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DYNAMICS OF RELAY SYSTEMS WITH HYSTERESIS AND HARMONIC PERTURBATION

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Key words: multidimensional system of ordinary differential equations; relay hysteresis; harmonic perturbation; decomposition; parametric matrix; subsystems; Jordan block; asymptotically stable periodic solution.

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Abstract. We consider a system of ordinary differential equations with a relay hysteresis and a harmonic perturbation. We propose an approach that allows one to decompose an n-dimensional system into one- and two-dimensional subsystems. The approach is illustrated by a numerical example for the system of dimension 3. As a result of the decomposition, a two-dimensional subsystem with non-trivial Jordan block in right-hand side is studied. For this subsystem we prove a theorem on the existence and uniqueness of an asymptotically stable solution with a period being multiple to period of the perturbation. Moreover, we show how to obtain this solution by tuning the parameters defining the relay. We also provide a supporting example in this regard.

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1 Introduction

It is well known in oscillation theory [1] that many results obtained for linear systems and nonlinear systems with continuous right-hand sides are impossible to use for systems with the nonlinearities being nonlinearized. Such nonlinearities are called "essential" ones (see [23]). In practice, by essentially nonlinear systems including relay systems with hysteresis, one describes numerous automatic control devices [32], in particular, devices with non-ideal relays, which are installed, for instance, on water crafts [7], [30]. Moreover, the mathematical models of these devices are multidimensional. Influence of hysteresis becomes actually important when the devices are utilized in high-precision engineering systems [33]. The readers are referred to monographs [4], [6], [24], [28], and [34] for general information about systems with hysteresis and their applications.

It is clear that, for specific parameters, one can first study these mathematical models by numerical methods, using a powerful computing technique and then perform simulations to support numerical results. At the same time, analytical results provide a basis for examining the multidimensional and essentially nonlinear automatic control systems. Even if the results are obtained for bounded domains in the system parameter space, they might be considered as a scientific background in numerical experiments. It is beyond any doubt that both theoretical and numerical investigations of the models with hysteresis provide more adequate results for applications.

Models with hysteresis were already surveyed in a number of works (see, e.g., [2], [5], [8], [17], [26], [27], [31], and [35]–[37]). From the latest papers, we draw attention to [13], [19], [20], [29], [40], and [41]. Notice that classical methods including methods of fitting, fixed points, and point mappings are still actively employed to investigate piecewise integrable systems, in particular, relay systems [9], [12], [21], [38], and [39].

Definitely, the use of analytical methods in research of n-dimensional nonlinear systems ($n > 2$) yields challenges. As is known, classical methods are usually based on a local approach to the research of system phase space. But even in this case, some numerical calculations are to be carried out. On the other hand, there is an opportunity to study the phase portraits of lower-dimensional systems for $n \leq 2$ in large, non-locally (see, e.g., [18]).

We consider a multidimensional system of ordinary differential equations with a non-ideal relay and a harmonic pertubation. The present study develops the results of the authors obtained, in particular, in [17], [18], [20], [21], and [37]–[39].

The aim of the paper is to propose a new approach for the research of the system under consideration by analytical methods. The approach involves a decomposition of the system into subsystems and the study of these subsystems. The decomposition allows to overcome difficulties related to higher dimensions and investigate multidimensional systems analytically, precisely and so fully as we can do it for systems of lower dimensions using phase planes [18].

The main idea of the approach is the following. For a system of dimension n, we select some domains in its parameter space. Then, in these domains, we reduce the system to subsystems of dimensions 1 and 2, using a nonsingular linear transformation. These subsystems are connected in such a way that one can integrate them consistently, one behind another, if consider some of them as inhomogeneous ones (see [22]), and therefore study them by well-known methods. As a result of the reduction, the system parameter space is decomposed into the direct sum of subspaces, and a number of dynamic behaviour types in these subspaces is put into a dynamic behaviour in the space. Thus, by investigating the subsystems, it is possible to obtain a certain knowledge not only about the dynamics of the system but about the structure of its parameter space, since the points in this space are associated with the various topological phase portraits.

Different approaches with the reduction of matrices to diagonal or Jordan forms are proposed in a number of publications, in particular in [14] for a delayed differential system. For systems with constant matrices in the linear part and relay hysteresis, nonsingular transformations that reduce matrices to the same forms were applied in the papers of authors as well (see, e.g., [38]). The novelty of the present study is the usage of a transformation matrix as the product of two matrices, one of which is parametric. Parameters in the transformation matrix give an opportunity to choose such system parameters to investigate multidimensional relay systems up to the end analytically. In contrast to the earlier works of authors, in this paper the form of the feedback vector depends on the eigenvalues of the system matrix as well as the form of the parametric matrix does.

The decomposition of the system into subsystems is presented in Section 3 and illustrated by Example 1 in Section 6. In Sections 4 and 5, we consider one of subsystems, namely, a system of dimension 2 with non-trivial Jordan block, called *basic system*. Section 5 is concerned with the existence and stability of periodic solutions to the basic system. We prove a theorem on the existence and uniqueness of an asymptotically stable periodic solution (Theorem 5.1). Example 2 in Section 6 shows that periodic oscillations take place in the dynamics of relay systems owing to the periodic perturbation. Interesting results concerning relay systems and perturbations, in particular influence of perturbations, can be found in [9]–[11]. These studies are dedicated to a new type of oscillation, the so-called unpredictable. Their results manifest that the main source for the unpredictable controllable behaviour in the dynamics is the relay perturbations.

