ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2024, Volume 15, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

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EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 15, Number 2 (2024), 61 – 74

CURVILINEAR PARALLELOGRAM IDENTITY AND MEAN-VALUE PROPERTY FOR A SEMILINEAR HYPERBOLIC EQUATION OF THE SECOND ORDER

V.I. Korzyuk, J.V. Rudzko

Communicated by V.I. Burenkov

Key words: hyperbolic equation, characteristics parallelogram, mean-value theorem, mean-value property.

AMS Mathematics Subject Classification: 35B05, 35L10, 35L71, 35C15.

Abstract. In this paper, we discuss some of important qualitative properties of solutions of second-order hyperbolic equations, whose coefficients of the terms involving the second-order derivatives are independent of the desired function and its derivatives. Solutions of these equations have a special property called curvilinear parallelogram identity (or mean-value property), which can be used to solve some initial-boundary value problems.

DOI: https://doi.org/10.32523/2077-9879-2024-15-2-61-74

1 Introduction

The terms "mean value theorem", "mean value property", "mean formula", and "mean value" are quite common in mathematics (e.g. real analysis, complex analysis, probability theory, partial differential equations) and physics. But they may pertain to diverse phenomena.

In the theory of partial differential equations mean value theorems for harmonic functions and solutions of various elliptic equations are best known. They include the classical mean value property for harmonic functions [12] and the results obtained in works [9, 7, 8, 27] for more general elliptic equations and elliptic operators. Similar theorems are formulated for (hypoelliptic) parabolic equations [16, 17, 18].

Such facts can be established not only for elliptic and parabolic equations but also for hyperbolic ones. Foremost, it should be noted the classical Asgeirson's mean value theorem [3, 6] for the ultrahyperbolic differential equation and the mean value theorem of Bitsadze and Nakhushev for the wave equation [2]. Spherical means can be used to solve initial-value problems as it is done in work [10] for the wave equation and the Darboux equation. Using a symbolic approach [28] several results [24, 22, 23, 30, 25, 31, 29, 26, 33, 32] associated with mean values of solutions of various differential equations were obtained in works of Polovinkin and Meshkov et al. It should also be said that in these works the parallelogram identity (parallelogram rule) for the wave equation (which the authors call 'difference mean-value formula') was generalized to the following cases: a (nonstrictly) hyperbolic equation with constant coefficients of the third-order [24], fourth-order [22], higher-order [32], an equation with constant coefficients and with the operator represented by the product of the first order hyperbolic operators and the second-order elliptic operators [29]. These results can be used to obtain analytical and numerical solutions to differential equations as it was done in [12, 14, 11, 20, 21]. However, these results are mainly given for equations with constant coefficients because of the methods used (Fourier transform, search for accompanying distribution with compact support).

Moreover, the characteristic parallelogram of differential equations has some applications in hydrodynamics [19].

In this paper, we derive the identity of a curvilinear characteristic parallelogram for a general semilinear second-order hyperbolic equation using the method of characteristics [12]. This identity can be considered as the mean value theorem in some sense.

2 Semilinear hyperbolic equation

In the domain $\Omega \subseteq \mathbb{R}^2$ of two independent variables $\mathbf{x} = (x_1, x_2) \in \Omega$ we consider the following semilinear hyperbolic equation of the second-order

$$Au(x_1, x_2) = f(x_1, x_2, u(x_1, x_2), \partial_{x_1} u(x_1, x_2), \partial_{x_2} u(x_1, x_2)), \tag{2.1}$$

where the operator A is defined as

$$Au(x_1, x_2) := a(x_1, x_2)\partial_{x_1}^2 u(x_1, x_2) + 2b(x_1, x_2)\partial_{x_1}\partial_{x_2} u(x_1, x_2) + c(x_1, x_2)\partial_{x_2}^2 u(x_1, x_2),$$

and is hyperbolic (this means $b^2(\mathbf{x}) - a(\mathbf{x})c(\mathbf{x}) > 0$ for any $x \in \Omega$).

Equation (2.1) has two families of characteristics: $\gamma_1(x_1, x_2)$ and $\gamma_2(x_1, x_2)$, which are the first integrals of the ordinary differential equation [12]

$$a(\mathbf{x})(\mathrm{d}x_2)^2 - 2b(\mathbf{x})\mathrm{d}x_1\mathrm{d}x_2 + c(\mathbf{x})(\mathrm{d}x_1)^2 = 0, \tag{2.2}$$

and solutions of the equation of characteristics [12]

$$a\left(\frac{\partial \gamma_i}{\partial x_1}\right)^2 + 2b\frac{\partial \gamma_i}{\partial x_1}\frac{\partial \gamma_i}{\partial x_2} + c\left(\frac{\partial \gamma_i}{\partial x_2}\right)^2 = 0, \quad i = 1, 2. \tag{2.3}$$

It is known [12] that equation (2.2), generally speaking, can be decomposed into two equations

$$a(\mathbf{x})\mathrm{d}x_2 - (b(\mathbf{x}) \pm \sqrt{b^2(\mathbf{x}) - a(\mathbf{x})c(\mathbf{x})})\mathrm{d}x_1 = 0, \text{ if } a(\mathbf{x}) \neq 0,$$

or

$$c(\mathbf{x})\mathrm{d}x_1 - (b(\mathbf{x}) \pm \sqrt{b^2(\mathbf{x}) - a(\mathbf{x})c(\mathbf{x})})\mathrm{d}x_2 = 0, \text{ if } c(\mathbf{x}) \neq 0,$$

or

$$\mathrm{d}x_1\mathrm{d}x_2 = 0, \text{ if } a(\mathbf{x}) = c(\mathbf{x}) = 0.$$

Therefore, we can assume that γ_1 and γ_2 are the first integrals of different differential equations and they are functionally independent since the Jacobian $\left|\dfrac{\partial(\gamma_1, \gamma_2)}{\partial(x_1, x_2)}\right|$ is nonzero [12].

