**ISSN (Print): 2077-9879**
**ISSN (Online): 2617-2658**

# Eurasian Mathematical Journal

**2024, Volume 15, Number 2**

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

# EURASIAN MATHEMATICAL JOURNAL




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13 Kazhymukan St
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EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 15, Number 2 (2024), 75 – 91

# AN EXISTENCE RESULT FOR A $(p(x), q(x))$-KIRCHHOFF TYPE SYSTEM WITH DIRICHLET BOUNDARY CONDITIONS VIA TOPOLOGICAL DEGREE METHOD

**S. Yacini, C. Allalou, K. Hilal**

Communicated by M.A. Ragusa

**Key words:** weak solutions, $(p(x), q(x))$-Kirchhoff type systeme, variable-exponent Sobolev spaces, topological degree methods.

**AMS Mathematics Subject Classification:** 35J35, 47H11, 47H30

**Abstract.** This paper focuses on the existence of at least one weak solution for a nonlocal elliptic system of $(p(x), q(x))$-Kirchhoff type with Dirichlet boundary conditions. The results are obtained by applying the topological degree method of Berkovits applied to an abstract Hammerstein equation associated to our system and also by the theory of the generalized Sobolev spaces.

**DOI: https://doi.org/10.32523/2077-9879-2024-15-2-75-91**

## 1 Introduction

The study of nonlinear boundary value problems involving variable exponents has received considerable attention in the last decades. This is motivated by the developments in elastic mechanics, electrorheological fluids, and image restoration [4, 7, 12, 13, 21, 32, 33].

In this work, we aim to prove the existence of a weak solution for the following nonlocal elliptic system of $(p(x), q(x))$-Kirchhoff type with the Dirichlet boundary conditions:

$$\begin{cases} \mathcal{T}_1(u) = \lambda h(x, u, \nabla u) + \mathcal{Q}(x)|u|^{r_1(x)-2}u & \text{in } \Omega, \\ \mathcal{T}_2(v) = \kappa g(x, v, \nabla v) + \mathcal{O}(x)|v|^{r_2(x)-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\mathcal{T}_1(u) = -\mathcal{N}_1\Big(\int_\Omega (A_1(x, \nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})dx\Big) \times \operatorname{div}\Big(a_1(x, \nabla u) + |\nabla u|^{p(x)-2}\nabla u\Big),$$

and

$$\mathcal{T}_2(v) = -\mathcal{N}_2\Big(\int_\Omega (A_2(x, \nabla v) + \frac{1}{q(x)}|\nabla v|^{q(x)})dx\Big) \times \operatorname{div}\Big(a_2(x, \nabla v) + |\nabla v|^{q(x)-2}\nabla v\Big).$$

Here and in the sequel, $\Omega$ designates a bounded open set in $\mathbb{R}^N (N \geq 2)$, with a Lipschitz boundary denoted by $\partial\Omega$. $p, q, r \in C_+(\overline{\Omega})$, $\lambda$ and $\kappa$ are two real parameters, $-\text{div } a_i(x, \nabla u)$ $(i = 1, 2)$ are Leray-Lions operators, $h, g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are two Carathéodory's functions that satisfy the assumption of growth, $\mathcal{Q}, \mathcal{O} \in L^\infty(\Omega)$ and $\mathcal{N}_i : \mathbb{R}^+ \to \mathbb{R}^+$ are functions that satisfy some conditions which will be stated later. As is well known, problem (1.1) is related to the stationary problem of a model presented by Kirchhoff in 1883 [16]. More precisely, Kirchhoff introduced a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

which extends the classical d'Alembert's wave equation that takes into account the effects of length changes of the string produced by transverse vibrations, the parameters in (1.2) have the following meanings: $h$ is the cross-section area, $E$ is the Young modulus, $\rho$ is the mass density, $L$ is the length of the string, and $\rho_0$ is the initial tension.

The Kirchhoff type equations involving variable exponent growth conditions have been a very interesting topic in recent years, it has been studied in many papers; we refer to [10, 11, 19, 23, 29] in which variational methods have been used to get the existence and multiplicity of solutions, on the other hand, many authors used the topological degree methods to prove the existence of solutions see for example (see, for example,[9, 24, 25, 27, 28]).

The purpose of this work is to study the existence of solutions to the problem (1.1) in the Sobolev spaces with variable exponents by using another approach based on the topological degree of Berkovits based on the Leray-Schauder principle, presented in [5, 6] for a class of demicontinuous operators of generalized $(S_+)$ type, and the theory of the variable-exponent Sobolev spaces.

This article is arranged as follows. In Section 2, we recall some basic facts about the variable exponent Lebesgue and Sobolev spaces and we introduce, some classes of operators of generalized $(S_+)$ type and the topological degree, while Section 3 is devoted to the existence of at least one weak solution for problem (1.1).

# 2 Preliminary results

## 2.1 The generalized Lebesgue-Sobolev spaces:

First, we introduce some definitions and basic properties of the Lebesgue-Sobolev spaces with variable exponents $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. In this context, we refer to [14, 18, 31] for more details.

Let us set $C_+(\overline{\Omega})=\Big\{p:p\in C(\overline{\Omega})$ and is such that $p(x)>1$ for all $x\in\overline{\Omega}\Big\}$.

For each $p\in C_+(\overline{\Omega})$, we define $p^+ := \max\big\{p(x),\ x\in\overline{\Omega}\big\}$ and $p^- := \min\big\{p(x),\ x\in\overline{\Omega}\big\}$.

For every $p\in C_+(\overline{\Omega})$, we define

$$L^{p(x)}(\Omega)=\Big\{v:\Omega\to\mathbb{R}\ \text{ is measurable and such that }\ \int_\Omega|v(x)|^{p(x)}dx<+\infty\Big\},$$

equipped with the Luxemburg norm given by

$$|v|_{p(x)}=\inf\Big\{\varepsilon>0,\ \int_\Omega\Big|\frac{v(x)}{\varepsilon}\Big|^{p(x)}dx\le 1\Big\},$$

$\big(L^{p(\cdot)}(\Omega),|\cdot|_{p(\cdot)}\big)$, we call it the generalized Lebesgue space, is a separable, and reflexive Banach space (see, [18]).

**Proposition 2.1** ([14]). *Set*

$$\varrho_{p(x)}(v)=\int_\Omega|v(x)|^{p(x)}dx,\ \forall\ v\in L^{p(x)}(\Omega),$$

*then,*

$$|v|_{p(x)} < 1(respectively = 1; > 1) \;\Leftrightarrow\; \varrho_{p(x)}(v) < 1(respectively = 1; > 1), \tag{2.1}$$

$$|v|_{p(x)} > 1 \Rightarrow |v|^{p^-}_{p(x)} \leq \varrho_{p(x)}(v) \leq |v|^{p^+}_{p(x)}, \tag{2.2}$$

$$|v|_{p(x)} < 1 \;\Rightarrow\; |v|^{p^+}_{p(x)} \leq \varrho_{p(x)}(v) \leq |v|^{p^-}_{p(x)}, \tag{2.3}$$

$$\lim_{n\to\infty} |v_k - v|_{p(x)} = 0 \;\Leftrightarrow\; \lim_{n\to\infty} \varrho_{p(x)}(v_k - v) = 0. \tag{2.4}$$

**Remark 1.** From (2.2) and (2.3), we can deduce the follwingin inequalities:

$$|v|_{p(x)} \leq \varrho_{p(x)}(v) + 1, \tag{2.5}$$

$$\varrho_{p(x)}(v) \leq |v|^{p^-}_{p(x)} + |v|^{p^+}_{p(x)}. \tag{2.6}$$

**Proposition 2.2** ([18])**.** *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, $\forall x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the Hölder-type inequality*

$$\Big|\int_\Omega uv\ dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.7}$$

**Remark 2.** If $k_1$, $k_2 \in C_+(\overline{\Omega})$ with $k_1(x) \leq k_2(x)$ for any $x \in \overline{\Omega}$ then, the embedding $L^{k_2(x)}(\Omega) \hookrightarrow L^{k_1(x)}(\Omega)$ is continuous.

$L^{p(x),q(x)}(\Omega)$ refers to the generalized Lebesgue space $L^{p(x)}(\Omega) \times L^{q(x)}(\Omega)$ equipped with the norm $\|\cdot\|_{p(x),q(x)}$ given by

$$\|(u, v)\|_{p(x),q(x)} = |u|_{p(x)} + |v|_{q(x)}, \quad \forall (u, v) \in L^{p(x),q(x)}(\Omega).$$

Now, we define the generalized Sobolev space $W^{1,p(x)}(\Omega)$, for all $p \in C_+(\overline{\Omega})$:

$$W^{1,p(x)}(\Omega) = \Big\{v \in L^{p(x)}(\Omega) \text{ such that } |\nabla v| \in L^{p(x)}(\Omega)\Big\},$$

equipped with the norm

$$|v|_{1,p(x)} = |v|_{p(x)} + |\nabla v|_{p(x)}.$$

We define $W_0^{1,p(\cdot)}(\Omega)$ as the subspace of $W^{1,p(\cdot)}(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $|\cdot|_{1,p(x)}$.

