2^{(k_1+k_2)(r-\frac{1}{p})} \left(\sum_{[2^{k_1-1}] \leqslant |m_1| < 2^{k_1}} \sum_{[2^{k_2-1}] \leqslant |m_2| < 2^{k_2}} |\Gamma(m_1,m_2)|^p \right)^{\frac{1}{p}} < \infty$$

if $\theta = \infty$. *Moreover* $\beta^r_{p,\theta}(f)$ *is equivalent to the norm* $\|f\|_{B^r_{p\theta}}$.

In the case $n = 1$ theorem was proved by P.I. Lizorkin [4]. The proof for $n = 2$ is similar.

3 Estimation of error of a cubature formula in the space $B^{\alpha}_{p\theta}[0,1]^2$

Let $f \in \mathbb{C}[0,1]^2$ be a periodic function with period 1 and $m \in \mathbb{N}$. The following cubature formula

$$T_{2^m}(f) = \sum_{\substack{k_1+k_2=m \\ k_1,k_2\geqslant 0}} \frac{1}{2^m} \sum_{r_1=0}^{2^{k_1}-1} \sum_{r_2=0}^{2^{k_2}-1} (-1)^{r_1+\mathrm{sgn}\,\mathrm{k}_1} f\left(\frac{r_1}{2^{k_1}}, \frac{r_2}{2^{k_2}}\right).$$

was introduced and studied in [6], [7]. In [7] it was shown that it is best possible in order in Sobolev spaces W^{α}_p and in Korobov spaces E^{α}.

Theorem 2. *Let*

$$f^{\pm\pm}(x,y) = \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} \sum_{s_1=0}^{2^{\nu_1-1}} \sum_{s_2=0}^{2^{\nu_2-1}} \Gamma(2^{\nu_1-1}+s_1, 2^{\nu_2-1}+s_2) \cdot E^{(s)\pm}_{2^{\nu_1-1}}(x) E^{(s)\pm}_{2^{\nu_2-1}}(x)$$

be an absolutely convergent series. Then

$$\left| T_{2^m}(f^{\pm\pm}) - \int_{[0,1]^2} f^{\pm\pm}(x,y)dxdy \right| \leqslant$$

$$\leqslant \frac{1}{2^m} \sum_{t=m+1}^{\infty} \min(t-m, m) \sum_{\substack{\nu_1+\nu_2=t \\ \nu_1,\nu_2\geqslant 0}} \sum_{s_1=0}^{2^{\nu_1-1}} \sum_{s_2=0}^{2^{\nu_2-1}} \Gamma(2^{\nu_1-1}+s_1, 2^{\nu_2-1}+s_2).$$

Proof. We prove the statement for f^{++}, the other cases being similar. Note that

$$T_{2^m}(f) = \frac{1}{2^m} \sum_{\substack{m_1+m_2=m \\ m_1,m_2\geqslant 0}} \sum_{r_1=0}^{2^{m_1}-1} \sum_{r_2=0}^{2^{m_2}-1} f(\frac{r_1}{2^{m_1}}, \frac{r_2}{2^{m_2}}) -$$

$$-\frac{1}{2^{m-1}} \sum_{\substack{m_1+m_2=m-1 \\ m_1,m_2\geqslant 0}} \sum_{r_1=0}^{2^{m_1}-1} \sum_{r_2=0}^{2^{m_2}-1} f(\frac{r_1}{2^{m_1}}, \frac{r_2}{2^{m_2}}).$$

Since the series is absolutely convergent, we have

$$\left|T_{2^m}(f^{\pm\pm}) - \Gamma(0,0)\right| = \left| \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} \sum_{s_1=0}^{2^{\nu_1-1}} \sum_{s_2=0}^{2^{\nu_2-1}} \Gamma(2^{\nu_1-1}+s_1, 2^{\nu_2-1}+s_2) \cdot \right.$$

$$\cdot \left[\sum_{m_1+m_2=m} \left(\frac{1}{2^{m_1}} \sum_{r_1=0}^{2^{m_1}} E^{(s)\pm}_{2^{\nu_1-1}}\left(\frac{r_1}{2^{m_1}}\right) \cdot \frac{1}{2^{m_2}} \sum_{r_2=0}^{2^{m_2}} E^{(s)\pm}_{2^{\nu_2-1}}\left(\frac{r_2}{2^{m_2}}\right) \right) - \right.$$