2 Statement of the problem

We study a complicated automatic control system the dynamics of which can be governed by the following n-dimensional system of ordinary differential equations

$$\dot{\overline{X}} = A_0\overline{X} + B_0(F(\sigma) + \psi(t)), \quad \sigma(t) = C_0\overline{X}(t). \tag{2.1}$$

Here A_0, B_0, and C_0 are $(n \times n)$, $(n \times m)$, and $(m \times n)$ matrices, respectively $(n \geq m)$; $\dot{\overline{X}}(t)$, $\overline{X}(t)$, and $\sigma(t)$ are $(n \times 1)$, $(n \times 1)$, and $(m \times 1)$ vectors. The nonlinear part of the system is described by the $(m \times 1)$ vector $F(\sigma)$; $\psi(t)$ is a $(m \times 1)$ vector of perturbations.

We also study the two-dimensional system, named *basic system*, of the form

$$\dot{X} = AX + B(F(\sigma) + \psi(t)). \tag{2.2}$$

Here X is the vector of system state such that $X = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2$, where the symbol T means the transposition operation, $A = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$, where $\lambda \in \mathbb{R}\backslash\{0\}$; $B = (1, 0)^{\mathrm{T}}$ or $B = (-1, 0)^{\mathrm{T}}$; $F(\sigma)$ is a scalar function and $\sigma = \gamma_1 x_1 + \gamma_2 x_2$, where γ_1, γ_2 are real constants; $\Gamma = (\gamma_1, \gamma_2)^{\mathrm{T}} \in \mathbb{R}^2$ is a nonzero vector; the scalar function $\psi(t)$ stands for a perturbation.

We consider systems (2.1) and (2.2) as models of automatic control systems in which $F(\sigma)$ is a relay-type control, and Γ is a feedback vector. We define $F(\sigma)$ as follows [20]: $F(\sigma) = m_1$ if $\sigma < l_2$, and $F(\sigma) = m_2$ if $\sigma > l_1$, where $m_1, m_2, l_1, l_2 \in \mathbb{R}$, $m_1 < m_2$, $l_1 < l_2$. If $\sigma(0) \leq l_1$ or $\sigma(0) \geq l_2$, then $F(\sigma(t))$ is single-valued. If $\sigma(0) \in (l_1, l_2)$, then $F(\sigma(t))$ is two-valued, therefore we need to specify either $F(\sigma(0)) = m_1$ or $F(\sigma(0)) = m_2$ and follow the positive spin in the plane (σ, F), namely, the value of $F(\sigma(t))$ is kept constant for all $t > 0$ until $\sigma(t)$ crosses the value l_2 from below or the value l_1 from above, respectively. At these instants (when $\sigma(t) = l_i$, $i = 1, 2$) the value of $F(\sigma(t))$ is changed to m_1 or m_2, respectively. Thus, $F(\sigma)$ describes the relay hysteresis with counterclockwise orientation in the plane (σ, F), its figure one can see, for example, in [26].

Notice that the research of systems with relay feedback is quite a task (see [3], [15], and [42]).

The general problem is to investigate system (2.1) completely analytically. The aim of this paper is to show how this problem can be studied by a decomposition of system (2.1) into subsystems of lower dimensions one of which is system (2.2). Also, we pose the problem about sufficient conditions under which system (2.2) has a unique periodic solution. To solve this problem we study system (2.2) in the particular case when $\gamma_2 = 0$, $\psi(t) = F_0 + k\sin(\omega t + \varphi)$, where F_0, $\varphi \in \mathbb{R}$ and k, $\omega \in \mathbb{R}_+$.

On the one hand, if $x_1 = x$, $x_2 = \dot{x}$, then system (2.2) can result from a transformation of the second-order equation in x. On the other hand, system (2.2) is a subsystem of system (2.1) for $n > 2$ after its decomposition. In view of that, we call system (2.2) *basic system*.

3 Decomposition of the n-dimensional system

System (2.1) with arbitrary $F(\sigma)$ and $\psi(t)$ cannot be fully investigated by qualitative methods of the theory of differential equations even for $n = 2$ (see [1]). That is why researchers often use a linear transformation of system (2.1) that reduces the matrix A_0 to diagonal or Jordan form.

It is known [25] that a nonsingular matrix of the transformation is not uniquely determined. Therefore, we look for the transformation $\overline{X}(t) = MX(t)$ with the matrix M being in the form $M = SQ$. Here Q is a nonsingular parametric matrix such that $M^{-1}A_0M = A_j$, where A_j has Jordan form (see [16]). Besides, S and Q are nonsingular matrices such that $S^{-1}A_0S = A_j$, $A_j = Q^{-1}A_jQ$.