**Proposition 2.3** ([15, 22])**.** *If the exponent $p(\cdot)$ satisfies the log-Hölder continuity condition, i.e. there is a constant $\alpha > 0$ such that for every $x,\ y \in \Omega,\ x \neq y$ with $|x - y| \leq \dfrac{1}{2}$ one has*

$$|p(x) - p(y)| \leq \frac{\alpha}{-\log|x - y|}, \tag{2.8}$$

*then we have the Poincaré inequality, i.e. the exists a constant $C > 0$ depending only on $\Omega$ and the function $p$ such that*

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)},\ \forall\ u \in W_0^{1,p(\cdot)}(\Omega). \tag{2.9}$$

In particular, the space $W_0^{1,p(x)}(\Omega)$ has the norm $\|v\|_{1,p(x)}$ which is equivalent to $|v|_{1,p(x)}$, defined by

$$\|v\|_{1,p(x)} = |\nabla v|_{p(x)}.$$

**Proposition 2.4** ([14, 18])**.** *The spaces* $\left(W^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\right)$ *and* $\left(W_0^{1,p(x)}(\Omega), \|\cdot\|_{1,p(x)}\right)$ *are separable and reflexive Banach spaces.*

*Furthermore, we have the compact embedding* $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ *(see [18]).*

**Remark 3.** The dual space of $W_0^{1,p(x)}(\Omega)$ denoted $W^{-1,p'(x)}(\Omega)$, is equipped with the norm

$$\|u\|_{-1,p'(x)} = \inf\left\{|u_0|_{p'(x)} + \sum_{i=1}^{N} |u_i|_{p'(x)}\right\},\ \forall u \in W^{-1,p'(x)}(\Omega)$$

where the infinimum is taken on all possible decompositions $u = u_0 - \mathrm{div}F$ with $u_0 \in L^{p'(x)}(\Omega)$ and $F = (u_1, \dots, u_N) \in (L^{p'(x)}(\Omega))^N$

In the sequel, the notation $\mathcal{X}^{1,p(x),q(x)}(\Omega)$ refers to the Orlicz-Sobolev space $W_0^{1,p(x)}(\Omega)\times W_0^{1,q(x)}(\Omega)$, equipped with th norm $\|(u,v)\| = \|(u,v)\|_{1,p(x),q(x)}$ given by

$$\|(u,v)\| = \|(u,v)\|_{1,p(x),q(x)} = \|u\|_{1,p(x)} + \|v\|_{1,q(x)},\ \forall (u,v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$$

$\left(\mathcal{X}^{1,p(x),q(x)}(\Omega)\right)^* = \mathcal{X}^{-1,p'(x),q'(x)}(\Omega)$ is the dual space of $\mathcal{X}^{1,p(x),q(x)}(\Omega)$, corresponding to the Orlicz-Sobolev space $W^{-1,p'(x)}(\Omega) \times W^{-1,q'(x)}(\Omega)$ equipped with the norme

$$\|(\varphi,\phi)\|_{-1,p'(x),q'(x)} = \|\varphi\|_{-1,p'(x)} + \|\varphi\|_{-1,q'(x)},\ \forall (\varphi,\phi) \in \mathcal{X}^{-1,p'(x),q'(x)}(\Omega).$$

The continuous pairing between $\mathcal{X}^{1,p(x),q(x)}(\Omega)$ and $\mathcal{X}^{-1,p'(x),q'(x)}(\Omega)$ is denoted by $\langle\cdot,\cdot\rangle_{1,p(x),q(x)}$ satisfying

$$\langle (u,v),(\varphi,\phi)\rangle_{1,p(x),q(x)} = \langle u,\varphi\rangle_{1,p(x)} + \langle v,\phi\rangle_{1,q(x)},$$

for all $(\varphi,\phi) \in \mathcal{X}^{-1,p'(x),q'(x)}(\Omega)$ and $(u,v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$.

## 2.2 Topological degree theory

Let $\mathcal{X}$ be a real separable and reflexive Banach space, $\mathcal{X}^*$ its dual space with dual pairing $\langle\cdot,\cdot\rangle$ and $\mathcal{D}$ be a nonempty subset of $\mathcal{X}$. Strong (weak) convergence is represented by the symbol $\to$ ($\rightharpoonup$), and let $\mathcal{O}$ be the collection of all bounded open sets in $\mathcal{X}$. The readers can find more information about the history of this theory in [1, 8, 25, 27, 17].

**Definition 1.** Let $\mathcal{Y}$ be a real Banach space. An operator $F : \mathcal{D} \subset \mathcal{X} \to \mathcal{Y}$ is said to be

1) bounded, if it takes any bounded set into a bounded set.

2) demicontinuous, if for any $(u_n) \subset \mathcal{D}$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.

3) compact, if it is continuous and the image of any bounded set is relatively compact.

**Definition 2.** A mapping $F : \mathcal{D} \subset \mathcal{X} \to \mathcal{X}^*$ is said to be

1) of type $(S_+)$, if for any sequence $(u_n) \subset \mathcal{D}$ with $u_n \rightharpoonup u$ and $\limsup_{n\to\infty} \langle Fu_n, u_n - u\rangle \le 0$, it follows that $u_n \to u$.

2) quasimonotone, if for any sequence $(u_n) \subset \mathcal{D}$ with $u_n \rightharpoonup u$, it follows that $\limsup_{n\to\infty} \langle Fu_n,\ u_n - u\rangle \ge 0$.

For any bounded operator $T : \mathcal{D}_1 \subset \mathcal{X} \to \mathcal{X}^*$ such that $\mathcal{D} \subset \mathcal{D}_1$ and for any operator $F : \mathcal{D} \subset \mathcal{X} \to \mathcal{X}$, we say that $F$ of type $(S_+)_T$, if for any sequence $(u_n) \subset \mathcal{D}$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup_{n\to\infty} \langle Fu_n,\ y_n - y\rangle \le 0$, we have $u_n \to u$.

**Remark 4** (see [30]). 1) If a mapping is compact in a set, then it is quasi-monotone in that set.

2) If the mapping is demicontinuous and satisfies the condition $(S_+)$ in a set, then it is quasi-monotone in that set.

In the sequel, we consider the following classes of operators :

$$\mathcal{F}_1(\mathcal{D}) := \Big\{F : \mathcal{D} \to \mathcal{X}^* \mid \ F \text{ is bounded, demicontinuous and of type } (S_+)\Big\},$$
$$\mathcal{F}_T(\mathcal{D}) := \Big\{F : \mathcal{D} \to \mathcal{X} \ \mid F \ \text{ is demicontinuous and of type } (S_+)_T\Big\},$$
$$\mathcal{F}_{T,B}(\mathcal{D}) := \{F : \mathcal{D} \to \mathcal{X}| \ F \ \text{ is bounded, demicontinuous and of class } (S_+)_T\}.$$

An operator $\mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})$ is called an essential inner map to $F$.

**Lemma 2.1** ([17]). *Let $T \in \mathcal{F}_1(\overline{G})$ be continuous and $S : D_S \subset \mathcal{X}^* \to \mathcal{X}$ be demicontinuous such that $T(\overline{G}) \subset D_s$, where $G \in \mathcal{O}$. Then the following statements are true:*

1) if *$S$ is quasimonotone, then $I + S \circ T \in \mathcal{F}_T(\overline{G})$, where $I$ denotes the identity operator,*

2) *if $S$ is of type $(S_+)$, then $S \circ T \in \mathcal{F}_T(\overline{G})$.*

**Definition 3.** Let $G \in \mathcal{O}$, $T \in \mathcal{F}_1(\overline{G})$ be continuous and consider the mappings $F, S : \overline{G} \subset \mathcal{X} \to \mathcal{X}^*$. The affine homotopy $\mathcal{H} : [0, 1] \times \overline{G} \to \mathcal{X}$, defined by

$$\mathcal{H}(t, u) := (1 - t)Fu + tSu \quad \text{ for all } \ (t, u) \in [0, 1] \times \overline{G},$$

is called an admissible affine homotopy with the common continuous essential inner map $T$.

**Lemma 2.2** ([17]). *If the mappings $F, S \in \mathcal{F}_T(\overline{G})$, then the affine homotopy $\mathcal{H} : [0, 1] \times \overline{G} \to \mathcal{X}$ defined in Definition 3 of type $(S_+)_T$.*

Now we give the Berkovits topological degree for a class of demicontinuous operators satisfying condition $(S_+)_T$ for more details, see [17].