$$\left. \left. - \sum_{m_1+m_2=m-1} \left(\frac{1}{2^{m_1}} \sum_{r_1=0}^{2^{m_1}} E^{(s)\pm}_{2^{\nu_1-1}}\left(\frac{r_1}{2^{m_1}}\right) \cdot \frac{1}{2^{m_2}} \sum_{r_2=0}^{2^{m_2}} E^{(s)\pm}_{2^{\nu_2-1}}\left(\frac{r_2}{2^{m_2}}\right) \right) \right] - \Gamma(0,0) \right| =$$

$$= \left| \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} \sum_{s_1=0}^{2^{\nu_1-1}} \sum_{s_2=0}^{2^{\nu_2-1}} \Gamma(2^{\nu_1-1}+s_1, 2^{\nu_2-1}+s_2) \cdot \right.$$
$$\left. \cdot \left[I_m(\nu_1,\nu_2) - I_{m-1}(\nu_1,\nu_2) \right] - \Gamma(0,0) \right|,$$

where

$$I_m(\nu_1,\nu_2) = \sum_{m_1+m_2=m} \left(\frac{1}{2^{m_1}} \sum_{r_1=0}^{2^{m_1}} E^{(s)\pm}_{2^{\nu_1-1}} \left(\frac{r_1}{2^{m_1}} \right) \cdot \frac{1}{2^{m_2}} \sum_{r_2=0}^{2^{m_2}} E^{(s)\pm}_{2^{\nu_2-1}} \left(\frac{r_2}{2^{m_2}} \right) \right).$$

We consider the difference $I_m(\nu_1,\nu_2) - I_{m-1}(\nu_1,\nu_2)$ separately for the cases in which $(\nu_1,\nu_2) = (0,0); (\nu_1,\nu_2) = (0,\nu_2), \nu_2 > 0; (\nu_1,\nu_2) = (\nu_1,0), \nu_1 > 0$ and $(\nu_1,\nu_2), \nu_1 > 0, \nu_2 > 0$.

Let $(\nu_1,\nu_2) = (0,0)$. Taking into account Lemma 1 we have

$$I_m(0,0) = \sum_{\substack{m_1+m_2=m \\ m_1,m_2 \geqslant 0}} d^{s_1}_{0m_1} d^{s_2}_{0m_2} = m+1,$$

hence

$$I_m(0,0) - I_{m-1}(0,0) = 1.$$

Next let $(\nu_1,\nu_2) = (0,\nu_2), \nu_2 > 0$. By Lemma 1 we get

$$I_m(0,\nu_2) = \sum_{\substack{m_1+m_2=m \\ m_1,m_2 \geqslant 0}} d^{s_2}_{\nu_2 m_2} = \sum_{m_2=0}^{m-1} d^{s_2}_{\nu_2 m_2}.$$

Taking into account that $\nu_2 \geqslant 1$ and the fact that by Lemma 1 $d^{s_2}_{\nu_2 m_2} = 0, \nu_2 \leqslant m_2$ we have

$$I_m(0,\nu_2) = \sum_{m_2=0}^{\min(m,\nu_2)-1} d^{s_2}_{\nu_2 m_2}.$$

Then

$$I_m(0,\nu_2) - I_{m-1}(0,\nu_2) = \sum_{m_2=\min(m-1,\nu_2)}^{\min(m,\nu_2)} d^{s_2}_{\nu_2 m} = \begin{cases} 0 & \text{for } \nu_2 < m\,, \\ d^{s_2}_{\nu_2 m} & \text{for } \nu_2 \geqslant m\,. \end{cases}$$

Similarly, if $\nu_1 > 0$ then

$$I_m(\nu_1,0) - I_{m-1}(\nu_1,0) = \begin{cases} 0 & \text{for } \nu_1 < m\,, \\ d^{s_2}_{\nu_1 m} & \text{for } \nu_1 \geqslant m\,. \end{cases}$$

Finally let $\nu_1 \geqslant 1, \nu_2 \geqslant 1$. By Lemma 1

$$I_m(\nu_1,\nu_2) = \sum_{\substack{m_1+m_2=m \\ m_1,m_2 \geqslant 0}} d_{\nu_1 m} d_{\nu_2 m} = \begin{cases} 0 & \text{for } \nu_1+\nu_2 \leqslant m\,, \\ \sum\limits_{\substack{m_1+m_2=m \\ m_1<\nu_1, m_2<\nu_2}} d_{\nu_1 m_1} d_{\nu_2 m_2} & \text{for } \nu_1+\nu_2 = m+t, t \geqslant 1\,. \end{cases}$$