If the curves γ_i, $i = 1, 2$, have a parametric representation $(x_1^{(i)}(t), x_2^{(i)}(t))$, where $x_j^{(i)}$, $j = 1, 2$, are some twice continuously differentiable functions, then the following equality holds [4]

$$a\left(Dx_2^{(i)}\right)^2 - 2bDx_1^{(i)}Dx_2^{(i)} + c\left(Dx_1^{(i)}\right)^2 = 0, \quad i = 1, 2,$$

where D is the ordinary differential operator.

3 Curvilinear characteristic parallelogram

Definition 1. Curvilinear characteristic parallelogram of hyperbolic differential equation (2.1) is the set $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1, l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1, r_2]\}$, where l_1, l_2, r_1, r_2 are some real numbers and γ_i, $i = 1, 2$ are two different functionally independent characteristics.

Remark 1. Definition 1 is well defined. It is known [1] that any other first integral of (2.2) has the form $q \circ \gamma_1$, where q is some continuously differentiable function. If $\gamma_1(\mathbf{x}) \in [l_1, l_2]$, then, due to the continuity of q, $q(\gamma_1(\mathbf{x})) \in q([l_1, l_2]) = [\widetilde{l}_1, \widetilde{l}_2]$. So the curvilinear characteristic parallelogram does not depend on considered characteristics.

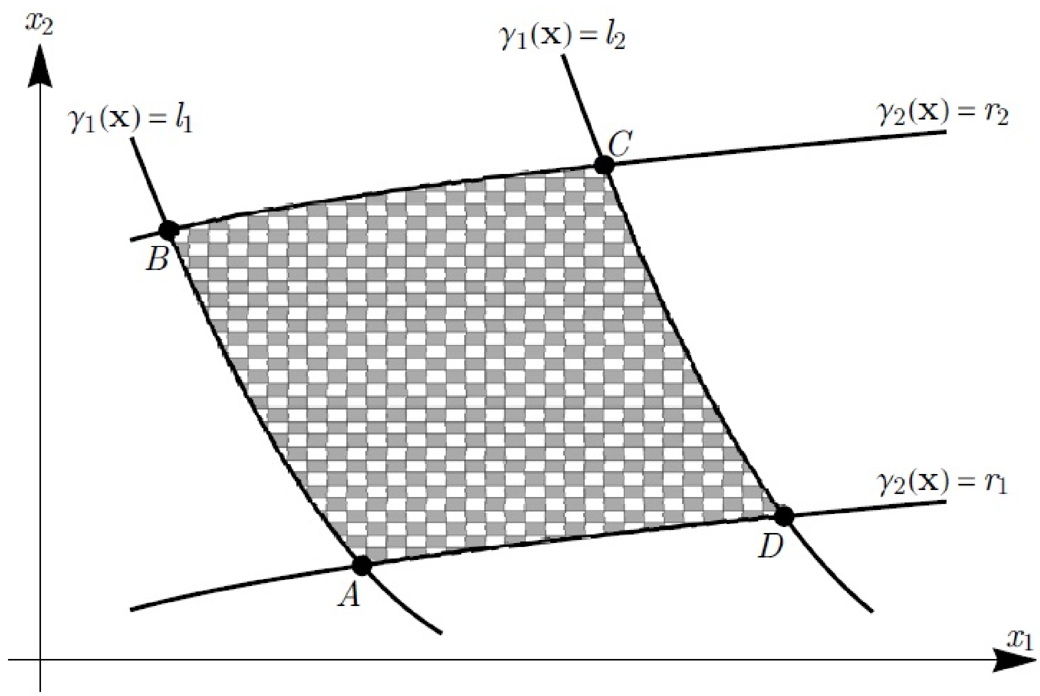

Fig. 1. Curvilinear characteristic parallelogram

Definition 2. Vertices of the curvilinear characteristic parallelogram $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1, l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1, r_2]\}$ are points $\mathbf{x}$ such that $\gamma_1(x) = l_i \wedge \gamma_2(x) = r_j$, $(i, j) \in \{1, 2\} \times \{1, 2\}$.

Remark 2. Definition 2 is well defined. We should show that $q \circ \gamma_1$, where q is some continuously differentiable function, maps $[l_1, l_2]$ into $[\widetilde{l}_1, \widetilde{l}_2]$ and $\partial([l_1, l_2])$ into $\partial([\widetilde{l}_1, \widetilde{l}_2])$. Obviously, if the function q is increasing or decreasing these properties must be true. But if the the function q does not satisfy these conditions, then there exists at least one point $l_0 \in (l_1, l_2)$ such that $q'(l_0) = 0$. Due to the continuity of q, there exists a point $\mathbf{x} \in \Pi$ such that $\gamma_1(\mathbf{x}) = l_0 \in (l_1, l_2)$ This implies

$$\left|\frac{\partial(q \circ \gamma_1, \gamma_2)}{\partial(x_1, x_2)}\right|(\mathbf{x}) = \begin{vmatrix} q'(\gamma_1(\mathbf{x}))\partial_{x_1}\gamma_1(\mathbf{x}) & q'(\gamma_1(\mathbf{x}))\partial_{x_2}\gamma_1(\mathbf{x}) \\ \partial_{x_1}\gamma_2(\mathbf{x}) & \partial_{x_2}\gamma_2(\mathbf{x}) \end{vmatrix} = 0 \text{ when } \gamma_1(\mathbf{x}) = l_0.$$

But we consider only characteristics with nonzero Jacobian. The statement is proved.

Definition 3. Opposite vertices of the curvilinear characteristic parallelogram $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1, l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1, r_2]\}$ are its vertices $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $\gamma_1(\mathbf{x}_1) \neq \gamma_1(\mathbf{x}_2)$ and $\gamma_2(\mathbf{x}_1) \neq \gamma_2(\mathbf{x}_2)$.

Point transformation of variables of the form $y_1 = \gamma_1(x_1, x_2)$, $y_1 = \gamma_2(x_1, x_2)$ is invertible [34], i.e. there is the inverse change of variables $x_1 = \gamma_1^{-1}(y_1, y_2)$, $x_2 = \gamma_2^{-1}(y_1, y_2)$.