**Theorem 2.1.** *There exists a unique degree function*

$$d\, :\, \mathcal{M} = \Big\{(F, G, h) \mid G \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{G}),\ \ F \in \mathcal{F}_T(\overline{G}),\ \ h \notin F(\partial G)\Big\} \longrightarrow \mathbb{Z}$$

*which satisfies the following properties.*

1) (Existence) *If $d(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in $G$.*

2) (Normalization) *For any $h \in F(G)$, we have $d(I, E, h) = 1$.*

3) (Additivity) *Let $F \in \mathcal{F}_{T,B}(\overline{G})$. If $G_1$ and $G_2$ are two disjoint open subsets of $G$ such that $h \notin F(\overline{G}\backslash(G_1 \cup G_2))$ then we have*

$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h).$$

4) (Homotopy invariance) *If $\mathcal{H}$: $[0, 1] \times \overline{G} \to \mathcal{X}$ is a bounded admissible affine homotopy with a common continuous essential inner map and $h$: $[0, 1] \to \mathcal{X}$ is a continuous path in $\mathcal{X}$ such that $h(t) \notin \mathcal{H}(t, \partial G)$ $\forall t \in [0, 1]$, then*

$$d(\mathcal{H}(t, \cdot), G, h(t)) = \text{constant} \ \ \text{for} \ \ \text{all} \ \ t \in [0, 1].$$

## 3 Assumptions and main results

In this section, we will discuss the existence of a weak solution to problem (1.1).

Let $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$. For almost every $x$ in $\Omega$ and $i = 1, 2$, we assume the following hypothesis: $a_i(x,\xi) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function, is the gradient with respect to $\xi$ of the mapping $A_i(x,\xi) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$, that is $a_i(x,\xi) = \nabla_\xi A_i(x,\xi)$, and is such that

$(M_1)$ $\quad A_i(x,0) = 0,$

$(M_2)$ $\quad \sigma|\xi|^{p(x)} \leq a_1(x,\xi)\cdot\xi \leq p(x)A_1(x,\xi)$ and $\iota|\xi|^{q(x)} \leq a_2(x,\xi)\cdot\xi \leq q(x)A_2(x,\xi),$

$(M_3)$ $\quad \big|a_1(x,\xi)\big| \leq \eta\big(\rho(x) + |\xi|^{p(x)-1}\big)$ and $\big|a_2(x,\xi)\big| \leq \beta\big(\theta(x) + |\xi|^{q(x)-1}\big),$

$(M_4)$ $\quad \big[a_i(x,\xi) - a_i(x,\xi')\big]\cdot(\xi - \xi') > 0,$

where $\sigma, \eta, \iota, \theta, \beta$ are some positive constants, $\rho(x)$ is a positive function belonging to $L^{p'(x)}(\Omega)$ and $\theta(x)$ is a positive function belonging to $L^{q'(x)}(\Omega)$, ($p'(x)$ is the conjugate exponent of $p(x)$).

$(H_1)$ $h : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying the following growth condition:

$$|h(x,\xi,\xi')| \leq \mu(\gamma(x) + |\xi|^{r_1(x)-1} + |\xi'|^{r_1(x)-1}),$$

where $\mu > 0$, $\gamma \in L^{p'(x)}(\Omega)$ and $1 \leq r_1^- \leq r_1(x) \leq r_1^+ < p^-$.

$(H_2)$ $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying the following growth condition:

$$|g(x,\xi,\xi')| \leq \alpha(e(x) + |\xi|^{r_2(x)-1} + |\xi'|^{r_2(x)-1}).$$

where $\alpha > 0$ and $e \in L^{q'(x)}(\Omega)$ and $1 \leq r_2^- \leq r_2(x) \leq r_2^+ < q^-$.

$(M_5)$ $\mathcal{N}_i$: $\mathbb{R}^+ \to \mathbb{R}^+$ $(i = 1, 2)$ are continuous and nondecreasing function, for which there exist two functions $l, j$ such that,

$$k_0 t^{l(x)-1} \leq \mathcal{N}_1(t) \leq k_1 t^{l(x)-1},$$
$$m_0 t^{j(x)-1} \leq \mathcal{N}_2(t) \leq m_1 t^{j(x)-1},$$

where $m_i, k_i$ $(i = 0, 1)$ are positive constants $l, j \in C_+(\overline{\Omega})$ $1 \leq l^- \leq l(x) \leq l^+ < p^-$, and $1 \leq j^- \leq j(x) \leq j^+ < q^-$.

Finally, we recall that the $\mathcal{Q}, \mathcal{O} \in L^\infty(\Omega)$ and $\mathcal{Q}(x), \mathcal{O}(x) > 0$ for almost every $x$ in $\Omega$.

The definition of a weak solution for problem(1.1) can be stated as follows:

**Definition 4.** A couple $(u, v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ is called a weak solution of (1.1) if

$$\begin{aligned}\langle f_p u, \varphi\rangle + \langle f_q v, \psi\rangle + \int_\Omega \mathcal{Q}(x)|u|^{r_1(x)-2}u\varphi(x)dx + \int_\Omega \mathcal{O}(x)|v|^{r_2(x)-2}v\psi(x)dx \\ = \int_\Omega \lambda h(x,u,\nabla u)\varphi(x)dx + \int_\Omega \kappa g(x,v,\nabla v)\psi(x)dx,\end{aligned}$$

where

$$\langle f_p u, \varphi\rangle = \mathcal{N}_1\Big(\int_\Omega (A_1(x,\nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\,dx\Big)\Big[\int_\Omega a_1(x,\nabla u)\nabla\varphi + \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi\Big],$$

and

$$\langle f_q v, \psi\rangle = \mathcal{N}_2\Big(\int_\Omega (A_2(x,\nabla v) + \frac{1}{q(x)}|\nabla v|^{q(x)})\,dx\Big)\Big[\int_\Omega a_2(x,\nabla v)\nabla\psi + \int_\Omega |\nabla v|^{q(x)-2}\nabla v\nabla\psi\Big],$$

for every $(\varphi, \psi) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$.

**Lemma 3.1** ([2]). *Let* $g \in L^{r(x)}(\Omega)$ *and* $(g_n) \subset L^{r(x)}(\Omega)$ *such that* $\sup_{n\in\mathbb{N}} \|g_n\|_{r(x)} < \infty$, *If* $g_n(x) \to g(x)$ *for almost every* $x \in \Omega$, *then* $g_n \rightharpoonup g$ *weakly in* $L^{r(x)}(\Omega)$.

**Lemma 3.2** ([2]). *Assume that* $(M_2)$-$(M_4)$ *hold. Let* $(u_m)_m$ *be a sequence in* $W_0^{1,n(x)}(\Omega)$ *such that* $u_m \rightharpoonup u$ *weakly in* $W_0^{1,n(x)}(\Omega)$ *and*

$$\int_\Omega \big[a(x, \nabla u_m) - a(x, \nabla u)\big] \nabla(u_m - u) dx \longrightarrow 0, \tag{3.1}$$

*then* $u_m \longrightarrow u$ *strongly in* $W_0^{1,n(x)}(\Omega)$.

Before giving our main result, we first give two important lemmas that will be used later. Let us consider the following functionals:

$$\begin{aligned}\mathcal{L}(u, v) &:= \widehat{\mathcal{N}_1}\Big(\mathcal{J}_1(u)\Big) + \widehat{\mathcal{N}_2}\Big(\mathcal{J}_2(v)\Big)\\ &:= \widehat{\mathcal{N}_1}\Big(\int_\Omega (A_1(x, \nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\, dx\Big) + \widehat{\mathcal{N}_2}\Big(\int_\Omega (A_2(x, \nabla v) + \frac{1}{q(x)}|\nabla v|^{q(x)}) dx\Big),\end{aligned}$$

for all $(u, v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$, where the functionals $\mathcal{J}_1 : W_0^{1,p(x)}(\Omega) \longrightarrow \mathbb{R}$ and $\mathcal{J}_2 : W_0^{1,q(x)}(\Omega) \longrightarrow \mathbb{R}$, are defined by
$\mathcal{J}_1(u) = \displaystyle\int_\Omega (A_1(x, \nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\, dx$ and $\mathcal{J}_2(v) = \displaystyle\int_\Omega (A_2(x, \nabla v) + \frac{1}{q(x)}|\nabla v|^{q(x)})\, dx$,
then $\mathcal{J}_1 \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$, and $\mathcal{J}_2 \in C^1(W_0^{1,q(x)}(\Omega), \mathbb{R})$, $\widehat{\mathcal{N}_i} : [0, +\infty[ \longrightarrow [0, +\infty[$ be the primitive of the functions $\mathcal{N}_i$ $(i = 1, 2)$, defned by

$$\widehat{\mathcal{N}_i}(t) = \int_0^t \mathcal{N}_i(\xi)\, d\xi.$$

On the other hand, we consider the functional $\mathbf{J} : \mathcal{X}^{1,p(x),q(x)}(\Omega) \to \mathbb{R}$ defined by:

$$\begin{aligned}\mathbf{J}(u, v) &= \mathcal{J}_1(u) + \mathcal{J}_2(v)\\ &= \int_\Omega (A_1(x, \nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\, dx + \int_\Omega (A_2(x, \nabla v) + \frac{1}{q(x)}|\nabla v|^{q(x)})\, dx,\end{aligned}$$

for all $(u, v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$, then $\mathbf{J} \in C^1\big(\mathcal{X}^{1,p(x),q(x)}(\Omega), \mathbb{R}\big)$ and,

$$\begin{aligned}\langle \mathbf{J}'(u, v)\, , (\varphi, \psi)\rangle &= \langle \mathcal{J}_1'(u, \varphi)\rangle + \langle \mathcal{J}_2'(v, \psi)\rangle\\ &= \int_\Omega a_1(x, \nabla u)\nabla\varphi\, dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi\, dx + \int_\Omega a_2(x, \nabla v)\nabla\psi\, dx\\ &\quad + \int_\Omega |\nabla v|^{q(x)-2}\nabla v \nabla\psi\, dx.\end{aligned}$$