Applying Corollary of Lemma 1 we obtain

$$\left|\sum_{\substack{m_1+m_2=m\\ m_1<\nu_1,m_2<\nu_2}} d_{\nu_1 m_1} d_{\nu_2 m_2}\right| \leqslant \sum_{\substack{m_1+m_2=m\\ m_1<\nu_1,m_2<\nu_2}} \frac{1}{2^{m_1}}\frac{1}{2^{m_2}} =$$

$$= \frac{1}{2^m}\sum_{\substack{m_1+m_2=m\\ m_1<\nu_1,m_2<\nu_2}} 1 \leqslant \frac{\min(t,m)}{2^m},$$

thus

$$|I_m(\nu_1,\nu_2)| \leqslant \begin{cases} 0 & \text{for } \nu_1+\nu_2 \leqslant m \ , \\ \frac{\min(t,m)}{2^m} & \text{for } \nu_1+\nu_2 = m+t, t\geqslant 1 \ . \end{cases}$$

By the above estimates we get

$$\left|T_{2^m}(f^{\pm\pm}) - \Gamma(0,0)\right| = \Bigg|\Bigg\{\sum_{\nu_1=1}^{\infty}\sum_{\nu_2=1}^{\infty}\sum_{s_1=0}^{2^{\nu_1-1}}\sum_{s_2=0}^{2^{\nu_2-1}} \Gamma(2^{\nu_1-1}+s_1, 2^{\nu_2-1}+s_2)\cdot$$

$$(I_m(\nu_1,\nu_2) - I_{m-1}(\nu_1,\nu_2)) + \sum_{\nu_1=1}^{\infty}\sum_{s_1=0}^{2^{\nu_1-1}} \Gamma(2^{\nu_1-1}+s_1,0)\cdot(I_m(\nu,0)-I_{m-1}(\nu,0)) +$$

$$+\sum_{\nu_2=1}^{\infty}\sum_{s_2=0}^{2^{\nu_2-1}} \Gamma(0,2^{\nu_2-1}+s_2)\cdot(I_m(0,\nu)-I_{m-1}(0,\nu))\Bigg\} +$$

$$+\Gamma(0,0)(I_m(0,0)-I_{m-1}(0,0)) - \Gamma(0,0)| =$$

$$= \Bigg|\sum_{\nu_1=1}^{\infty}\sum_{\nu_2=1}^{\infty}\sum_{s_1=0}^{2^{\nu_1-1}}\sum_{s_2=0}^{2^{\nu_2-1}} \Gamma(2^{\nu_1-1}+s_1, 2^{\nu_2-1}+s_2)\cdot (I_m(\nu_1,\nu_2) - I_{m-1}(\nu_1,\nu_2)) +$$

$$+\sum_{\nu_1=1}^{\infty}\sum_{s_1=0}^{2^{\nu_1-1}} \Gamma(2^{\nu_1-1}+s_1,0)\cdot(I_m(\nu,0)-I_{m-1}(\nu,0)) +$$

$$+\sum_{\nu_2=1}^{\infty}\sum_{s_2=0}^{2^{\nu_2-1}} \Gamma(0,2^{\nu_2-1}+s_2)\cdot(I_m(0,\nu)-I_{m-1}(0,\nu))\Bigg| \leqslant$$

$$\leqslant \frac{1}{2^m}\sum_{t=0}^{\infty}\sum_{\substack{\nu_1+\nu_2=t+m\\ \nu_1\geqslant 1,\nu_2\geqslant 1}}\sum_{s_1=0}^{2^{\nu_1-1}}\sum_{s_2=0}^{2^{\nu_2-1}} |\Gamma(2^{\nu_1}+s_1,2^{\nu_2}+s_2)|\min(t,m) +$$

$$+\frac{1}{2^m}\sum_{\nu_1=m+1}^{\infty}\sum_{s_1=0}^{2^{\nu_1-1}} |\Gamma(2^{\nu_1-1}+s_1,0)| + \frac{1}{2^m}\sum_{\nu_2=m+1}^{\infty}\sum_{s_2=0}^{2^{\nu_2-1}} |\Gamma(0,2^{\nu_2-1}+s_2)| \leqslant$$

$$\leqslant \frac{1}{2^m}\sum_{t=m+1}^{\infty}\min(t-m,m)\sum_{\substack{\nu_1+\nu_2=t\\ \nu_1\geqslant 0,\nu_2\geqslant 0}}\sum_{s_1=0}^{2^{\nu_1-1}}\sum_{s_2=0}^{2^{\nu_2-1}} \Gamma(2^{\nu_1}+s_1,2^{\nu_2}+s_2).$$