Next, we show how to find the matrices Q and Q^{-1} if

$$A_j = \begin{pmatrix} \Lambda_1 & 0 & \dots & 0 \\ 0 & \Lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Lambda_k \end{pmatrix}.$$

Here Λ_i are the r_i-order block diagonal matrix with the Jordan block K_{ij} corresponding to the eigenvalue λ_i $(i = \overline{1,k})$, $\sum\limits_{i=1}^{k} r_i = n$ (r_i is the multiplicity of λ_i). Then

$$Q = \begin{pmatrix} Q_1 & 0 & \dots & 0 \\ 0 & Q_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & Q_k \end{pmatrix},$$

where Q_i are also block diagonal matrix with the block Q_{ij} having the same dimension as K_{ij} has. To establish the form of Q_{ij}, we consider K_{ij} of order q corresponding to λ_i, i.e.

$$K_{ij} = \begin{pmatrix} \lambda_i & 0 & \dots & 0 & 0 \\ 1 & \lambda_i & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \lambda_i \end{pmatrix}.$$

Note that K_{ij} can be written down as follows: $K_{ij} = \lambda_i I + H$, where I is the identity matrix, the matrix H is of the form

$$H = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

The block Q_{ij} has the form $Q_{ij} = \alpha_0 I + \alpha_1 H + \alpha_2 H^2 + \ldots + \alpha_{q-1} H^{q-1}$, where α_0, α_1, $\alpha_2, \ldots,$ α_{q-1} are nonzero real numbers. The blocks K_{ij} and Q_{ij} are commutative. Therefore Λ_i and Q_i are commutative and $QA_j = A_jQ$.

To find out Q^{-1}, it suffices to find Q_{ij}^{-1}. Note that

$$Q_{ij}^{-1} = \beta_0 I + \beta_1 H + \beta_2 H^2 + \ldots + \beta_{q-1} H^{q-1},$$

where β_0, β_1, β_2, $\ldots$, β_{q-1} are the solution to the system that follows from the equality $Q_{ij}Q_{ij}^{-1} = I$ taking into account that H^q is a zero matrix.

Thus, after the transformation, system (2.1) acquires the form

$$\dot{X}(t) = A_j X(t) + \overline{B}_M(F(\sigma) + \psi(t)), \quad \sigma(t) = \overline{C}_M X(t), \tag{3.1}$$

where $\overline{B}_M = Q^{-1}S^{-1}B_0$, $\overline{C}_M = C_0 SQ$.

If we consider the elements of B_0 and C_0 as the parameters of system (2.1) for tuning, then Q in the relation $M = SQ$ allows one to simplify and expand the choice of these elements for system (2.1) to be investigated up to the end analytically.

4 Study of the two-dimensional subsystem

Consider system (2.2) with $B = (-1, 0)^{\mathrm{T}}$, namely,

$$\begin{cases} \dot{x}_1 = \lambda x_1 - (F(\sigma) + \psi(t)), \\ \dot{x}_2 = x_1 + \lambda x_2, \end{cases} \tag{4.1}$$

where $\sigma = \gamma_1 x_1$.

Remark 1. If n is even and $n > 3$, then the subsystems in the form (4.1) can also be obtained by the transformation of the initial system with the result that

$$A_j = \begin{pmatrix} \lambda_1 & 0 & & & \\ 1 & \lambda_1 & & & \\ & & \ddots & & \\ & & & \lambda_i & 0 \\ & & & 1 & \lambda_i \end{pmatrix},$$

where $i = n/2$.

Next we examine system (4.1) analytically for the purpose of obtaining the conditions on the system parameters such that there exist solutions with periods being other than period of the perturbation and studying the properties of these solutions.

To analyse the solutions to system (4.1) and its phase space, we use Cauchy's form for the solution representation. Thus, for the first equation of (4.1), we have

$$x_1(t) = e^{\lambda t} x_1^0 - \int\limits_0^t e^{-\lambda(\tau - t)}(F(\sigma) + \psi(\tau))d\tau,$$

where $x_1^0 = x_1(0)$.

Multiplying the latter equation by γ_1 and taking into account $F(\sigma) = m_i$ $(i = 1, 2)$, $\psi(\tau) = F_0 + k\sin(\omega\tau + \varphi)$, we come to the expression

$$\sigma(t) = \gamma_1 x_1(t) = e^{\lambda t}\gamma_1 x_1^0 - \gamma_1 \int\limits_0^t e^{-\lambda(\tau - t)}(m_i + F_0 + k\sin(\omega\tau + \varphi))d\tau.$$

Note that

$$\int\limits_0^t e^{-\lambda(\tau - t)}(m_i + F_0 + k\sin(\omega\tau + \varphi))d\tau$$

$$= e^{\lambda t}\left((m_i + F_0)\left(-\frac{e^{-\lambda\tau}}{\lambda}\right) - \frac{\lambda\sin(\omega\tau + \varphi) + \omega\cos(\omega\tau + \varphi)}{\lambda^2 + \omega^2} k e^{-\lambda\tau}\right)\Bigg|_0^t.$$

After integrating, we have

$$\sigma(t) = \left(\sigma_0 - \frac{\gamma_1 \overline{m}_i}{\lambda} - \frac{\gamma_1 k}{\sqrt{\lambda^2 + \omega^2}}\sin(\varphi + \delta)\right)e^{\lambda t} + \frac{\gamma_1 \overline{m}_i}{\lambda} + \frac{\gamma_1 k}{\sqrt{\lambda^2 + \omega^2}}\sin(\omega t + \varphi + \delta).$$

Here $\sigma_0 = \sigma(0)$, $\overline{m}_i = m_i + F_0$ $(i = 1, 2)$, and $\delta = \arctan(\omega/\lambda) + \pi q$ ($q = 0$ if $\lambda > 0$ and $q = 1$ if $\lambda < 0$). Consequently, $x_1(t) = \sigma(t)/\gamma_1$ and $x_1^0 = \sigma_0/\gamma_1$. Integrating the second equation in (4.1) provided that x_1 is known, we have

$$x_2(t) = x_2^0 e^{\lambda t} + \left(\frac{\sigma_0}{\gamma_1} - \frac{\overline{m}_i}{\lambda} - \frac{k}{\sqrt{\lambda^2 + \omega^2}}\sin(\varphi + \delta)\right) t e^{\lambda t}$$

$$-\frac{\overline{m}_i}{\lambda^2}(1-e^{\lambda t})+\frac{ke^{\lambda t}}{\lambda^2+\omega^2}\sin(\varphi+2\delta)+\frac{1}{\lambda^2+\omega^2}\sin(\omega t+\varphi+2\delta), \tag{4.2}$$

where $x_2^0 = x_2(0)$.