It is obvious that the functional $\mathcal{L}$ is defined and continuously Gâteaux differentiable and whose Gâteaux derivative at the point $(u, v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ is the functional $\mathcal{F} := \mathcal{L}'(u, v) \in (\mathcal{X}^{1,p(x),q(x)}(\Omega))^*$ given by

$$\langle \mathcal{L}'(u, v), (\varphi, \psi)\rangle = \langle \mathcal{F}(u, v), (\varphi, \psi)\rangle = \langle f_p u, \varphi\rangle + \langle f_q v, \psi\rangle.$$

**Lemma 3.3.** *Suppose that hypotheses* $(M_1)$-$(M_5)$ *hold, then*

*i)* $\mathcal{F}$ *is continuous, bounded, strictly monotone operator.*

*ii)* $\mathcal{F}$ *is a mapping of type* $(S_+)$.

*Proof.* i) It is obvious that $\mathcal{F}$ is continuous because $\mathcal{F}$ is the Fréchet derivative of $\mathcal{L}$. Now, we verify that $\mathcal{F}$ is bounded. For all $(u,v)$ and $(\varphi,\psi) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ we have,

$$\begin{aligned}|\langle \mathcal{F}(u,v),(\varphi,\psi)\rangle| \leq &\Big|\mathcal{N}_1\Big(\mathcal{J}_1(u)\Big)\Big[\int_\Omega a_1(x,\nabla u)\nabla\varphi\,dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx\Big]\Big| \\ &+\Big|\mathcal{N}_2\Big(\mathcal{J}_2(v)\Big)\Big[\int_\Omega a_2(x,\nabla v)\nabla\psi\,dx + \int_\Omega |\nabla v|^{q(x)-2}\nabla v\nabla\psi dx\Big]\Big|.\end{aligned}$$

Applying $(M_5)$ and Hölder's inequality, from the last inequality, it follows that

$$\begin{aligned}|\langle \mathcal{F}(u,v),(\varphi,\psi)\rangle| \leq\, &k_1\big(\mathcal{J}_1(u)\big)^{l(x)-1}\Big[\int_\Omega |a_1(x,\nabla u)\nabla\varphi|dx + \int_\Omega |\nabla u|^{p(x)-1}|\nabla\varphi|dx\Big] \\ &+ m_1\big(\mathcal{J}_2(v)\big)^{j(x)-1}\Big[\int_\Omega |a_2(x,\nabla v)\nabla\psi|dx + \int_\Omega |\nabla v|^{q(x)-1}|\nabla\psi|dx\Big] \\ \leq\, &C_1\Big(\Big(\int_\Omega A_1(x,\nabla u)dx\Big)^{l(x)-1} + \Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^{l(x)-1}\Big) \\ &\times\Big[|a_1(x,\nabla u)|_{p'(x)}|\nabla\varphi|_{p(x)} + |\nabla u^{p(x)-1}|_{p'(x)}|\nabla\varphi|_{p(x)}\Big] \\ &+ C_2\Big(\Big(\int_\Omega A_2(x,\nabla v)dx\Big)^{j(x)-1} + \Big(\int_\Omega |\nabla v|^{q(x)}dx\Big)^{j(x)-1}\Big) \\ &\times\Big[|a_2(x,\nabla v)|_{q'(x)}|\nabla\psi|_{q(x)} + |\nabla v^{q(x)-1}|_{q'(x)}|\nabla\psi|_{q(x)}\Big].\end{aligned}$$

Bearing (2.5) and (2.6) in mind, we obtain

$$\begin{aligned}|\langle \mathcal{F}(u,v),(\varphi,\psi)\rangle| \leq\, &\mathrm{C}_3\Big(\Big(\int_\Omega A_1(x,\nabla u)dx\Big)^{l(x)-1} + \|u\|_{1,p(x)}^{p^-(l(x)-1)} + \|u\|_{1,p(x)}^{p^+(l(x)-1)}\Big) \\ &\times\Big[|a_1(x,\nabla u)|_{p'(x)} + \varrho_{p'(x)}(\nabla u^{p(x)-1}) + 1\Big]\|\varphi\|_{1,p(x)} \\ &+ \mathrm{C}_4\Big(\Big(\int_\Omega A_2(x,\nabla v)dx\Big)^{j(x)-1} + \|v\|_{1,q(x)}^{q^-(j(x)-1)} + \|v\|_{1,q(x)}^{q^+(j(x)-1)}\Big) \\ &\times\Big[|a_2(x,\nabla v)|_{q'(x)} + \varrho_{q'(x)}(\nabla v^{q(x)-1}) + 1\Big]\|\psi\|_{1,q(x)} \\ \leq\, &C_5\Big(\Big(\int_\Omega A_1(x,\nabla u)dx\Big)^{l(x)-1} + \|u\|_{1,p(x)}^{p^-(l(x)-1)} + \|u\|_{1,p(x)}^{p^+(l(x)-1)}\Big) \\ &\times\Big[|a_1(x,\nabla u)|_{p'(x)} + \|u\|_{1,p(x)}^{p^-} + \|u\|_{1,p(x)}^{q^+} + 1\Big]\|\varphi\|_{1,p(x)} \\ &+ C_6\Big(\Big(\int_\Omega A_2(x,\nabla v)dx\Big)^{j(x)-1} + \|v\|_{1,q(x)}^{q^-(j(x)-1)} + \|v\|_{1,q(x)}^{q^+(j(x)-1)}\Big) \\ &\times\Big[|a_2(x,\nabla v)|_{q'(x)} + \|v\|_{1,q(x)}^{q^-} + \|v\|_{1,q(x)}^{q^+} + 1\Big]\|\psi\|_{1,q(x)},\end{aligned}$$

where $C_1,...,C_6 > 0$ are independent of $u$ and $v$.

By $(M_1)$, we have for any $x\in\Omega$, $\xi\in\mathbb{R}^n$ and $(i=1,2)$,

$$A_i(x,\xi) = \int_0^1 \frac{d}{ds}A_i(x,s\xi)ds = \int_0^1 a_i(x,s\xi)\xi ds,$$

by combining $(M_3)$, Fubini's theorem and Young's inequality, we have

$$\begin{aligned}\int_\Omega A_1(x,\nabla u)dx = \int_\Omega\int_0^1 a_1(x,s\nabla u)\nabla u ds\,dx &= \int_0^1\Big[\int_\Omega a_1(x,s\nabla u)\nabla u dx\Big]ds \\ &\leq \int_0^1\Big[c_0\int_\Omega \big|a_1(x,s\nabla u)\big|^{p'(x)}dx + c_1\int_\Omega |\nabla u|^{p(x)}dx\Big]ds\end{aligned}$$

$$\leq \int_0^1 \Big[c_2 \int_\Omega |\rho(x)|^{p'(x)} + |s\,\nabla u|^{p(x)}\,dx + c_1 \int_\Omega |\nabla u|^{p(x)} dx\Big] ds$$
$$\leq c_3 + c_4 \varrho_{p(x)}(\nabla u)$$
$$\leq c_3 + c_4\big(\|u\|_{1,p(x)}^{p-} + \|u\|_{1,p(x)}^{p+}\big)$$
$$\leq c_5\Big(\|u\|_{1,p(x)}^{p-} + \|u\|_{1,p(x)}^{p+} + 1\Big), \tag{3.2}$$

where $c_0, ..., c_5 > 0$ are independent of $u$ and $v$.