□

Theorem 3. *Let*

$$f(x,y)=\sum_{\nu_1=-\infty}^{\infty}\sum_{\nu_2=-\infty}^{\infty}\Gamma(\nu_1,\nu_2)E_{\nu_1,\nu_2}(x,y)$$

be an absolutely convergent series. Then

$$\left|T_{2^m}(f)-\int_{[0,1]^2}f(x,y)dxdy\right|\leqslant\frac{1}{2^m}\sum_{t=m+1}^{\infty}\min(t-m,m)\times$$

$$\times\left[\sum_{\substack{\nu_1+\nu_2=t\\ \nu_i\geqslant 0}}\sum_{s_1=0}^{2^{\nu_1-1}}\sum_{s_2=0}^{2^{\nu_2-1}}|\Gamma(2^{\nu_1-1}+s_1,2^{\nu_2-1}+s_2)|+|\Gamma(-2^{\nu_1-1}-s_1,-2^{\nu_2-1}-s_2)|+\right.$$

$$\left.+\sum_{\substack{\nu_1+\nu_2=t\\ \nu_1\geqslant 1,\nu_2\geqslant 1}}\sum_{s_1=0}^{2^{\nu_1-1}}\sum_{s_2=0}^{2^{\nu_2-1}}|\Gamma(-2^{\nu_1-1}-s_1,2^{\nu_2-1}+s_2)|+|\Gamma(2^{\nu_1+1}-s_1,-2^{\nu_2-1}-s_2)|\right].$$

Proof. We decompose f in the following way

$$f=f^{++}+f^{+-}+f^{-+}+f^{--},\tag{2}$$

where

$$f^{++}=\sum_{\nu_1=-\infty}^{\infty}\sum_{\nu_2=-\infty}^{\infty}\Gamma^{++}(\nu_1,\nu_2)E_{\nu_1,\nu_2}(x,y),$$

$$f^{+-}=\sum_{\nu_1=-\infty}^{\infty}\sum_{\nu_2=-\infty}^{\infty}\Gamma^{+-}(\nu_1,\nu_2)E_{\nu_1,\nu_2}(x,y),$$

$$f^{-+}=\sum_{\nu_1=-\infty}^{\infty}\sum_{\nu_2=-\infty}^{\infty}\Gamma^{-+}(\nu_1,\nu_2)E_{\nu_1,\nu_2}(x,y),$$

$$f=\sum_{\nu_1=-\infty}^{\infty}\sum_{\nu_2=-\infty}^{\infty}\Gamma^{--}(\nu_1,\nu_2)E_{\nu_1,\nu_2}(x,y),$$

$$\Gamma^{++}(\nu_1,\nu_2)=\begin{cases}\Gamma(\nu_1,\nu_2), & \nu_1\geqslant 0,\nu_2\geqslant 0,\\ 0, & \text{otherwise },\end{cases}$$

$$\Gamma^{+-}(\nu_1,\nu_2)=\begin{cases}\Gamma(\nu_1,\nu_2), & \nu_1\geqslant 1,\nu_2\leqslant -1,\\ 0, & \text{otherwise },\end{cases}$$

$$\Gamma^{-+}(\nu_1,\nu_2)=\begin{cases}\Gamma(\nu_1,\nu_2), & \nu_1\leqslant -1,\nu_2\geqslant 1\\ 0, & \text{otherwise },\end{cases}$$

$$\Gamma^{--}(\nu_1,\nu_2)=\begin{cases}\Gamma(\nu_1,\nu_2), & \nu_1\leqslant 0,\nu_2\leqslant 0,(\nu_1,\nu_2)\neq(0,0)\\ 0, & \text{otherwise }.\end{cases}$$

Applying Theorem 2 to each function in decompozition (2) we get the statement of the theorem. □

Let $m \in \mathbb{N}, \nu \in \mathbb{Z}_+^2$,

$$\rho(\nu) = \{s = (s_1, s_2) \in \mathbb{N}^2 : \mathrm{sgn}\,(\nu_\mathrm{i} - 1)2^{\nu_\mathrm{i}-2} < |\mathrm{s_i}| \leqslant 2^{\nu_\mathrm{i}-1},\ \ \mathrm{i} = 1, 2\}.$$

The set

$$G_m = \{r \in \mathbb{Z}^2 : (|r_1|, |r_2|) \in \bigcup_{\nu_1+\nu_2 \leqslant m+1} \rho(\nu)\} \setminus \{\pm[2^{k_1-1}], \pm[2^{k_2-1}], k_1 + k_2 = m + 1\}$$

is called steplike hyperbolic cross of the order m.