Lemma 3.1. *Let $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1, l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1, r_2]\}$ be a curvilinear characteristic parallelogram and the conditions $a \in C^2(\Pi)$, $b \in C^2(\Pi)$, $c \in C^2(\Pi)$, and $f \in C^1(\Pi \times \mathbb{R}^3)$ be satisfied. The function u belongs to the class $C^2(\Pi)$ and satisfies equation (2.1) if and only if it can be represented as*

$$\begin{aligned}
u(\mathbf{x}) &= g_1(\gamma_1(\mathbf{x})) + g_2(\gamma_2(\mathbf{x})) \\
&+ \int_{l^{(0)}}^{\gamma_1(\mathbf{x})} dz_1 \int_{r^{(0)}}^{\gamma_2(\mathbf{x})} \frac{1}{2\left(a\partial_{x_1}\gamma_1\partial_{x_1}\gamma_2 + b\left(\partial_{x_2}\gamma_2\partial_{x_1}\gamma_1 + \partial_{x_2}\gamma_1\partial_{x_1}\gamma_2\right) + c\partial_{x_2}\gamma_1\partial_{x_2}\gamma_2\right)\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)} \\
&\times \Big[f\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z}), u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right),\right. \\
&\left.\partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\right) \\
&- A\gamma_1\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\left(\partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\partial_{y_1}\gamma_1^{-1}(\mathbf{z})\right. \\
&\left.+ \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\partial_{y_1}\gamma_2^{-1}(\mathbf{z})\right) \\
&- A\gamma_2\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\left(\partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\partial_{y_2}\gamma_1^{-1}(\mathbf{z})\right. \\
&\left.+ \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\partial_{y_2}\gamma_2^{-1}(\mathbf{z})\right)\Big] dz_2,
\end{aligned} \tag{3.1}$$

where $l^{(0)} \in [l_1, l_2]$, $r^{(0)} \in [r_1, r_2]$, and the functions g_1, g_2 belong to the classes $C^2(\mathfrak{D}(g_1))$, $C^2(\mathfrak{D}(g_2))$ respectively.

Proof. Let a function $u \in C^2(\Pi)$ satisfy equation (2.1). Making the nonlinear nondegenerate change of independent variables $y_1 = \gamma_1(x_1, x_2)$, $y_1 = \gamma_2(x_1, x_2)$ and denoting $u(x_1, x_2) = v(y_1, y_2)$ we obtain a new differential equation

$$\begin{aligned}
&2\left(a\partial_{x_1}\gamma_1\partial_{x_1}\gamma_2 + b\left(\partial_{x_2}\gamma_2\partial_{x_1}\gamma_1 + \partial_{x_2}\gamma_1\partial_{x_1}\gamma_2\right) + c\partial_{x_2}\gamma_1\partial_{x_2}\gamma_2\right)\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right) \\
&\times \partial_{y_1}\partial_{y_2}v(\mathbf{y}) + A\gamma_1\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right)\partial_{y_1}v(\mathbf{y}) + A\gamma_2\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right)\partial_{y_2}v(\mathbf{y}) \\
&= f\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y}), u\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right), \partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right),\right. \\
&\left.\partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right)\right) = f\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y}), v(\mathbf{y}), \partial_{y_1}v(\mathbf{y})\partial_{x_1}\gamma_1\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right)\right. \\
&+ \partial_{y_2}v(\mathbf{y})\partial_{x_1}\gamma_2\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right), \partial_{y_1}v(\mathbf{y})\partial_{x_2}\gamma_1\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right) \\
&\left.+ \partial_{y_2}v(\mathbf{y})\partial_{x_2}\gamma_2\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right)\right)
\end{aligned}$$

Let us integrate it twice to obtain the equality

$$\begin{aligned}
v(\mathbf{y}) &= g_1(\mathbf{y}) + g_2(\mathbf{y}) \\
&+ \int_{l^{(0)}}^{y_1} dz_1 \int_{r^{(0)}}^{y_2} \frac{1}{2\left(a\partial_{x_1}\gamma_1\partial_{x_1}\gamma_2 + b\left(\partial_{x_2}\gamma_2\partial_{x_1}\gamma_1 + \partial_{x_2}\gamma_1\partial_{x_1}\gamma_2\right) + c\partial_{x_2}\gamma_1\partial_{x_2}\gamma_2\right)\left(\gamma_1^{-1}(\mathbf{y}), \gamma_2^{-1}(\mathbf{y})\right)} \\
&\times \Big[f\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z}), u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right),\right. \\
&\left.\partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\right) \\
&- A\gamma_1(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z}))\partial_{y_1}v(\mathbf{z}) - A\gamma_2(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z}))\partial_{y_2}v(\mathbf{z})\Big] dz_2,
\end{aligned}$$

Returning to the variables x_1 and x_2 we obtain equation (3.1). This also implies that the functions g_j belong to the class $C^2(\mathfrak{D}(g_1))$, $j = 1, 2$.

Substituting representations (3.1) into equation (2.1), we verify that the function u satisfies this equation in Π. □

Remark 3. Under some additional conditions on the functions f, a, b, c, g_1, g_2, we can show the solvability of integro-differential equation (3.1) using the methods proposed in the works [5, 13, 35].

For the convenience of further presentation, we introduce the notation

$$
\begin{aligned}
&\beta = 2\left(a\partial_{x_1}\gamma_1\partial_{x_1}\gamma_2 + b\left(\partial_{x_2}\gamma_2\partial_{x_1}\gamma_1 + \partial_{x_2}\gamma_1\partial_{x_1}\gamma_2\right) + c\partial_{x_2}\gamma_1\partial_{x_2}\gamma_2\right),\\
&K(\mathbf{z},p,q,r) = f(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z}),p,q,r)\\
&-A\gamma_1(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z}))(q\partial_{y_1}\gamma_1^{-1}(\mathbf{z}) + r\partial_{y_1}\gamma_2^{-1}(\mathbf{z}))\\
&-A\gamma_2(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z}))(q\partial_{y_2}\gamma_1^{-1}(\mathbf{z}) + r\partial_{y_2}\gamma_2^{-1}(\mathbf{z})),\\
&\widetilde{K}(\mathbf{z},p,q,r) = (\beta(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})))^{-1}K(\mathbf{z},p,q,r)
\end{aligned}
$$