The same reasoning is used to prove that,

$$\int_\Omega A_2(x, \nabla v) dx \leq c_6\Big(\|v\|_{1,q(x)}^{q-} + \|v\|_{1,q(x)}^{q+} + 1\Big).$$

From $(M_3)$, we can easily show that $|a_1(x, \nabla u)|_{p'(x)}$ and $|a_2(x, \nabla v)|_{q'(x)}$ are bounded for all $(u, v)$ in $\mathcal{X}^{1,p(x),q(x)}(\Omega)$ . Therefore,

$$|\langle \mathcal{F}(u, v), (\varphi, \psi)\rangle| \leq C_7 \Big(\|\varphi\|_{1,p(x)} + \|\psi\|_{1,q(x)}\Big),$$

where $C_7 > 0$ is independent of $\phi$ and $\psi$. Hence, the operator $\mathcal{F}$ is bounded.

Next, we prove that the operator $\mathcal{F}$ is coercive. For each $(u, v) \in \mathcal{X}^{1,p(x,q(x))}(\Omega)$, we have

$$\frac{\langle \mathcal{F}(u, v), (u, v)\rangle}{\|(u, v)\|} = \frac{\mathcal{N}_1\big(\mathcal{J}_1(u)\big) \Big[\int_\Omega a_1(x, \nabla u)\nabla u + \int_\Omega |\nabla u|^{p(x)} dx\Big]}{\|(u, v)\|}$$
$$+ \frac{\mathcal{N}_2\big(\mathcal{J}_2(v)\big) \Big[\int_\Omega a_2(x, \nabla v)\nabla v + \int_\Omega |\nabla v|^{q(x)} dx\Big]}{\|(u, v)\|}.$$

From $(M_2)$ and $(M_5)$, we obtain

$$\frac{\langle \mathcal{F}(u, v), (u, v)\rangle}{\|(u, v)\|} \geq k_0 \frac{\Big(\int_\Omega (A_1(x, \nabla u) + \frac{1}{p^+}|\nabla u|^{p(x)})\,dx\Big)^{l(x)-1} \Big[\sigma \int_\Omega |\nabla u|^{p(x)} + \int_\Omega |\nabla u|^{p(x)} dx\Big]}{\|(u, v)\|}$$
$$+ m_0 \frac{\Big(\int_\Omega (A_2(x, \nabla v) + \frac{1}{q^+}|\nabla v|^{q(x)})\,dx\Big)^{j(x)-1} \Big[\iota \int_\Omega |\nabla v|^{q(x)} + \int_\Omega |\nabla v|^{q(x)} dx\Big]}{\|(u, v)\|}$$
$$\geq k_0 \frac{\Big(\frac{\sigma}{p^+} \int_\Omega |\nabla u|^{p(x)} + \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)})\,dx\Big)^{l(x)-1} \Big[\sigma \int_\Omega |\nabla u|^{p(x)} + \int_\Omega |\nabla u|^{p(x)} dx\Big]}{\|(u, v)\|}$$
$$+ m_0 \frac{\Big(\frac{\iota}{q^+} \int_\Omega |\nabla v|^{q(x)} + \frac{1}{q^+} \int_\Omega |\nabla v|^{q(x)})\,dx\Big)^{j(x)-1} \times \Big[(1+\iota) \int_\Omega |\nabla v|^{q(x)}\,dx\Big]}{\|(u, v)\|}$$
$$\geq k_0 \frac{\Big(\frac{\sigma}{p^+} \int_\Omega |\nabla u|^{p(x)} + \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)})\,dx\Big)^{l(x)-1} \Big[\sigma \int_\Omega |\nabla u|^{p(x)} + \int_\Omega |\nabla u|^{p(x)} dx\Big]}{\|(u, v)\|}$$
$$+ \geq m_0 \frac{\Big(\frac{\iota}{q^+} \int_\Omega |\nabla v|^{q(x)} + \frac{1}{q^+} \int_\Omega |\nabla v|^{q(x)})\,dx\Big)^{j(x)-1} \times \Big[(1+\iota) \int_\Omega |\nabla v|^{q(x)}\,dx\Big]}{\|(u, v)\|}$$
$$\geq C_1 \frac{\|u\|_{1,p(x)}^{\gamma l(x)} + \|v\|_{1,q(x)}^{\beta j(x)}}{\|(u, v)\|}$$
$$\geq C_1 \frac{\|u\|_{1,p(x)}^{\gamma l^-} + \|v\|_{1,q(x)}^{\beta j^-}}{\|u\|_{1,p(x)} + \|v\|_{1,q(x)}},$$

where $C_1 > 0$ is independent of $u$ and $v$, $\gamma = \begin{cases} p^- & \text{if} \quad \|u\| \leq 1 \\ p^+ & \text{if} \quad \|u\| \geq 1. \end{cases}$ and $\beta = \begin{cases} q^- & \text{if} \quad \|v\|_{1,a(x)} \leq 1 \\ q^+ & \text{if} \quad \|v\|_{1,q(x)} \geq 1. \end{cases}$

Since $\lim\limits_{x+y\longrightarrow\infty} \frac{x^s+y^t}{x+y} = +\infty$ for $s,t > 1$, then $\lim\limits_{\|(u,v)\|\to\infty} \dfrac{\langle \mathcal{F}(u,v)\rangle}{\|(u,v)\|} = \infty$.

Next, we prove that $\mathcal{F}$ is a strictly monotone operator, we show first the monotonicity of $\mathcal{J}'_i$ $(i = 1,2)$. Using $(M_4)$ and taking into account the following inequality (see [20]), for all $x, y \in \mathbb{R}^N$,

$$(|x|^{p-2}x - |y|^{p-2}y)(x-y) \cdot (|x|^p + |y|^p)^{\frac{2-p}{p}} \geq (p-1)|x-y|^p \quad \text{if} \quad 1 < p < 2,$$
$$(|x|^{p-2}x - |y|^{p-2}y) \cdot (x-y) \geq (\frac{1}{2})^p |x-y|^p \quad \text{if} \qquad p \geq 2,$$

we obtain, for all $(u_1, v_1), (u_2, v_2) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ with $(u_1, v_1) \neq (u_2, v_2)$, that

$$\langle \mathcal{J}_1'(u_1) - \mathcal{J}_1'(u_2), u_1 - u_2\rangle > 0 \text{ and } \langle \mathcal{J}_2'(v_1) - \mathcal{J}_2'(v_2), v_1 - v_2\rangle > 0,$$

which implies that $\mathcal{J}_1'$, $\mathcal{J}_2'$ are strictly monotone.

Thus, by [30, Proposition 25.10 ], $\mathcal{J}_i$ are strictly convex. Furthermore, as $\mathcal{N}_i \quad (i = 1,2)$ are nondecreasing, then $\widehat{\mathcal{N}_i}$ are convex in $\mathbb{R}^+$. So, for each $(u_1, v_1), (u_2, v_2) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ with $(u_1, v_1) \neq (u_2, v_2)$, and every $s, t \in (0,1)$ with $s + t = 1$, we have

$$\widehat{\mathcal{N}_1}(\mathcal{J}_1(su_1 + tu_2)) < \widehat{\mathcal{N}_1}(s\mathcal{J}_1(u_1) + t\mathcal{J}_1(u_2)) \leq s\widehat{\mathcal{N}_1}(\mathcal{J}_1(u_1)) + t\widehat{\mathcal{N}_1}(\mathcal{J}_1(u_2)),$$

and

$$\widehat{\mathcal{N}_2}(\mathcal{J}_2(sv_1 + tv_2)) < \widehat{\mathcal{N}_2}(s\mathcal{J}_2(v_1) + t\mathcal{J}_2(v_2)) \leq s\widehat{\mathcal{N}_2}(\mathcal{J}_2(v_1)) + t\widehat{\mathcal{N}_2}(\mathcal{J}_2(v_2)).$$

This proves that $\mathcal{L} = \widehat{\mathcal{N}_1}\big(\mathcal{J}_1\big) + \widehat{\mathcal{N}_2}\big(\mathcal{J}_2\big)$ is strictly convex. Since $\mathcal{L}'(u,v) = \mathcal{F}(u,v)$ for all $(u,v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$, finally, we infer that $\mathcal{F}$ is strictly monotone on $\big(\mathcal{X}^{1,p(x),q(x)}(\Omega)\big)^*$.

ii) Now, we verify that the operator $\mathcal{F}$ is of type $(S_+)$. Assume that

$$\begin{cases} (u_n, v_n) \rightharpoonup (u,v) & \text{in } \mathcal{X}^{1,p(x),q(x)}(\Omega) \\ \limsup\limits_{n\to\infty} \langle \mathcal{F}(u_n, v_n), (u_n - u, v_n - v)\rangle \leq 0. \end{cases} \tag{3.3}$$

We will show that $(u_n, v_n) \to (u,v)$ in $\mathcal{X}^{1,p(x),q(x)}(\Omega)$. By the strict monotonicity of $\mathcal{F}$ we get,

$$\limsup_{n\to\infty} \langle \mathcal{F}(u_n, v_n) - \mathcal{F}(u,v),\ (u_n - u, v_n - v)\rangle = \lim_{n\to\infty} \langle \mathcal{F}(u_n, v_n) - \mathcal{F}(u,v),\ (u_n - u, v_n - v)\rangle = 0.$$