Let $f \in B^\alpha_{p\theta}[0, 1]^2$ and

$$E_{G_m}(f)_{B^\alpha_{p\theta}} = \inf_{\{a_{k_1k_2}\}} \|f - \sum_{k \in G_m} a_{k_1k_2} e^{2\pi i(k_1x+k_2y)}\|_{B^\alpha_{p\theta}}.$$

Theorem 4. *Let $\alpha > 1, 1 < p < \infty, 1 \leqslant \theta \leqslant \infty, 1/\theta + 1/\theta' = 1$, then for any function $f \in B^\alpha_{p\theta}[0, 1]^2$*

$$\left|T_{2^m}(f) - \int_{[0,1]^2} f(x, y)dxdy\right| \leqslant c_{\alpha,p,\theta} \frac{m^{\frac{1}{\theta'}}}{2^{m\alpha}} E_{G_m}(f)_{B^\alpha_{p\theta}},$$

where $c_{\alpha,p,\theta} > 0$ is independent of m and f.

Proof. Without loss of generality it is possible to assume that

$$f(x, y) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \Gamma(k_1, k_2)E_{k_1k_2}(x, y).$$

By Theorem 3 we get

$$\left|T_{2^m}(f) - \int_{[0,1]^2} f(x, y)dxdy\right| \leqslant$$

$$\leqslant \frac{1}{2^m} \sum_{t=m+1}^{\infty} (t - m) \sum_{\substack{\nu_1+\nu_2=t \\ \nu_1 \geqslant 0, \nu_2 \geqslant 0}} \sum_{s_1=0}^{2^{\nu_1-1}} \sum_{s_2=0}^{2^{\nu_2-1}} |\Gamma(2^{\nu_1-1} + s_1, 2^{\nu_2-1} + s_2)| \leqslant$$

$$\leqslant \frac{1}{2^m} \sum_{t=m+1}^{\infty} 2^{\frac{t}{p'}} (t - m) \cdot \sum_{\substack{\nu_1+\nu_2=t \\ \nu_1 \geqslant 0, \nu_2 \geqslant 0}} \left(\sum_{s_1=0}^{2^{\nu_1-1}} \sum_{s_2=0}^{2^{\nu_2-1}} |\Gamma(2^{\nu_1-1} + s_1, 2^{\nu_2-1} + s_2)|^p \right)^{\frac{1}{p}} =$$

$$= \frac{1}{2^m} \sum_{t=m+1}^{\infty} 2^{t(\alpha-\frac{1}{p})} 2^{t(\frac{1}{p'}-(\alpha-\frac{1}{p}))} (t - m) \times$$

$$\times \sum_{\substack{\nu_1+\nu_2=t \\ \nu_1 \geqslant 0, \nu_2 \geqslant 0}} \left(\sum_{s_1=0}^{2^{\nu_1-1}} \sum_{s_2=0}^{2^{\nu_2-1}} |\Gamma(2^{\nu_1-1} + s_1, 2^{\nu_2-1} + s_2)|^p \right)^{\frac{1}{p}} \leqslant$$

$$\leqslant \frac{1}{2^m}\left(\sum_{t=m+1}^{\infty} 2^{t(1-\alpha)\theta'}(t-m)^{\theta'}\sum_{\substack{\nu_1+\nu_2=t\\ \nu_i\geqslant 0}} 1^{\theta'}\right)^{\frac{1}{\theta'}}\times$$

$$\times\left(\sum_{t=m+1}^{\infty} 2^{t(\alpha-\frac{1}{p})\theta}\sum_{\substack{\nu_1+\nu_2=t\\ \nu_1\geqslant 0,\nu_2\geqslant 0}}\left(\sum_{s_1=0}^{2^{\nu_1-1}}\sum_{s_2=0}^{2^{\nu_2-1}}|\Gamma(2^{\nu_1-1}+s_1,2^{\nu_2-1}+s_2)|^p\right)^{\frac{\theta}{p}}\right)^{\frac{1}{\theta}}\leqslant$$