4 Curvilinear parallelogram identity

Theorem 4.1. *Let a function u belong to the class $C^2(\Omega)$ and be a solution to hyperbolic equation (2.1), where $a \in C^2(\Omega)$, $b \in C^2(\Omega)$, $c \in C^2(\Omega)$, and $f \in C^1(\Omega\times\mathbb{R}^3)$. Then for any curvilinear characteristic parallelogram $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1,l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1,r_2]\} \subseteq \Omega$ with vertices $A(\gamma_1^{-1}(l_1,r_1),\gamma_2^{-1}(l_1,r_1))$, $B(\gamma_1^{-1}(l_1,r_2),\gamma_2^{-1}(l_1,r_2))$, $C(\gamma_1^{-1}(l_2,r_2),\gamma_2^{-1}(l_2,r_2))$, $(\gamma_1^{-1}(l_2,r_1),\gamma_2^{-1}(l_2,r_1))$, the following equality holds*

$$
\begin{aligned}
&u(A) - u(B) + u(C) - u(D)\\
&\quad = \int_{l_1}^{l_2} dz_1 \int_{r_1}^{r_2} \widetilde{K}\Big(\mathbf{z}, u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right),\\
&\qquad \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\Big)dz_2.
\end{aligned}
\tag{4.1}
$$

Proof. According to Lemma 3.1, the function u is representable in the form

$$
\begin{aligned}
&u(\mathbf{x}) = g_1(\gamma_1(\mathbf{x})) + g_2(\gamma_2(\mathbf{x}))\\
&\quad + \int_{l_1}^{\gamma_1(\mathbf{x})} dz_1 \int_{r_1}^{\gamma_2(\mathbf{x})} \widetilde{K}\Big(\mathbf{z}, u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right),\\
&\qquad \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\Big)dz_2.
\end{aligned}
\tag{4.2}
$$

where $g_i \in C^2(\mathfrak{D}(g_i))$, $i = 1,2$. Using expression (4.2) we calculate

$$
\begin{aligned}
&u(A) = g_1(l_1) + g_2(r_1), u(B) = g_1(l_1) + g_2(r_2), u(D) = g_1(l_2) + g_2(r_1),\\
&u(C) = g_1(l_2) + g_2(r_2)\\
&+ \int_{l_1}^{l_2} dz_1 \int_{r_1}^{r_2} \widetilde{K}\Big(\mathbf{z}, u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right),\\
&\quad \partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\Big)dz_2.
\end{aligned}
\tag{4.3}
$$

Substituting representations (4.3) into (4.1) we obtain the correct equality. □

Theorem 4.2. *Let functions $u \in C^2(\Omega)$, $a \in C^2(\Omega)$, $b \in C^2(\Omega)$, $c \in C^2(\Omega)$, $f \in C^1(\Omega\times\mathbb{R}^3)$, and the condition $b^2(\mathbf{x}) - a(\mathbf{x})c(\mathbf{x}) > 0$ be satisfied, where $\Omega \subseteq \mathbb{R}^2$. If for any curvilinear characteristic parallelogram $\Pi = \{\mathbf{x} \mid \gamma_1(\mathbf{x}) \in [l_1,l_2] \wedge \gamma_2(\mathbf{x}) \in [r_1,r_2]\} \subseteq \Omega$ with vertices $A(\gamma_1^{-1}(l_1,r_1),\gamma_2^{-1}(l_1,r_1))$, $B(\gamma_1^{-1}(l_1,r_2),\gamma_2^{-1}(l_1,r_2))$, $C(\gamma_1^{-1}(l_2,r_2),\gamma_2^{-1}(l_2,r_2))$, $(\gamma_1^{-1}(l_2,r_1),\gamma_2^{-1}(l_2,r_1))$, where γ_i, $i = 1,2$ are solutions of equations (2.2) and γ_i^{-1} are defined as before, equality (4.1) is satisfied, then the function u is a solution to equation (2.1).*

Proof. Let $l_2 = l + l_1$, $r_2 = r + r_1$. So, we can write the coordinates of points A, B, C and D in the form

$$A(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1)),\ B(\gamma_1^{-1}(l_1, r + r_1), \gamma_2^{-1}(l_1, r + r_1)),$$
$$C(\gamma_1^{-1}(l + l_1, r + r_1), \gamma_2^{-1}(l + l_1, r + r_1)),\ D(\gamma_1^{-1}(l + l_1, r_1), \gamma_2^{-1}(l + l_1, r_1)).$$

Let us consider the expression

$$\frac{u(A) - u(B)}{r} = \frac{u(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1)) - u(\gamma_1^{-1}(l_1, r + r_1), \gamma_2^{-1}(l_1, r + r_1))}{r} \xrightarrow[r\to 0]{} \xrightarrow[r\to 0]{} -\partial_r u(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1)).$$

In the same way

$$\frac{u(C) - u(D)}{r} \xrightarrow[r\to 0]{} \partial_r u(\gamma_1^{-1}(l_1 + l, r_1), \gamma_2^{-1}(l_1 + l, r_1)).$$

Now since

$$\frac{\partial_r u(\gamma_1^{-1}(l_1 + l, r_1), \gamma_2^{-1}(l_1 + l, r_1)) - \partial_r u(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1))}{l} \xrightarrow[l\to 0]{} \xrightarrow[l\to 0]{} \partial_l \partial_r u(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1)),$$

we obtain $\lim\limits_{(r,l)\to(0,0)} (lr)^{-1}(u(A) - u(B) + u(C) - u(D)) = \partial_l \partial_r u(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1))$. Similarly, we get

$$\lim_{(r,l)\to(0,0)} \frac{1}{lr} \int\limits_{l_1}^{l+l_1} dz_1 \int\limits_{r_1}^{r+r_1} \widetilde{K}\Big(\mathbf{z}, u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_2} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\Big) dz_2 =$$
$$= \widetilde{K}\Big(\mathbf{z} = (l_1, r_1), u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_2} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\Big).$$