Then,

$$\lim_{n\to\infty} \langle \mathcal{F}(u_n, v_n),\ (u_n - u, v_n - v)\rangle = 0.$$

Therefore,

$$\lim_{n\to\infty} \langle f_p(u_n), u_n - u\rangle + \langle f_q(v_n), v_n - v\rangle = 0.$$

Since $f_p$ and $f_q$ are monotone,

$$\lim_{n\to\infty} \langle f_p(u_n), u_n - u\rangle = 0 \text{ and } \lim_{n\to\infty} \langle f_q(v_n), v_n - v\rangle = 0. \tag{3.4}$$

which means that

$$\lim_{n\to\infty} \mathcal{N}_1\Big(\mathcal{J}_1(u)\Big) \Big[ \int_\Omega a_1(x, \nabla u_n)\nabla(u_n - u) + \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n - u) dx \Big] = 0, \tag{3.5}$$
$$\lim_{n\to\infty} \mathcal{N}_2\Big(\mathcal{J}_2(v)\Big) \Big[ \int_\Omega a_2(x, \nabla v_n)\nabla(v_n - v) + \int_\Omega |\nabla u_n|^{q(x)-2}\nabla v_n \nabla(v_n - v) dx \Big] = 0.$$

By (3.2), we infer that $\mathcal{J}_1(u_n)$ and $\mathcal{J}_2(v_n)$ are bounded.

As $\mathcal{N}_1$ is continuous, up to a subsequence there is $y, z \geq 0$ such that

$$\begin{aligned} \mathcal{N}_1(\mathcal{J}_1(u_n)) &\longrightarrow \mathcal{N}_1(y) \geq k_0 y^{l(x)-1} \qquad \text{as} \quad n \to \infty, \\ \mathcal{N}_2(\mathcal{J}_2(v_n)) &\longrightarrow \mathcal{N}_2(z) \geq m_0 z^{j(x)-1} \qquad \text{as} \quad n \to \infty. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we get

$$\lim_{n\to\infty} \int_\Omega a_1(x, \nabla u_n)\nabla(u_n - u)dx + \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx = 0.$$

Using the continuous embedding $W_0^{1,r(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$, we have

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx = 0 \text{ and } \lim_{n\to\infty} \int_\Omega |\nabla v_n|^{q(x)-2}\nabla v_n(\nabla v_n - \nabla v)dx = 0.$$

Then,

$$\lim_{n\to\infty} \int_\Omega a_1(x, \nabla u_n)\nabla(u_n - u)dx = 0. \text{ and } \lim_{n\to\infty} \int_\Omega a_2(x, \nabla v_n)\nabla(v_n - v)dx = 0.$$

In the light of Lemma 3.2, we obtain

$$(u_n, v_n) \longrightarrow (u, v) \quad \text{strongly in } \mathcal{X}^{1,p(x),q(x)}(\Omega),$$

which implies that $\mathcal{F}$ is of type $(S_+)$.

$\square$

**Lemma 3.4.** *Assume that assumptions $(H_1)$ and $(H_2)$ hold, then the operator*

$$\mathcal{S} : \mathcal{X}^{1,p(x),q(x)}(\Omega) \longrightarrow \big(\mathcal{X}^{1,p(x),q(x)}(\Omega)\big)^*, \textit{ defined for all } (\varphi, \psi) \in \mathcal{X}^{1,p(x),q(x)}(\Omega) \textit{ by}$$
$$\begin{aligned} \langle \mathcal{S}(u, v), (\varphi, \psi)\rangle = &-\lambda \int_\Omega h(x, u, \nabla u)\varphi dx - \kappa \int_\Omega g(x, v, \nabla v)\psi dx \\ &+ \int_\Omega \mathcal{Q}(x)|u|^{r_1(x)-2}u\varphi(x)dx + \int_\Omega \mathcal{O}(x)|v|^{r_2(x)-2}v\psi(x)dx, \end{aligned}$$

*where $\lambda, \kappa \in \mathbb{R}$, is compact.*

*Proof.* In order to prove this lemma, we proceed in three steps.
**Step 1.** Let us define the operator $\Psi : \mathcal{X}^{1,p(x),q(x)}(\Omega) \to L^{p'(x),q'(x)}(\Omega)$ by

$$\Psi(u, v) := \big(\mathcal{Q}(x)|u|^{r_1(x)-2}u\,,\, \mathcal{O}(x)|v|^{r_2(x)-2}v\big),$$

that is for all $(\varphi, \psi) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ by

$$\langle \Psi(u, v), (\varphi, \psi)\rangle = \int_\Omega \mathcal{Q}(x)|u|^{r_1(x)-2}u\varphi dx + \int_\Omega \mathcal{O}(x)|v|^{r_2(x)-2}v\psi dx.$$

We will show that $\Psi$ is bounded and continuous.

It is clear that $\Psi$ is continuous. Next, we prove that $\Psi$ is bounded. Let $(u, v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$. Since $r_1^+ < p^- < p(x)$ and $r_2^+ < q^- < q(x)$, then

$$\begin{aligned} |\Psi(u, v)|_{p'(x),q'(x)} &= |\mathcal{Q}(x)|u|^{r_1(x)-2}u|_{p'(x)} + |\mathcal{O}(x)|v|^{r_2(x)-2}v|_{q'(x)} \\ &\leq \varrho_{p'(x)}(\mathcal{Q}(x)|u|^{p(x)-2}u) + \varrho_{q'(x)}(\mathcal{O}(x)|v|^{q(x)-2}v) + 2 \\ &= \int_\Omega |\mathcal{Q}(x)|u|^{p(x)-2}u|^{p'(x)}dx + \int_\Omega |\mathcal{O}(x)|v|^{q(x)-2}v|^{q'(x)}dx + 2 \\ &\leq \int_\Omega |\mathcal{Q}(x)|^{p'(x)}|u|^{p(x)}dx + \int_\Omega |\mathcal{O}(x)|^{q'(x)}|v|^{q(x)}dx + 2 \\ &\leq \|\mathcal{Q}^{p'+}\|_\infty \varrho_{p(x)}(u) + \|\mathcal{O}^{q'+}\|_\infty \varrho_{q(x)}(v) + 2 \\ &\leq C_1\Big(|u|_{p(x)}^{p+} + |u|_{p(x)}^{p-} + |v|_{q(x)}^{q+} + |v|_{q(x)}^{q-}\Big) \\ &\leq C_2\Big(\|u\|_{1,p(x)}^{p+} + \|u\|_{1,p(x)}^{p-} + \|v\|_{1,q(x)}^{q+} + |v\|_{1,q(x)}^{q-}\Big), \end{aligned}$$

where $C_1, C_2 > 0$ are independent of $u, v$. Consequently, $\Psi$ is bounded on $\mathcal{X}^{1,p(x),q(x)}(\Omega)$.
**Step 2.** Let us define the operator $\varsigma : \mathcal{X}^{1,p(x),q(x)}(\Omega) \to L^{p'(x),q'(x)}(\Omega)$ by

$$\varsigma(u,v) := \big( - \lambda h(x,u,\nabla u), -\kappa g(x,v,\nabla v)\big),$$

that is for $(\varphi,\psi) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$, by

$$\langle \varsigma(u,v), (\varphi,\psi)\rangle = -\lambda \int_\Omega h(x,u,\nabla u)\varphi dx - \kappa \int_\Omega g(x,v,\nabla v)\psi dx.$$

We will show that $\varsigma$ is bounded. Let $(u,v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$, then

$$\begin{aligned}
|\varsigma(u,v)|_{p'(x),q'(x)} &\le |\lambda h(x,u,\nabla u)|_{p'(x)} + |\kappa g(x,v,\nabla v)|_{q'(x)} \\
&= \int_\Omega |\lambda h(x,u,\nabla u)|^{p'(x)} dx + \int_\Omega |\kappa g(x,v,\nabla v)|^{q'(x)} dx + 2 \\
&\le \big(|\lambda|^{p+} + |\lambda|^{p-}\big) \int_\Omega \Big|\mu\big(\gamma(x) + |u|^{r_1(x)-1} + |\nabla u|^{r_1(x)-1}\big)\Big|^{p'(x)} dx \\
&+ \big(|\kappa|^{q+} + |\kappa|^{q-}\big) \int_\Omega \Big|\alpha\big(e(x) + |v|^{r_2(x)-1} + |\nabla v|^{r_2(x)-1}\big)\Big|^{q'(x)} dx \\
&\le C_1 \int_\Omega (|u|^{p(x)} + |\nabla u|^{p(x)}) dx + C_2 \int_\Omega (|v|^{q(x)} + |\nabla v|^{q(x)}) dx \\
&\le C_3\Big(\|u\|_{1,p(x)}^{p+} + \|u\|_{1,p(x)}^{p-}\Big) + C_4\Big(\|v\|_{1,q(x)}^{q+} + \|v\|_{1,q(x)}^{q-}\Big) \\
&\le C_5\Big(\|u\|_{1,p(x)}^{p+} + \|u\|_{1,p(x)}^{p-} + \|v\|_{1,q(x)}^{q+} + \|v\|_{1,q(x)}^{q-}\Big),
\end{aligned}$$

where $C_1, ..., C_5 > 0$ are independent of $u$ and $v$. Therefore, $\varsigma$ is bounded.