Put

$$\Phi(\sigma_0,\overline{m}_i,t)=\left(\frac{\sigma_0}{\gamma_1}-\frac{\overline{m}_i}{\lambda}-\frac{k}{\sqrt{\lambda^2+\omega^2}}\sin(\varphi+\delta)\right)te^{\lambda t}$$

$$-\frac{\overline{m}_i}{\lambda^2}(1-e^{\lambda t})+\frac{ke^{\lambda t}}{\lambda^2+\omega^2}\sin(\varphi+2\delta)+\frac{1}{\lambda^2+\omega^2}\sin(\omega t+\varphi+2\delta). \tag{4.3}$$

Let τ_1 be the time for moving of representative point from the switching line H_2 ($\gamma_1 x_1 = l_2$) to H_1 ($\gamma_1 x_1 = l_1$) and τ_2 transition time from H_1 to H_2. Further we seek for τ_1, τ_2, using the following initial and boundary conditions for $\sigma(t)$:
if $t \in [0,\tau_1]$, then $\sigma_0 = l_2$, $\overline{m}_i = \overline{m}_2$, $\sigma(\tau_1) = l_1$;
if $t \in [0,\tau_2]$, then $\sigma_0 = l_1$, $\overline{m}_i = \overline{m}_1$, $\sigma(\tau_2) = l_2$.

Hence, we come to the transcendental equations with respect to τ_1, τ_2

$$l_1-\frac{\gamma_1\overline{m}_2}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\omega\tau_1+\varphi+\delta)$$

$$=\left(l_2-\frac{\gamma_1\overline{m}_2}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\varphi+\delta)\right)e^{\lambda\tau_1}, \tag{4.4}$$

$$l_2-\frac{\gamma_1\overline{m}_1}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\omega\tau_2+\varphi+\delta)$$

$$=\left(l_1-\frac{\gamma_1\overline{m}_1}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\varphi+\delta)\right)e^{\lambda\tau_2}. \tag{4.5}$$

Now we write out the sufficient conditions for the existence of positive roots τ_1, τ_2. Equations (4.4), (4.5) have solutions $\tau_1 > 0$, $\tau_2 > 0$ if the following inequalities hold:

$$\lambda<0,\quad \gamma_1>0,\quad \overline{m}_1<0,\quad \overline{m}_2>0,\quad l_1<0,\quad l_2>0,$$

$$l_2-\frac{\gamma_1\overline{m}_2}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\varphi+\delta)>0,\quad l_1-\frac{\gamma_1\overline{m}_2}{\lambda}>-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}},$$

$$l_1-\frac{\gamma_1\overline{m}_1}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\varphi+\delta)<0,\quad l_2-\frac{\gamma_1\overline{m}_1}{\lambda}<\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}. \tag{4.6}$$

Equations (4.4), (4.5) have also solutions $\tau_1 > 0$, $\tau_2 > 0$ under the conditions below

$$\lambda<0,\quad \gamma_1<0,\quad \overline{m}_1<0,\quad \overline{m}_2>0,\quad l_1<0,\quad l_2>0,$$

$$l_2-\frac{\gamma_1\overline{m}_2}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\varphi+\delta)>0,\quad l_1-\frac{\gamma_1\overline{m}_2}{\lambda}>\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}},$$

$$l_1-\frac{\gamma_1\overline{m}_1}{\lambda}-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}\sin(\varphi+\delta)<0,\quad l_2-\frac{\gamma_1\overline{m}_1}{\lambda}<-\frac{\gamma_1 k}{\sqrt{\lambda^2+\omega^2}}. \tag{4.7}$$

5 Existence of asymptotically stable periodic solutions

The existence and uniqueness of a solution are established by the the following theorem.

Theorem 5.1. *Let inequalities* (4.6) *or* (4.7) *be true, equation* (4.4) *have the least solution (or a unique one)* τ_1 *such that* $\tau_1 = 2\pi\nu_1/\omega$, $\nu_1 \in \mathbb{N}$, *and equation* (4.5) *the least solution (or a unique one)* τ_2 *such that* $\tau_2 = 2\pi\nu_2/\omega$, $\nu_2 \in \mathbb{N}$. *Then system* (4.1) *has a unique asymptotically stable* T_f*-periodic solution with* $T_f = \tau_1 + \tau_2$.

Proof. If inequalities (4.6) or (4.7) are satisfied, then equations (4.4), (4.5) have solutions $\tau_1 > 0$ and $\tau_2 > 0$, respectively. But equation (4.4) as well as equation (4.5) can have more than one solution. By τ_i, $i = 1, 2$, we denote transition time from one switching line to the other one. From this, it follows that τ_i is the least solution (or a unique one).

By assumption, $\gamma_2 = 0$. That is why the switching lines H_1 and H_2 are orthogonal to the Ox_1-axis on the plane (x_1Ox_2). By construction of equations (4.4), (4.5), transition time τ_i depends on $x_1(0)$ but does not depend on $x_2(0)$. This implies that the value τ_i is independent of the initial state of representative point on H_i.