Thus

$$\lim_{(r,l)\to(0,0)} \frac{1}{lr}\Bigg(u(A) - u(B) + u(C) - u(D) - \int\limits_{l_1}^{l+l_1} dz_1 \int\limits_{r_1}^{r+r_1} \widetilde{K}\Big(\mathbf{z}, u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_2} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\Big) dz_2 \Bigg)$$
$$= \lim_{(r,l)\to(0,0)} \frac{u(A) - u(B) + u(C) - u(D)}{lr}$$
$$- \lim_{(r,l)\to(0,0)} \frac{1}{lr} \int\limits_{l_1}^{l+l_1} dz_1 \int\limits_{r_1}^{r+r_1} \widetilde{K}\Big(\mathbf{z}, u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_2} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\Big) dz_2 = \partial_l \partial_r u(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1))$$
$$- \frac{K\left(\mathbf{z} = (l_1, r_1), u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_1} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right), \partial_{x_2} u\left(\gamma_1^{-1}(\mathbf{z}), \gamma_2^{-1}(\mathbf{z})\right)\right)}{\beta(\gamma_1^{-1}(l_1, r_1), \gamma_2^{-1}(l_1, r_1))}.$$

This means that the function u satisfies at the point

$$(\gamma_1^{-1}(\mathbf{z} = (y_1 = l_1, y_2 = r_1)), \gamma_2^{-1}(\mathbf{z}))$$

the differential equation

$$
\begin{aligned}
&\beta(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z}))\partial_{y_1}\partial_{y_2}u(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z}))\\
&= f\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z}),u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right),\right.\\
&\left.\partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right),\partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\right)\\
&- A\gamma_1\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\left(\partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\partial_{y_1}\gamma_1^{-1}(\mathbf{z})\right.\\
&\left.+\partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\partial_{y_1}\gamma_2^{-1}(\mathbf{z})\right)\\
&- A\gamma_2\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\left(\partial_{x_1}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\partial_{y_2}\gamma_1^{-1}(\mathbf{z})\right.\\
&\left.+\partial_{x_2}u\left(\gamma_1^{-1}(\mathbf{z}),\gamma_2^{-1}(\mathbf{z})\right)\partial_{y_2}\gamma_2^{-1}(\mathbf{z})\right),
\end{aligned}
\tag{4.4}
$$

where $x_1=\gamma_1^{-1}(y_1,y_2)$, $x_2=\gamma_2^{-1}(y_1,y_2)$. By virtue of the arbitrariness of $\Pi\subseteq\Omega$, equality (4.4) is true for any point $(x_1=\gamma_1^{-1}(\mathbf{z}=(l_1,r_1)),x_2=\gamma_2^{-1}(\mathbf{z}=(l_1,r_1)))\in\Omega$.

Making the change of variables $x_1=\gamma_1^{-1}(y_1,y_2)$, $x_2=\gamma_2^{-1}(y_1,y_2)$ in equation (4.4), we obtain equation (2.1). □

Note that formula (4.1) can be considered as a kind of a mean value theorem.

5 Applications

5.1 Wave equation

Let us consider $Au(x_1,x_2)=\partial_{x_1}^2u(x_1,x_2)-a^2\partial_{x_2}^2u(x_1,x_2)$, where $a>0$ (for definiteness). Then we have $\gamma_1(x_1,x_2)=x_2-ax_1$, $\gamma_2(x_1,x_2)=x_2+ax_1$, $\gamma_1^{-1}(y_1,y_2)=(y_2-y_1)/(2a)$, $\gamma_2^{-1}(y_1,y_2)=(y_1+y_2)/2$, $A\gamma_1\equiv0$, $A\gamma_2\equiv0$.

5.1.1 Parallelogram identity

Let $f\equiv0$. In this case, formula (4.1) transforms to

$$
\begin{aligned}
&u\left(\frac{r_1-l_1}{2a},\frac{l_1+r_1}{2}\right)-u\left(\frac{r_2-l_1}{2a},\frac{l_1+r_2}{2}\right)\\
&+u\left(\frac{r_2-l_2}{2a},\frac{l_2+r_2}{2}\right)-u\left(\frac{r_1-l_2}{2a},\frac{l_2+r_1}{2}\right)=0,
\end{aligned}
\tag{5.1}
$$

where l_1, l_2, r_1 and r_2 are some real numbers. Equality (5.1) is the well-known parallelogram identity for the wave equation.

5.1.2 Goursat problem

Let us consider the Goursat problem [15]

$$
\begin{cases}
(\partial_{x_1}^2-a^2\partial_{x_2}^2)u(\mathbf{x})=f(\mathbf{x}), & 0<x_1,-ax_1<x_2<ax_1,\\
u(x_1,x_2=ax_1)=\phi^{(1)}(x_1), & u(x_1,x_2=-ax_1)=\phi^{(1)}(x_2),\quad x_1>0,
\end{cases}
\tag{5.2}
$$

where $f\in C^1(\{\mathbf{x}\mid 0\leqslant x_1,-ax_1\leqslant x_2\leqslant ax_1\})$, $\phi^{(1)}\in C^2([0,\infty))$, $\phi^{(2)}\in C^2([0,\infty))$ and $\phi^{(1)}(0)=\phi^{(2)}(0)$. We can write the classical solution of (5.2) using formula (4.1). If we select $C(x_1,x_2)$, $B\left(\frac{ax_1+x_2}{2a},\frac{ax_1+x_2}{2}\right)$, $D\left(\frac{ax_1-x_2}{2a},\frac{x_2-ax_1}{2}\right)$, $A(0,0)$ and apply (4.1), then we obtain

$$
u(x_1,x_2)=u(C)=\phi^{(1)}\left(\frac{ax_1+x_2}{2a}\right)+\phi^{(2)}\left(\frac{ax_1-x_2}{2a}\right)-\phi^{(1)}(0)
$$

$$-\frac{1}{4a^2}\int\limits_{0}^{x_2-ax_1} dy_1 \int\limits_{0}^{x_2+ax_1} f\left(\frac{y_2-y_1}{2a}, \frac{y_1+y_2}{2}\right) dy_2, \quad 0 < x_1, -ax_1 < x_2 < ax_1.$$

5.1.3 Mixed problem

Let us consider the first mixed problem [12]

$$\begin{cases} (\partial_{x_1}^2 - a^2\partial_{x_2}^2)u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in (0,\infty)\times(0,\infty), \\ u(0,x_2) = \phi(x_1), \quad \partial_{x_1}u(0,x_2) = \psi(x_2), \quad x_1 > 0, \\ u(x_1,0) = \mu(x_1), \quad x_2 > 0, \end{cases} \tag{5.3}$$

where $f \in C^1([0,\infty)\times[0,\infty))$, $\phi \in C^2([0,\infty))$, $\psi \in C^1([0,\infty))$, $\mu \in C^2([0,\infty))$.