Next, we show that $\varsigma$ is continuous. Let $(u_n, v_n) \to (u,v)$ in $\mathcal{X}^{1,p(x),q(x)}(\Omega)$ then,$(u_n, v_n) \to (u,v)$ in $L^{p(x),q(x)}(\Omega)$ and $(\nabla u_n, \nabla v_n) \to (\nabla u, \nabla v)$ in $(L^{p(x),q(x)}(\Omega))^N$. Then

$$\begin{aligned}
\|\varsigma(u_n,v_n) - \varsigma(u,v)\|_{p'(x),q'(x)} &= \|\lambda\big(f(x,u_n,\nabla u_n) - f(x,u,\nabla u)\big)\|_{p'(x)} \\
&\quad + \|\kappa\big(h(x,v_n,\nabla v_n) - h(x,v,\nabla v)\big)\|_{q'(x)}.
\end{aligned}$$

First, we prove that

$$\lim_{n\to+\infty} \|\lambda\big(h(x,u_n,\nabla u_n) - h(x,u,\nabla u)\big)\|_{p'(x)} = 0.$$

By Proposition 2.4, it is equivalent to prove that

$$\lim_{n\to+\infty} \varrho_{p'(x)}\Big(\lambda\big(h(x,u_n,\nabla u_n) - h(x,u,\nabla u)\big)\Big) = 0.$$

Since $u_n \to u$ in $L^{p(x),q(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in$(L^{p(x),q(x)}(\Omega))^N$. Then, there exist a subsequence still denoted by $(u_n)$ and $\delta$ in $L^{p(x)}$ and $\Upsilon$ in $(L^{p(x)}(\Omega))^N$ such that

$$u_n(x) \to u(x) \ \text{ and } \ \nabla u_n(x) \to \nabla u(x), \tag{3.7}$$

$$|u_n(x)| \le \delta(x) \ \text{ and } \ |\nabla u_n(x)| \le \Upsilon(x), \tag{3.8}$$

for almost every $x \in \Omega$ and all $n \in \mathbb{N}$. Thus, from assumption $(H_1)$ and (3.7), we have

$$h(x,u_n,\nabla u_n) \to h(x,u,\nabla u) \,\text{as}\, n \to \infty, \ \text{ for almost every } x \ \in \Omega,$$

by (3.8) and $(H_1)$, we can deduce

$$|h(x,u_n(x),\nabla u_n(x))| \le \mu(\gamma(x) + |\delta(x)|^{p(x)-1} + |\Upsilon(x)|^{p(x)-1}),$$

for almost every $x \in \Omega$ and for all $n \in \mathbb{N}$. Taking into account that

$$\gamma(x) + |\delta(x)|^{p(x)-1} + |\Upsilon(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

by applying Lebesgue's theorem, we have

$$\lim_{n\to+\infty} \varrho_{p'(x)}\Big(\lambda h(x, u_n, \nabla u_n) - \lambda h(x, u, \nabla u)\Big) = 0.$$

The same reasoning is used to prove that

$$\lim_{n\to+\infty} \varrho_{q'(x)}\Big(\kappa g(x, v_n, \nabla v_n) - \kappa g(x, v, \nabla v)\Big) = 0.$$

We conclude that $\zeta$ is continuous.

**Step 3.** Since the embedding $i : \mathcal{X}^{1,p(x),q(x)}(\Omega) \to L^{p(x),q(x)}(\Omega)$ is compact, then the adjoint operator $i^* : L^{p'(x),q'(x)}(\Omega) \to \big(\mathcal{X}^{1,p(x),q(x)}(\Omega)\big)^*$ is also compact. Hence, the compositions $i^* \circ \Psi : \mathcal{X}^{1,p(x),q(x)}(\Omega) \to \big(\mathcal{X}^{1,p(x),q(x)}(\Omega)\big)^*$ and $i^* \circ \zeta : \mathcal{X}^{1,p(x),q(x)}(\Omega) \to \big(\mathcal{X}^{1,p(x),q(x)}(\Omega)\big)^*$ are compact, that means $\mathcal{S} = i^* \circ \Psi + i^* \circ \zeta$ is compact. With this last step the proof of Lemma 3.4 is completed. □

Our main result is the following existence theorem.

**Theorem 3.1.** *Assume that assumptions $(M_1)$-$(M_5)$ and $(H_1)$,$(H_2)$ are satisfied. Then problem (1.1), admits at least one weak solution $(u, v)$ in $\mathcal{X}^{1,p(x),q(x)}(\Omega)$.*

*Proof.* The couple $(u, v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ is a weak solution of (1.1) if and only if

$$\mathcal{F}(u, v) = -\mathcal{S}(u, v), \tag{3.9}$$

where $\mathcal{F}, \mathcal{S}$ are defined as in Lemmas 3.3 and 3.4, respectively by

$$\begin{aligned} &\mathcal{F} : \mathcal{X}^{1,p(x),q(x)}(\Omega) \longrightarrow \big(\mathcal{X}^{1,p(x),q(x)}(\Omega)\big)^* \\ &\langle \mathcal{F}(u, v), (\varphi, \psi)\rangle = \langle f_p u, \varphi\rangle + \langle f_q v, \psi\rangle, \end{aligned}$$

and

$$\begin{aligned} &\mathcal{S} : \mathcal{X}^{1,p(x),q(x)}(\Omega) \longrightarrow \big(\mathcal{X}^{1,p(x),q(x)}(\Omega)\big)^* \\ &\langle \mathcal{S}(u, v), (\varphi, \psi)\rangle = -\lambda \int_\Omega h(x, u, \nabla u)\varphi dx - \kappa \int_\Omega g(x, v, \nabla v)\psi dx \\ &\qquad + \int_\Omega \mathcal{Q}(x)|u|^{r_1(x)-2} u\varphi(x) dx + \int_\Omega \mathcal{O}(x)|v|^{r_2(x)-2} v\psi(x) dx. \end{aligned}$$

By Lemma 3.3, the operator $\mathcal{F}$ is continuous, bounded, strictly monotone and of class $(S_+)$, therefore, by the Minty-Browder Theorem (see [30]), the inverse operator

$$\mathcal{T} := \mathcal{F}^{-1} : (\mathcal{X}^{1,p(x),q(x)}(\Omega))^* \to \mathcal{X}^{1,p(x),q(x)}(\Omega),$$

$$\mathcal{T}(\phi, \psi) = (T_p\phi, T_q\psi),$$

is also bounded, continuous, strictly monotone, and of class $(S_+)$. The operator $\mathcal{T}$ is such that

$$\mathcal{T}(\phi, \psi) = (u, v) \text{ if and only if } \quad (\phi, \psi) = \mathcal{F}(u, v).$$

Consequently, following Zeidler's terminology [30], equation (3.9) is equivalent to the following abstract Hammerstein equation

$$(u, v) = \mathcal{T}(\phi, \psi) \text{ and } (\phi, \psi) + \mathcal{S} \circ \mathcal{T}(\phi, \psi) = 0, \tag{3.10}$$

for all $(u, v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ and $(\phi, \psi) \in (\mathcal{X}^{1,p(x),q(x)}(\Omega))^*$. To say that a couple $(u, v) \in \mathcal{X}^{1,p(x),q(x)}(\Omega)$ is a solution to (3.9) is equivalent to say that $(\phi, \psi)$ is a dual solution of (3.10). Then to solve (3.9) it suffices to solve (3.10), and we will apply the Berkovits topological degree introduced in Section 2.2. To do this, we, first, claim that the set

$$B := \Big\{(\phi, \psi) \in (\mathcal{X}^{1,p(x),q(x)}(\Omega))^* : \exists\, t \in [0,1] \ \text{ such that } \ (\phi, \psi) + t\mathcal{S}o\mathcal{T}(\phi, \psi) = 0\Big\}.$$

is bounded. To verify this, we show that the set $\Big\{\mathcal{T}(\phi, \psi)|\ (\phi, \psi) \in B\Big\}$ is bounded. Indeed, taking into account that

$$\|\mathcal{T}(\phi, \psi)\|_{1,p(x),q(x)} = \|(u, v)\|_{1,p(x),q(x)} = \|\nabla u\|_{p(x)} + \|\nabla v\|_{q(x)}.$$