Next, we set the initial and boundary conditions for $x_2(t)$ in the case when representative point goes from the point $(l_2/\gamma_1, x_2^0)^{\mathrm{T}} \in H_2$ to the point $(l_1/\gamma_1, x_2^1)^{\mathrm{T}} \in H_1$

$$x_2^0 = x_2(0),\ \overline{m}_i = \overline{m}_2,\ \sigma_0 = l_2,\ x_2^1 = x_2(\tau_1)$$

and from the point $(l_1/\gamma_1, x_2^1)^{\mathrm{T}} \in H_1$ to the point $(l_2/\gamma_1, x_2^2)^{\mathrm{T}} \in H_2$

$$x_2^1 = x_2(0),\ \overline{m}_i = \overline{m}_1,\ \sigma_0 = l_1,\ x_2^2 = x_2(\tau_2).$$

Using (4.2), we obtain

$$x_2(\tau_1) = x_2^0 e^{\lambda\tau_1} + \Phi(l_2, \overline{m}_2, \tau_1),$$
$$x_2(\tau_2) = x_2^1 e^{\lambda\tau_2} + \Phi(l_1, \overline{m}_1, \tau_2),$$

where Φ is defined by (4.3).

Now we write out the point map of the line H_2 into itself in the form

$$x_1^2 = l_2/\gamma_1,\ x_2^2 = x_2^0 e^{\lambda(\tau_1+\tau_2)} + \Theta(\tau_1, \tau_2), \tag{5.1}$$

where $\Theta(\tau_1, \tau_2) = e^{\lambda\tau_2}\Phi(l_2, \overline{m}_2, \tau_1) + \Phi(l_1, \overline{m}_1, \tau_2)$.

By (5.1), we mean the point map such that representative point goes from any initial point $(l_2/\gamma_1, x_2^0)^{\mathrm{T}} \in H_2$ to any point $(l_1/\gamma_1, x_2^1)^{\mathrm{T}} \in H_1$ in τ_1 and from the latter point to the point $(l_2/\gamma_1, x_2^2)^{\mathrm{T}} \in H_2$ in τ_2. Therefore, we consider a set of the points $(l_2/\gamma_1, x_2^0)^{\mathrm{T}} \in H_2$ that are mapped into the line H_2 in T_f, where $T_f = \tau_1 + \tau_2$, by virtue of the solution to system (4.1). Note that (5.1) gives the first return map to the layer between the two hysteretic regimes. Clearly, for obtained τ_1 and τ_2, the value $\Theta(\tau_1, \tau_2)$ is constant.

Consider the second return map and so on. Since $\tau_i = 2\pi\nu_i/\omega$, we have $\sin(\eta) = \sin(\omega(\tau_1+\tau_2)+\eta)$ and $\sin(\omega\tau_i+\eta) = \sin(\omega(\tau_i+\tau_1+\tau_2)+\eta)$, where $\eta = \varphi+\delta$ or $\eta = \varphi+2\delta$. Hence, equations (4.4), (4.5) are kept their forms. It means that τ_i and hence $\Theta(\tau_1, \tau_2)$ are kept constant. Therefore relations (5.1) are met for the following return maps. Under conditions above, there exists a fixed point $(x_1^{\text{fix}}, x_2^{\text{fix}})^{\mathrm{T}}$ of the map defined by (5.1) such that $(x_1^{\text{fix}}, x_2^{\text{fix}})^{\mathrm{T}} = (l_2/\gamma_1, x_2^2)^{\mathrm{T}} = (l_2/\gamma_1, x_2^0)^{\mathrm{T}}$. From (5.1), we obtain $x_2^{\text{fix}} = \Theta(\tau_1, \tau_2)/(1 - e^{\lambda(\tau_1+\tau_2)})$. Moreover, if $\Theta > 0$, then x_2^{fix} is positive, but if $\Theta < 0$, then it is negative.

According to (4.6) or (4.7), we have $\lambda < 0$. Then the fixed point is asymptotically stable. Indeed, we have $x_2(t) = x_2^0 e^{\lambda t} + \Theta(\tau_1, \tau_2)$ for $t \geq 0$. For any $\delta > 0$, take any point $\tilde{x}_2^0$ such that $0 < |x_2^0 - \tilde{x}_2^0| < \delta$. Then $\tilde{x}_2(t) = \tilde{x}_2^0 e^{\lambda t} + \Theta(\tau_1, \tau_2)$ and hence

$$|x_2(t) - \tilde{x}_2(t)| = \left|x_2^0 - \tilde{x}_2^0\right| e^{\lambda t} < \delta e^{\lambda t} \to 0 \ \text{ as } \ t \to \infty.$$

The fixed point corresponds to a closed trajectory on the plane (x_1Ox_2). Thus, the representative point of the solution attains the switching line H_i every time in the same transition time τ_i and returns to the same fixed point in phase space along the same trajectory. This means that system (4.1) has an asymptotically stable T_f-periodic solution with $T_f = \tau_1 + \tau_2$. In view of the compressed map, this solution is unique. □

Theorem 5.1 gives the condition $T_f = nT$, where $n = \nu_1+\nu_2$, $T = 2\pi/\omega$, under which system (4.1) has a T_f-periodic solution.

Now we show consideration of how to obtain the parameters of $F(\sigma)$ satisfying (4.4), (4.5) such that system (4.1) has a periodic solution with period T_f being multiple to period T of the perturbation.