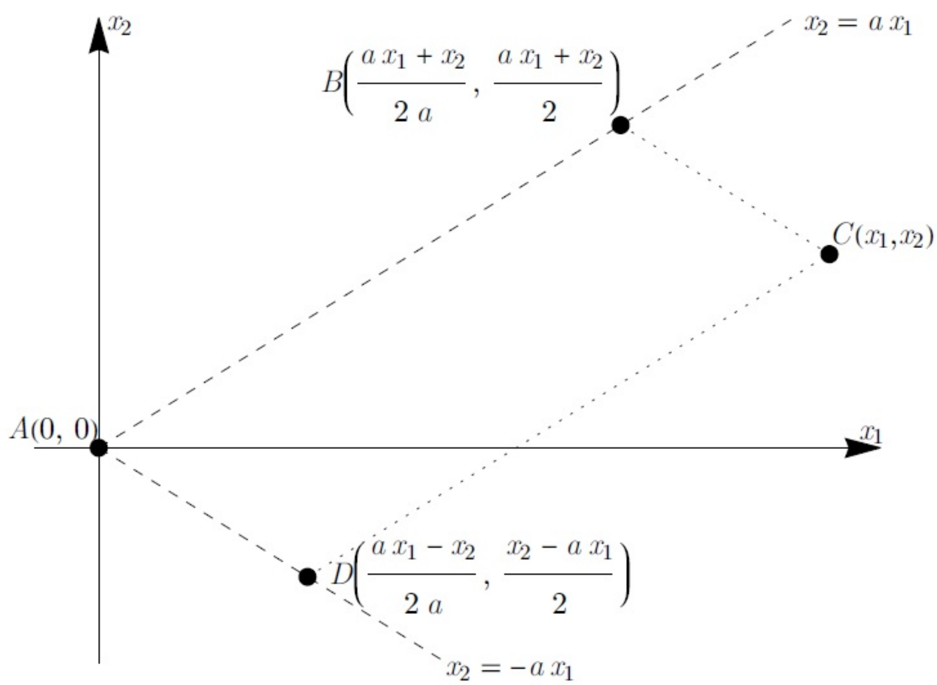

Fig. 2. To the Goursat problem (5.2).

If $x_2 - ax_1 > 0$, then the solution of (5.3) at the point (x_1, x_2) can be defined by d'Alembert formula

$$\begin{aligned} u(x_1,x_2) = \frac{\phi(x_2-ax_1)+\phi(x_2+ax_1)}{2} + \frac{1}{2a}\int\limits_{x_2-ax_1}^{x_2+ax_1} \psi(\xi)\,d\xi + \\ + \frac{1}{2a}\int\limits_{0}^{x_1} d\tau \int\limits_{x_2-a(x_1-\tau)}^{x_2+a(x_1-\tau)} f(\tau,\xi)\,d\xi, \quad x_2 - ax_1 > 0, x_1 > 0, x_2 > 0. \end{aligned} \tag{5.4}$$

If $x_2 - ax_1 < 0$, then we can use parallelogram identity (4.1) to derive the solution of (5.3) at the point (x_1, x_2). We can select $C(x_1, x_2)$, $B\left(x_1 - \frac{x_2}{a}, 0\right)$, $D\left(\frac{x_2}{a}, ax_1\right)$, $A(0, ax_1 - x_2)$, apply (4.1) and obtain

$$u(x_1,x_2)=\mu\Big(x_1-\frac{x_2}{a}\Big)+\frac{\phi(ax_1+x_2)-\phi(ax_1-x_2)}{2}+\frac{1}{2a}\int\limits_{ax_1-x_2}^{ax_1+x_2}\psi(\xi)\,d\xi$$
$$+\frac{1}{2a}\int\limits_0^{\frac{x_2}{a}}d\tau\int\limits_{ax_1-x_2+a\tau}^{ax_1+x_2-a\tau}f(\tau,\xi)\,d\xi-\frac{1}{4a^2}\int\limits_{ax_1-x_2}^{x_2-ax_1}dy_1\int\limits_{ax_1-x_2}^{ax_1+x_2}f\Big(\frac{y_2-y_1}{2a},\frac{y_2+y_1}{2}\Big)\,dy_2, \tag{5.5}$$
$x_2-ax_1<0, x_1>0, x_2>0.$

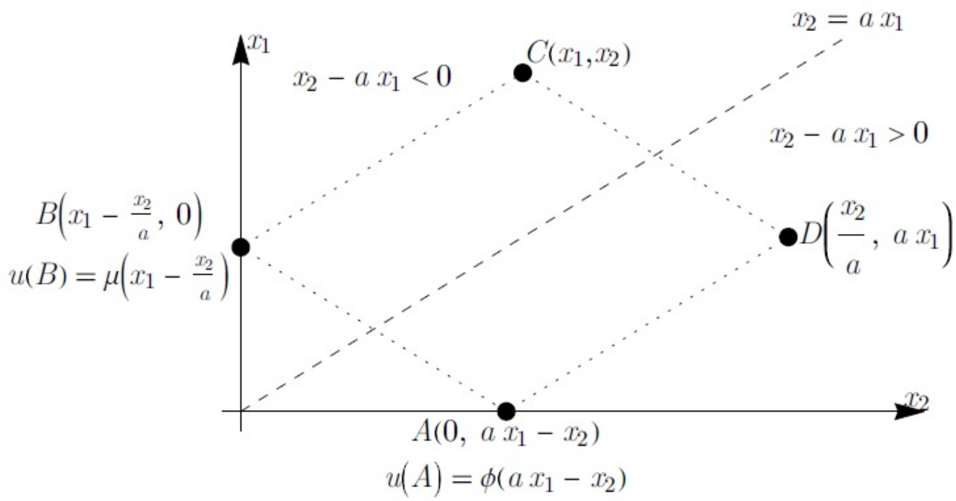

Fig. 3. To the first mixed problem (5.3).