We denote $\mathtt{D} = \mathcal{X}^{1,p(x),q(x)}(\Omega) \cap \mathcal{T}(B)$ and define

$$\begin{aligned}
\mathtt{D}_1 &= \Big\{(u, v) \in \mathtt{D}\Big|\, 1 \geq \|\nabla u\|_{p(x)},\ \|\nabla v\|_{q(x)}\Big\},\\
\mathtt{D}_2 &= \Big\{(u, v) \in \mathtt{D}\Big|\, 1 < \|\nabla u\|_{p(x)},\ \|\nabla v\|_{q(x)}\Big\},\\
\mathtt{D}_3 &= \Big\{(u, v) \in \mathtt{D}\Big|\, 1 < \|\nabla u\|_{p(x)} \text{ and } \|\nabla v\|_{q(x)} < 1\Big\},\\
\mathtt{D}_4 &= \Big\{(u, v) \in \mathtt{D}\Big|\, 1 > \|\nabla u\|_{p(x)} \text{ and } \|\nabla v\|_{q(x)} > 1\Big\}.
\end{aligned}$$

Then we have the following cases:
<u>*First case.*</u> If $(u, v) \in \mathtt{D}_1$, then $\|\mathcal{T}(\phi, \Psi)\|_{1,p(x),q(x)}$ is bounded by definition.
<u>*Second case.*</u> If $(u, v) \in \mathtt{D}_2$, we deduce from (2.2), $(M_2)$-$(M_3)$ that

$$\begin{aligned}
\|\mathcal{T}(\phi, \psi)\|_{1,p(x),q(x)} &\leq \|\nabla u\|_{p(x)}^{p-} + \|\nabla v\|_{q(x)}^{q-} \leq \varrho_{p(x)}(\nabla u) + \varrho_{q(x)}(\nabla v)\\
&\leq \frac{1}{\sigma}\int_\Omega |\nabla u|^{p(x)}dx + \frac{1}{\iota}\int_\Omega |\nabla v|^{q(x)}dx\\
&\leq \int_\Omega a_1(x, \nabla u)\nabla u\, dx + \int_\Omega a_2(x, \nabla v)\nabla v\, dx\\
&\leq max\{\frac{1}{\sigma}, \frac{1}{\iota}\}\ \langle \mathcal{F}(u, v),\ (u, v)\rangle_{1,p(x),q(x)}\\
&= -t\ max\{\frac{1}{\sigma}, \frac{1}{\iota}\}\ \langle \mathcal{S}o\mathcal{T}(\phi, \psi),\ \mathcal{T}(\phi, \psi)\rangle_{1,p(x),q(x)}.
\end{aligned}$$

Moreover, by assumptions $(H_1)$-$(H_2)$, Young's inequality and bearing (2.7), (2.6) in mind, we obtain

$$\begin{aligned}
\|\mathcal{T}(\phi, \psi)\|_{1,p(x),q(x)} &\leq C_1\Big(\int_\Omega \lambda f(x, u, \nabla u)u dx + \int_\Omega \kappa h(x, v, \nabla v)v dx + \int_\Omega \lambda \mathcal{Q}(x)|u|^{r_1(x)}dx\\
&\quad + \int_\Omega \kappa\, \mathcal{O}(x)|v|^{r_2(x)}dx\Big)\\
&\leq C_2\Big[\varrho_{p(x)}(u) + \varrho_{q(x)}(v) + \int_\Omega \mu(\gamma(x) + |u|^{r_1(x)-1} + |\nabla u|^{r_1(x)-1})u\, dx\\
&\quad + \int_\Omega \alpha(e(x) + |v|^{r_2(x)-1} + |\nabla v|^{r_2(x)-1})v\, dx\Big]\\
&\leq C_3\Big[\rho_{p(x)}(u) + \rho_{q(x)}(v) + \int_\Omega \gamma(x)u dx + \int_\Omega |u|^{p(x)-1}u\, dx + \int_\Omega |\nabla u|^{p(x)-1}u\, dx\\
&\quad + \int_\Omega e(x)v\, dx + \int_\Omega |\nabla v|^{q(x)-1}v\, dx + \int_\Omega |v|^{q(x)-1}v\, dx\Big]\\
&\leq C_4\Big[|u|_{p(x)}^{p-} + |u|_{p(x)}^{p+} + |v|_{q(x)}^{q-} + |v|_{q(x)}^{q+} + |\gamma|_{p'(x)}|u|_{p(x)} + |e|_{q'(x)}|v|_{q(x)}\\
&\quad + C_5\,\varrho_{p(x)}(\nabla u) + C_6\,\varrho_{p(x)}(u) + C_7\,\varrho_{q(x)}(\nabla v) + C_8\,\varrho_{q(x)}(v)\Big]\\
&\leq C_9\big[\|u\|_{p(x)}^{p-} + \|u\|_{p(x)}^{p+} + \|v\|_{q(x)}^{q-} + \|v\|_{q(x)}^{q+}\big],
\end{aligned}$$

where $C_1,...,C_9>0$ are independent of $u$, $v$. Hence, $\|\mathcal{T}(\phi,\psi)\|_{1,p(x),q(x)}$ is bounded.
*Third case.* If $(u,v)\in \mathtt{D}_3$, then

$$\begin{aligned}\|\mathcal{T}(\phi,\psi)\|_{1,p(x),q(x)} &= \|\nabla u\|_{p(x)}+\|\nabla v\|_{q(x)}\\ &\le \|\nabla u\|_{p(x)}^{p-}+1\le 1+\|\nabla u\|_{p(x)}^{p-}+\|\nabla v\|_{q(x)}^{q+}\\ &\le \varrho_{p(x)}(\nabla u)+\varrho_{q(x)}(\nabla v)+1.\end{aligned}$$

From here, we proceed in the same manner as in the prior case to arrive at the conclusion that $\|\mathcal{T}(\phi,\psi)\|_{1,p(x),q(x)}$ is bounded.
*Fourth case.* Similarly to the previous case, if $(u,v)\in\mathtt{D}_4$ inversing the positions of $u$ and $v$, we get that $\Big\{\mathcal{T}(\phi,\psi):(\phi,\psi)\in B\Big\}$ is bounded.

On the other hand, we have that the operator $\mathcal{S}$ is bounded. Thus, thanks to (3.10), we have that the set $B$ is bounded in $(\mathcal{X}^{1,p(x),q(x)}(\Omega))^*$. Consequently, there exists $R>0$ such that

$$\|(\phi,\psi)\|_{1,p'(x),q'(x)}<R \quad\text{for all}\quad (\phi,\psi)\in B.$$

Hence, it follows that

$$(\phi,\psi)+t\mathcal{S}\circ\mathcal{T}(\phi,\psi)\neq 0 \quad\text{for all}\quad (\phi,\psi)\in\partial B_R(0) \text{ and } t\in[0,1].$$

Moreover, $\mathcal{S}$ is compact, then it is known that $\mathcal{S}$ is continuous, quasimonotne and by Lemma 2.1, we conclude that

$$I+\mathcal{S}\circ\mathcal{T}\in\mathcal{F}_{\mathcal{T}}(\overline{B_R(0)}) \quad\text{and}\quad I=\mathcal{F}\circ\mathcal{T}\in\mathcal{F}_{\mathcal{T}}(\overline{B_R(0)}).$$

Since $I,\mathcal{S}$ and $\mathcal{T}$ are bounded, then

$$I+\mathcal{S}\circ\mathcal{T}\in\mathcal{F}_{\mathcal{T},B}(\overline{B_R(0)}) \quad\text{and}\quad I=\mathcal{F}o\mathcal{T}\in\mathcal{F}_{\mathcal{T},B}(\overline{B_R(0)}).$$

Consequently, the homotopy

$$\begin{aligned}\mathcal{H}:[0,1]\times\overline{B_R(0)} &\to (\mathcal{X}^{1,p(x),q(x)}(\Omega))^*\\ (t,\phi,\psi)&\mapsto \mathcal{H}(t,\phi,\psi):=(\phi,\psi)+t\mathcal{S}\circ\mathcal{T}(\phi,\psi)\end{aligned}$$

is such that $\mathcal{H}\in\mathcal{F}_{\mathcal{T},B}(\overline{B_R(0)})$, and thanks to the homotopy invariance and normalization property of the degree $d$, seen in Theorem 2.1, we obtain

$$d(I+\mathcal{S}\circ\mathcal{T},B_R(0),0)=d(I,B_R(0),\ 0)=1\neq 0,$$

which implies that there exists $(\phi,\psi)\in B_R(0)$ satisfying the equality

$$(\phi,\psi)+\mathcal{S}\circ\mathcal{T}(\phi,\psi)=0.$$

Finally, we conclude that $(u,v)=\mathcal{T}(\phi,\psi)$ is a weak solutions of (1.1). $\square$

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Received: 12.01.2024