$$\leqslant c_1\frac{m^{\frac{1}{\theta'}}}{2^{m\alpha}}\left(\sum_{t=m+1}^{\infty} 2^{t(\alpha-\frac{1}{p})\theta}\sum_{\substack{\nu_1+\nu_2=t\\ \nu_1\geqslant 0,\nu_2\geqslant 0}}\left(\sum_{s_1=0}^{2^{\nu_1-1}}\sum_{s_2=0}^{2^{\nu_2-1}}|\Gamma(2^{\nu_1-1}+s_1,2^{\nu_2-1}+s_2)|^p\right)^{\frac{\theta}{p}}\right)^{\frac{1}{\theta}}, \quad (3)$$

where $c_1>0$ is independent of f and m.

By Theorem 1 that the last bracket in (3) is equivalent to the norm $\|f-S_{G_m}(f)\|_{B^{\alpha}_{p\theta}}$, where

$$S_{G_m}(f)=\sum_{k\in G_m} a_{k_1k_2}e^{2\pi i(k_1x+k_2y)}.$$

Thus, we have

$$\left|T_{2^m}(f)-\int_{[0,1]^2} f(x,y)dxdy\right|\leqslant c_2\frac{m^{\frac{1}{\theta'}}}{2^{m\alpha}}\|f-S_{G_m}(f)\|_{B^{\alpha}_{p\theta}}, \quad (4)$$

where $c_2>0$ are independent of f and m.

By Theorem 1 we have

$$E_{G_m}(f)_{B^{\alpha}_{p\theta}}=\inf_{\{a_{k_1k_2}\}}\|f-\sum_{k\in G_m} a_{k_1k_2}e^{2\pi i(k_1x+k_2y)}\|_{B^{\alpha}_{p\theta}}=$$

$$=\inf_{\{a_{k_1k_2}\}}\|f-T_{G_m}\|_{B^{\alpha}_{p\theta}}\geqslant$$

$$\geqslant c_3\left(\sum_{t=0}^{m} 2^{t(\alpha-\frac{1}{p})\theta}\sum_{\substack{\nu_1+\nu_2=t\\ \nu_1\geqslant 0,\nu_2\geqslant 0}}\left(\sum_{s_1=[2^{\nu_1-1}]}^{2^{\nu_1}-1}\sum_{s_2=[2^{\nu_2-1}]}^{2^{\nu_2}-1}|\Gamma(s_1,s_2;f-T_{G_m})|^p\right)^{\frac{\theta}{p}}+\right.$$

$$\left.+\sum_{t=m+1}^{\infty} 2^{t(\alpha-\frac{1}{p})\theta}\sum_{\substack{\nu_1+\nu_2=t\\ \nu_1\geqslant 0,\nu_2\geqslant 0}}\left(\sum_{s_1=[2^{\nu_1-1}]}^{2^{\nu_1}-1}\sum_{s_2=[2^{\nu_2-1}]}^{2^{\nu_2}-1}|\Gamma(s_1,s_2;f-T_{G_m})|^p\right)^{\frac{\theta}{p}}\right)^{\frac{1}{\theta}}\geqslant$$

$$\geqslant c_3\left(\sum_{t=m+1}^{\infty} 2^{t(\alpha-\frac{1}{p})\theta}\sum_{\substack{\nu_1+\nu_2=t\\ \nu_1\geqslant 0,\nu_2\geqslant 0}}\left(\sum_{s_1=[2^{\nu_1-1}]}^{2^{\nu_1}-1}\sum_{s_2=[2^{\nu_2-1}]}^{2^{\nu_2}-1}|\Gamma(s_1,s_2;f)|^p\right)^{\frac{\theta}{p}}\right)^{\frac{1}{\theta}}\geqslant$$

$$\geqslant c_4\|f-S_{G_m}(f)\|_{B^{\alpha}_{p\theta}},$$

where $c_3,c_4>0$ are independent of f and m. This inequality together with (4) imply the statement. □

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Esmuhanbet Smailov
Institut of Applied Mathematics
28 A Universitetskaya St,
100028 Karagandy, Kazakhstan
E-mail: esmailov@mail.ru

Nazerke Tleukhanova
Faculty of Mathematics and Information Technology
L.N. Gumilyov Eurasian National University
5 Munaitpasov St,
010008 Astana, Kazakhstan
E-mail: tleukhanova@rambler.ru

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