Using representations (5.4) and (5.5), we can easily derive necessary and sufficient matching conditions $\mu(0)=\phi(0)$, $\mu'(0)=\psi(0)$ and $\mu''(0)=a^2\phi''(0)+f(0,0)$ under which the solution u of the first mixed problem (5.3) will be classical.

5.2 Nonlinear wave equation

For convenience, further in this chapter we will present equations in divergence form. Let us consider $Au(x_1,x_2)=\partial_{x_1}\partial_{x_2}u(x_1,x_2)$. Then we have $\gamma_1(x_1,x_2)=x_1$, $\gamma_2(x_1,x_2)=x_2$, $\gamma_1^{-1}(y_1,y_2)=y_1$, $\gamma_2^{-1}(y_1,y_2)=y_2$, $A\gamma_1\equiv0$, $A\gamma_2\equiv0$.

5.2.1 Darboux problem

Let us consider the second Darboux problem for a nonlinear wave equation in divergence form [11]

$$\begin{cases}\partial_{x_1}\partial_{x_2}u(\mathbf{x})+\lambda g(\mathbf{x},u(\mathbf{x}))=f(\mathbf{x}), & 0<x_1, \alpha x_1<x_2<\beta x_1,\\ u(x_1,x_2=\alpha x_1)=u(x_1,x_2=\beta x_1)=0, & x_1>0,\end{cases} \tag{5.6}$$

where $\lambda\in\mathbb{R}$, $0<\alpha<1<\beta<\infty$, $f\in C^1(\{\mathbf{x}\mid 0\leqslant x_1\wedge\alpha x_1\leqslant x_2\leqslant\beta x_1\})$, $g\in C^1(\{\mathbf{x}\mid 0\leqslant x_1\wedge\alpha x_1\leqslant x_2\leqslant\beta x_1\}\times\mathbb{R})$, $|g(x_1,x_2,z)|\leqslant L_1+L_2|z|$, $L_1\geqslant0$, $L_2\geqslant0$.

We want to obtain an expression for the classical solution u of problem (5.6) at the point $P_0(x_1,x_2)$. Let us denote by $P_1M_0P_0N_0$ the characteristic parallelogram, whose vertices N_0 and M_0 lie, respectively, on the segments $x_2=\alpha x_1$ and $x_2=\beta x_1$, that is: $N_0:=(x_1,\alpha x_1)$, $M_0:=(\beta^{-1}x_2,x_2)$,

$P_1 := (\beta^{-1}x_2, \alpha x_1)$. Since $P_1 \in \{\mathbf{x} \mid 0 < x_1 \wedge \alpha x_1 < x_2 < \beta x_1\}$, we construct analogously the characteristic parallelogram $P_2M_1P_1N_1$ whose vertices N_1 and M_1 lie, respectively, on the segments $x_2 = \alpha x_1$ and $x_2 = \beta x_1$. Continuing this process, we obtain the characteristic parallelogram $P_{i+1}M_iP_iN_i$ for which $N_i \in \{\mathbf{x} \mid x_2 = \alpha x_1\}$, $M_i \in \{\mathbf{x} \mid x_2 = \beta x_1\}$, and $N_i := \left(x_1^{(i)}, \alpha x_1^{(i)}\right)$, $M_i := \left(\beta^{-1}x_2^{(i)}, x_2\right)$, $P_{i+1} := \left(\beta^{-1}x_2^{(i)}, \alpha x_1^{(i)}\right)$ if $P_i := \left(x_1^{(i)}, x_2^{(i)}\right)$.

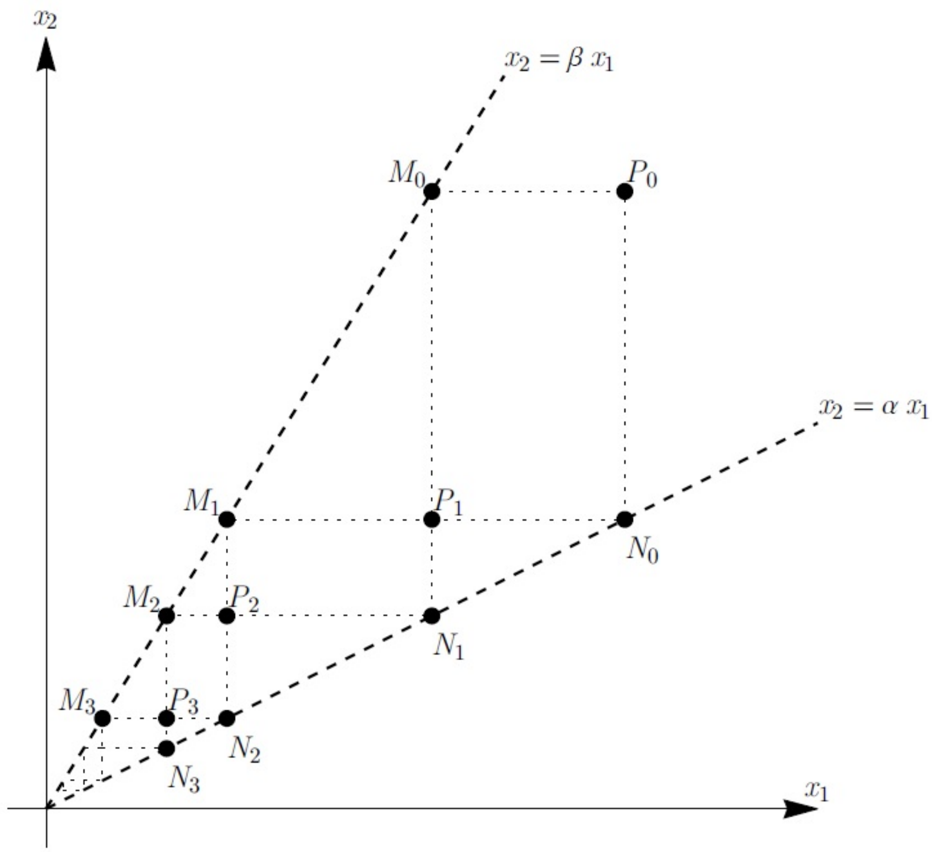

Fig. 4. To the second Darboux problem (5.6).