Let the parameters F_0, k, ω, and φ of the function $\psi(t)$ be given. Let a system of automatic control possess a periodic mode, for example, with $\tau_i = 2\pi\nu_i/\omega$ for some $\nu_i \in \mathbb{N}$. Then from (4.5) we obtain

$$l_2 - l_1 e^{2\pi\nu_2\lambda/\omega} = \gamma_1 \left(\frac{m_1 + F_0}{\lambda} + \frac{k\sin(\varphi+\delta)}{\sqrt{\lambda^2+\omega^2}} \right) \left(1 - e^{2\pi\nu_2\lambda/\omega}\right). \tag{5.2}$$

Relation (5.2) associates the parameters l_1, l_2, m_1 and can be used for their tuning. From (4.4) we express

$$m_2 = \frac{\lambda\left(l_1 - l_2 e^{2\pi\nu_1\lambda/\omega}\right)}{\gamma_1\left(1 - e^{2\pi\nu_1\lambda/\omega}\right)} - \frac{\lambda k\sin(\varphi+\delta)}{\sqrt{\lambda^2+\omega^2}} - F_0. \tag{5.3}$$

The parameter m_2 is defined by (5.3) under the proper choice of the parameters l_1, m_1, and l_2 according to (5.2) for some ν_i determining τ_i.

Corollary. *Let the characteristic equation of system* (4.1) *have the root $\lambda < 0$ corresponding to Jordan block, F_0, k, ω, and φ be given, $\tau_i = 2\pi\nu_i/\omega$, $i = 1, 2$, $\nu_i \in \mathbb{N}$. Also, let* (5.2), (5.3) *defining the parameters γ_1, l_i, m_i, $i = 1, 2$, of the function $F(\sigma)$ be fulfilled for some ν_1, ν_2 such that $\tau_1 + \tau_2 = nT$, $n \in \mathbb{N}$, $T = 2\pi/\omega$. Besides, let these parameters satisfy* (4.6) *or* (4.7). *Then system* (4.1) *has a unique asymptotically stable T_f-periodic solution with $T_f = nT$, $n \geq 2$, provided that τ_1 is the least solution (or a unique one) of equation* (4.4) *and τ_2 is the least solution (or a unique one) of equation* (4.5) *for chosen parameters.*

Stability or unstability of periodic solutions depends on the type of the iterative process that is defined by formula (5.1) containing the multiplier $e^{\lambda(\tau_1+\tau_2)}$ with $\lambda < 0$ or $\lambda > 0$.

Remark 2. If $\lambda = 0$ in case of the considered Jordan block, then it is necessary to explore the system

$$\begin{cases} \dot{x}_1 = F(\sigma) + \psi(t), \\ \dot{x}_2 = x_1 + F(\sigma) + \psi(t), \end{cases}$$

which can be successfully integrated when $\gamma_1 \neq 0$, $\gamma_2 = 0$.

6 Examples

To illustrate the decomposition approach given in Section 3, we provide the following example for system (2.1) with B_0 being a (3×1) matrix, C_0 being a (1×3) matrix, and $F(\sigma)$, $\psi(t)$ being scalar.

Example 1. Consider the system with the parameters

$$\begin{cases} \dot{\overline{x}}_1 = -6\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 + b_{11}(F(\sigma)+\psi(t)), \\ \dot{\overline{x}}_2 = -2\overline{x}_1 - 2\overline{x}_2 + b_{21}(F(\sigma)+\psi(t)), \\ \dot{\overline{x}}_3 = -2\overline{x}_3 + b_{31}(F(\sigma)+\psi(t)), \end{cases}$$

where $\sigma = c_{11}\overline{x}_1 + c_{12}\overline{x}_2 + c_{13}\overline{x}_3$. Here $A_0 = \begin{pmatrix} -6 & 2 & 2 \\ -2 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. The eigenvalues of A_0 are $\lambda_{1,2} = -4$, $\lambda_3 = -2$. The excess of the matrix $(-4I - A_0)$ is equal to 1, i.e. the (2×2) Jordan block corresponds to the eigenvalues $\lambda_{1,2} = -4$. The Jordan block A_j has the form $A_j = \begin{pmatrix} -4 & 0 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. There exists a nonsingular matrix S such that $SA_j = A_0 S$. Next we point out one of the possible matrices S meeting this condition. Let $S = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, then $S^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -0.5 & -0.5 \\ 0 & 0 & 1 \end{pmatrix}$. Now we need to obtain the matrices Q and Q^{-1}. The matrix A_j has the block form $A_j = \begin{pmatrix} A_1 & 0 \\ 0 & -2 \end{pmatrix}$, where A_1 is a (2×2) matrix. The matrix Q has the same form as A_j has, i.e. $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$, where Q_1 is a (2×2) matrix, Q_2 is a number. At this, $Q_1 = \alpha_0 I + \alpha_1 H$, $Q_2 = \alpha_2$, where α_0, α_1, α_2 are the real parameters such that $\det Q \neq 0$, $H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Thus, $Q = \begin{pmatrix} \alpha_0 & 0 & 0 \\ \alpha_1 & \alpha_0 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}$ and Q is commutative with A_j. Since $\det Q \neq 0$, we set $\alpha_0 \neq 0$, $\alpha_2 \neq 0$. In addition, we put $\alpha_1 \neq 0$. Then $Q^{-1} = \begin{pmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{pmatrix}$, where $Q_1^{-1} = \beta_0 I + \beta_1 H = \begin{pmatrix} \beta_0 & 0 \\ \beta_1 & \beta_0 \end{pmatrix}$, $Q_2^{-1} = \alpha_2^{-1}$. From the relation $Q_1 Q_1^{-1} = I$, we obtain the parameters β_0, β_1 such that $\beta_0 = \alpha_0^{-1}$, $\beta_1 = -\alpha_1/\alpha_0^2$. Thus, the matrices Q and Q^{-1} are found.