By virtue of (4.1) and (5.6) we have

$$\begin{aligned} u(P_i) &= u(M_i) + u(N_i) - u(P_{i+1}) + \iint\limits_{P_{i+1}M_iP_iN_i} [f(\mathbf{z}) - \lambda g(\mathbf{z}, u(\mathbf{z}))]\, d\mathbf{z} \\ &= -u(P_{i+1}) + \iint\limits_{P_{i+1}M_iP_iN_i} [f(\mathbf{z}) - \lambda g(\mathbf{z}, u(\mathbf{z}))]\, d\mathbf{z}, \quad i \in \mathbb{N} \cup \{0\}. \end{aligned}$$

Thus it follows that

$$\begin{aligned} u(x_1, x_2) = u(P_0) &= \iint\limits_{P_1M_0P_0N_0} [f(\mathbf{z}) - \lambda g(\mathbf{z}, u(\mathbf{z}))]\, d\mathbf{z} - u(P_1) \\ &= u(P_2) + \iint\limits_{P_1M_0P_0N_0} [f(\mathbf{z}) - \lambda g(\mathbf{z}, u(\mathbf{z}))]\, d\mathbf{z} - \iint\limits_{P_2M_1P_1N_1} [f(\mathbf{z}) - \lambda g(\mathbf{z}, u(\mathbf{z}))]\, d\mathbf{z} \\ &= (-1)^n u(P_n) + \sum_{i=0}^{n-1} (-1)^i \iint\limits_{P_{i+1}M_iP_iN_i} [f(\mathbf{z}) - \lambda g(\mathbf{z}, u(\mathbf{z}))]\, d\mathbf{z}. \end{aligned}$$

Clearly that $\lim_{n\to\infty} u(P_n) = u\left(\lim_{n\to\infty} P_n\right) = u(0,0) = 0$. Hence, passing to the limit, as $n \to \infty$, we obtain the following integral representation

$$u(x_1, x_2) = \sum_{i=0}^{\infty}(-1)^i \iint\limits_{P_{i+1}M_iP_iN_i} [f(\mathbf{z}) - \lambda g(\mathbf{z}, u(\mathbf{z}))]\, d\mathbf{z}. \tag{5.7}$$

The further solution of problem (5.6) is connected with the study of the solvability of equation (5.7), and it is given in the work [11]. And it turns out that under the conditions specified in the formulation of problem (5.6), it has a unique classical solution. But we still notice that in the linear case (i.e., when $\lambda = 0$), formula (5.7) transforms into

$$u(x_1, x_2) = \sum_{i=0}^{\infty}(-1)^i \iint\limits_{P_{i+1}M_iP_iN_i} f(\mathbf{z})d\mathbf{z}, \tag{5.8}$$

The series in the right-hand side of equality (5.8) is uniformly and absolutely convergent [11]. So, in the linear case, there is a solution u of (5.6) written in the explicit analytic form (5.8).

5.3 Linear second-order hyperbolic equation

As in the prevoius subsection, we consider $Au(x_1, x_2) = \partial_{x_1}\partial_{x_2}u(x_1, x_2)$. Then we have $\gamma_1(x_1, x_2) = x_1$, $\gamma_2(x_1, x_2) = x_2$, $\gamma_1^{-1}(y_1, y_2) = y_1$, $\gamma_2^{-1}(y_1, y_2) = y_2$, $A\gamma_1 \equiv 0$, $A\gamma_2 \equiv 0$.

5.3.1 Goursat problem

Let us consider the Goursat problem for a linear second-order hyperbolic equation [12]

$$\begin{cases} \partial_{x_1}\partial_{x_2}u(\mathbf{x}) + a(\mathbf{x})\partial_{x_1}u(\mathbf{x}) + b(\mathbf{x})\partial_{x_2}u(\mathbf{x}) + c(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}), \quad x_1^{(0)} < x_1, x_2^{(0)} < x_2, \\ u(x_1 = x_1^{(0)}, x_2) = \phi(x_2), \quad x_2 > x_2^{(0)}, \\ u(x_1, x_2 = x_2^{(0)}) = \psi(x_1), \quad x_1 > x_1^{(0)}, \end{cases} \tag{5.9}$$

where $f \in C(\{\mathbf{x} \mid x_1^{(0)} \leqslant x_1 \wedge x_2^{(0)} \leqslant x_2\})$, $\phi \in C^2([x_2^{(0)}, \infty))$, $\psi \in C^1([x_1^{(0)}, \infty))$ and $\phi(x_2^{(0)}) = \psi(x_1^{(0)})$. We can write the classical solution of (5.9) using formula (4.1). If we select $C(x_1, x_2)$, $B\left(x_1^{(0)}, x_2\right)$, $D\left(x_1, x_2^{(0)}\right)$, $A\left(x_1^{(0)}, x_2^{(0)}\right)$ and apply (4.1), then we obtain

$$\begin{aligned} u(\mathbf{x}) = u(C) &= \phi(x_2) + \psi(x_1) - \psi(x_2^{(0)}) \\ &+ \int_{x_1^{(0)}}^{x_1} dy_1 \int_{x_2^{(0)}}^{x_2} [f(\mathbf{y}) - a(\mathbf{y})\partial_{x_1}u(\mathbf{y}) - b(\mathbf{y})\partial_{x_2}u(\mathbf{y}) - c(\mathbf{y})u(\mathbf{y})]dy_2. \end{aligned} \tag{5.10}$$

A representation of the solution in the form of integro-differential equation (5.10) is obtained. Under the conditions specified in the formulation of problem (5.9), equation (5.10) will be solvable [12] and the function u will have the required smoothness. This proves the solvability of problem (5.9).

6 Conclusion

In the paper, the property of the characteristic parallelogram for the wave equation is generalized to the case of a semilinear hyperbolic equation of the second order. This identity connects not only the values of points at the vertices of the parallelogram but also the continuum of function values on the parallelogram, in contrast to the linear cases with constant coefficients considered earlier. It is shown how the obtained results, combined with other methods, can be used to solve various mixed problems.

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Received: 22.04.2022