Further we use the transformation with the matrix in the form $M = SQ$. Put $B_M = S^{-1}B_0$, where $B_0 = (b_{11} \quad b_{21} \quad b_{31})^{\mathrm{T}}$. Then $B_M = \begin{pmatrix} -b_{11} + b_{21} + b_{31} \\ b_{11} - 0.5b_{21} - 0.5b_{31} \\ b_{31} \end{pmatrix}$ and

$$\overline{B}_M = Q^{-1}S^{-1}B_0 = Q^{-1}B_M = \begin{pmatrix} \frac{1}{\alpha_0}(-b_{11} + b_{21} + b_{31}) \\ \frac{\alpha_1+\alpha_0}{\alpha_0^2}b_{11} - \frac{2\alpha_1+\alpha_0}{2\alpha_0^2}b_{21} - \frac{2\alpha_1+\alpha_0}{2\alpha_0^2}b_{31} \\ \frac{b_{31}}{\alpha_2} \end{pmatrix} = \begin{pmatrix} b_{11}^M \\ b_{21}^M \\ b_{31}^M \end{pmatrix}.$$

Put $\overline{C}_M = C_0SQ = C_MQ$, where $C_0 = (c_{11} \quad c_{12} \quad c_{13})$. Subsequently,

$$C_M = (c_{11} + 2c_{12} \quad 2c_{11} + 2c_{12} \quad -c_{12} + c_{13}),$$

$$\overline{C}_M = ((\alpha_0 + 2\alpha_1)c_{11} + 2(\alpha_0 + \alpha_1)c_{12} \quad \alpha_0(2c_{11} + 2c_{12}) \quad \alpha_2(-c_{12} + c_{13})) = (c_{11}^M \quad c_{12}^M \quad c_{13}^M).$$

Next, we point out two sets of the parameters under which the considered system acquires the canonical form. Suppose $b_{21}^M = b_{31}^M = 0$, $c_{11}^M \neq 0$, and $c_{12}^M = c_{13}^M = 0$. In addition, first let $b_{11}^M = 1$. Then

$$\begin{cases} b_{11}^M = \frac{1}{\alpha_0}(-b_{11} + b_{21} + b_{31}) = 1, \\ b_{21}^M = \frac{\alpha_1+\alpha_0}{\alpha_0^2}b_{11} - \frac{2\alpha_1+\alpha_0}{2\alpha_0^2}b_{21} - \frac{2\alpha_1+\alpha_0}{2\alpha_0^2}b_{31} = 0, \\ b_{31}^M = \frac{b_{31}}{\alpha_2} = 0. \end{cases} \tag{6.1}$$

From the last equation of system (6.1), we have $b_{31} = 0$. From the first two equations, it follows that $b_{11} = \alpha_0 + 2\alpha_1$, $b_{21} = 2(\alpha_0 + \alpha_1)$. If b_{11} and b_{21} are other than zero in the initial system, then, except for the conditions $\alpha_0 \neq 0$, $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, we obtain the two additional conditions $\alpha_0 \neq -2\alpha_1$ and $\alpha_0 \neq -\alpha_1$.

Now, in addition, let $b_{11}^M = -1$. Then $b_{11} = -(\alpha_0 + 2\alpha_1)$, $b_{21} = -2(\alpha_0 + \alpha_1)$. Put $\alpha_0 \neq -2\alpha_1$, $\alpha_0 \neq -\alpha_1$. For $\overline{C}_M$, we have

$$\begin{cases} c_{11}^M = (\alpha_0 + 2\alpha_1)c_{11} + 2(\alpha_0 + \alpha_1)c_{12} \neq 0, \\ c_{12}^M = 2\alpha_0(c_{11} + c_{12}) = 0, \\ c_{13}^M = \alpha_2(-c_{12} + c_{13}) = 0. \end{cases} \tag{6.2}$$

Since $\alpha_0 \neq 0$ and $\alpha_2 \neq 0$, we have $c_{11} = -c_{12}$ and $c_{12} = c_{13}$. From the first equation of system (6.2), we obtain $-\alpha_0 c_{11} \neq 0$. It follows from here that we may choose any number other than zero as c_{11}. Here $c_{12} = c_{13} = -c_{11}$.

Thus, in the considered system, the feedback vector can consist of all nonzero elements. It is obvious that for other matrices A_0 and S, the numerical coefficients in (6.1) and (6.2) are others, but the main sense does not change. Introducing the parameters α_0 and α_1 expands the number of options for choosing the elements in B_0 and C_0 such that, for $b_{11}^M = -1$, the initial system is reduced to the two-dimensional system (basic system)

$$\begin{cases} \dot{x}_1 = -4x_1 - (F(\sigma) + \psi(t)), \\ \dot{x}_2 = x_1 - 4x_2, \end{cases} \tag{6.3}$$

and the one-dimensional system

$$\dot{x}_3 = -2x_3,$$

where $\sigma = c_{11}^M x_1$. We have the vector Γ such that $\Gamma = (\gamma_1, 0)^{\mathrm{T}} = (c_{11}^M, 0)^{\mathrm{T}}$. The systems can be successfully integrated and investigated analytically.

Consider system (6.3). Setting the parameters of $F(\sigma)$, $\psi(t)$ and using Theorem 5.1, we can establish whether there exists a periodic solution with period T_f. However, setting the parameters of $\psi(t)$ and using Corollary, we can determine the parameters of $F(\sigma)$ under which system (6.3) has a solution with T_f given. Next we solve the latter task.

Example 2. In system (6.3) we have $\lambda = -4 < 0$. Put $F_0 = 1$, $k = 0.02$, $\omega = 2$, and $\varphi = -1$. Then $T = 2\pi/\omega = \pi$. Also, put $\nu_1 = \nu_2 = 1$. Then $\tau_1 = \tau_2 = \pi$ and $T_f = 2\pi$ are given.

According to (4.6), we take $\gamma_1 = 1 > 0$, $l_1 = -100 < 0$, and $m_1 = -100$ ($\overline{m}_1 = m_1 + F_0 = -99 < 0$). Using (5.2), we obtain $l_2 \approx 24.754011 > 0$. From (5.3) we calculate $m_2 \approx 399.019526$ and hence $\overline{m}_2 = m_2 + F_0 \approx 400.019526 > 0$.

Now we need to verify the inequalities in (4.6). So, we have

$$l_2 - \gamma_1 \left(\frac{\overline{m}_2}{\lambda} + \frac{k \sin(\varphi + \delta)}{\sqrt{\lambda^2 + \omega^2}} \right) \approx 124.754446 > 0,$$

$$l_1 - \gamma_1 \left(\frac{\overline{m}_2}{\lambda} - \frac{k}{\sqrt{\lambda^2 + \omega^2}} \right) \approx 0.009354 > 0,$$

$$l_1 - \gamma_1 \left(\frac{\overline{m}_1}{\lambda} + \frac{k \sin(\varphi + \delta)}{\sqrt{\lambda^2 + \omega^2}} \right) \approx -124.754446 < 0,$$

$$l_2 - \gamma_1 \left(\frac{\overline{m}_1}{\lambda} + \frac{k}{\sqrt{\lambda^2 + \omega^2}} \right) \approx -0.0004607 < 0.$$

Inequalities (4.6) are true.

Under the values of parameters stated above, consider equation (4.4) for τ_1 and equation (4.5) for τ_2. Feasible solution analysis makes it possible to assert that $\tau_1 = \pi$ is the least solution of equation (4.4) and $\tau_2 = \pi$ is a unique solution of equation (4.5) on the interval $(0, 2\pi)$. Therefore all the conditions of Corollary hold.

We thus come to the conclusion that there exists a unique asymptotically stable $2T$-periodic solution to the system with $\gamma_1 = 1$, $l_1 = -100$, $m_1 = -100$, $l_2 \approx 24.754011$, and $m_2 \approx 399.019526$. This task is solved.

To compare the dynamics of the system in both nonautonomous and autonomous cases, consider the system with $\psi(t) \equiv 0$ in the vector form $\dot{X} = AX + Bm_i$ $(i = 1, 2)$. On the plane (x_1Ox_2), the autonomous system has the stability centerpoints $X_i = -A^{-1}Bm_i$ with coordinates $(25, 6.25)$ and $(-99.754881, -24.938720)$. As is known, if both centerpoints of stability lay out of the ambiguity zone, then the autonomous system would have at least one periodic solution with two switch points. As we see, the point $(-99.754881, -24.938720)$ lies in the interior of the ambiguity zone $-100 \leq x_1 \leq 24.754011$ $(l_1/\gamma_1 \leq x_1 \leq l_2/\gamma_1)$. This means that the autonomous system $(\psi(t) \equiv 0)$ has no periodic solutions. Hence the perturbation $\psi(t) = 1 + 0.02\sin(2t - 1)$ has a strong effect on the system. Thus, the oscillatory process exists in the system only due to the periodic perturbation.

7 Conclusion

We have proposed the approach for the study of the multidimensional system with a non-ideal relay and a harmonic perturbation by analytical methods. It consists in the decomposition of the system into subsystems of dimensions 1 and 2 and the research these subsystems. The decomposition is based on the transformation with the parametric matrix that enables one to choose system parameters.

The main object for the research by analytical methods is a two-dimensional subsystem with the linear part being reduced to non-trivial Jordan block. We have obtained the sufficient conditions on the parameters under which the subsystem possesses a unique asymptotically stable periodic solution (see Theorem 5.1). These conditions are expressed as inequalities, which has allowed us to set the relay parameters such that there exists the solution with given period (see Corollary). Moreover, we have pointed out the parameters such that a periodic solution takes place in the dynamics of the subsystem with a harmonic perturbation but it does not in the dynamics of the same subsystem with no perturbations (see Example 2).

In the future, the results of Theorem 5.1 can be used to perform numerical simulations of the n-dimensional system dynamics, since theoretical investigations together with simulations provide more reliable results for applications. Also, the present study is worthwhile to develop for the system with the matrix A having complex eigenvalues and the truncated Fourier series (the sum of a constant and sine functions with different but commensurable periods) as the function of perturbations. Moreover, the proposed approach can be applied to a class of systems broader than the one considered in this study, namely, to systems with the nonlinearity being a monotone function of a non-ideal relay.

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