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ESTIMATE OF THE BEST CONSTANT OF DISCRETE HARDY-TYPE INEQUALITY WITH MATRIX OPERATOR SATISFYING THE OINAROV CONDITION

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Key words: Hardy-type inequality, weight sequence, space of sequences, matrix operator, Oinarov condition.

AMS Mathematics Subject Classification: 26D15.

Abstract. This paper studies the weighted inequality of Hardy-type in discrete form for matrix operators satisfying the Oinarov condition. Necessary and sufficient conditions on the weight sequences under which the Hardy-type inequality holds were found in [13] for the case $1 < p \leq q < \infty$, in [14] for the case $1 < q < p < \infty$, and in [15] for the case $0 < p \leq q < \infty$, $0 < p \leq 1$. In this paper, we extend the result of [13] with a two-sided estimate of the inequality constant.

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1 Introduction

For arbitrary non-negative sequences $f = \{f_i\}_{i=1}^{\infty}$ the modern form of the discrete Hardy-type inequality can be written as follows:

$$\left(\sum_{i=1}^{\infty} u_i^q \left(\sum_{j=1}^{i} a_{i,j} f_j\right)^q\right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{\infty} v_i^p f_i^p\right)^{\frac{1}{p}}, \tag{1.1}$$

where $u = \{u_i\}_{i=1}^{\infty}$ and $v = \{v_i\}_{i=1}^{\infty}$ are weight sequences of positive real numbers, and

$$(Af)_i = \sum_{j=1}^{i} a_{i,j} f_j \tag{1.2}$$

is a matrix operator with the kernel $a := \{a_{i,j}\}_{i,j=1}^{\infty}$, $i \geq j$, such that $a_{i,j} \geq 0$ for $i \geq j \geq 1$ and $C > 0$ depends only on p, q, u, v, and a.

In the case $a_{i,j} \equiv 1$, the problem of finding necessary and sufficient conditions on the weight sequences $u = \{u_i\}_{i=1}^{\infty}$ and $v = \{v_i\}_{i=1}^{\infty}$ such that inequality (1.1) holds for any non-negative sequences $f = \{f_i\}_{i=1}^{\infty}$ has been solved for all possible relations between the parameters $0 < p < \infty$ and $0 < q < \infty$ (see [1, 2, 3, 4, 6, 8]).

Suppose that $a_{i,j} \geq 0$ for $i \geq j \geq 1$ and there exists a number $d > 1$ such that

$$\frac{1}{d}(a_{i,k} + a_{k,j}) \leq a_{i,j} \leq d(a_{i,k} + a_{k,j}), \quad \forall\ i \geq k \geq j \geq 1. \tag{1.3}$$

This condition is an analogue of the Oinarov condition for kernels of integral operators introduced in [5] and [12]. Characterizations of the validity of inequality (1.1) for the operators satisfying discrete Oinarov condition (1.3) were found in [13] for the case $1 < p \le q < \infty$, in [14] for the case $1 < q < p < \infty$ and in [15] for the case $0 < p \le q < \infty$, $0 < p \le 1$.

In [12], the integral weighted Hardy-type inequality for the operator satisfying the Oinarov condition was characterized in the case $1 < p \le q < \infty$. In 2021, in paper [9] this result was extended with a two-sided estimate of the inequality constant. Since estimates of the best constants of Hardy-type inequalities have important applications in the oscillation theory of differential inequalities, paper [9] has got many citations over the past two years. In this paper, motivated by the development in the continuous case, we aim to find a two-sided estimate of the best constant $C > 0$ in inequality (1.1) also in the case $1 < p \le q < \infty$. The obtained result will be used to establish the oscillatory properties of difference equations.

Let $l_{p,v}$ denote the space of all sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers whose norm $\|f\|_{p,v} \equiv \|vf\|_p = \left(\sum\limits_{i=1}^{\infty} |v_i f_i|^p\right)^{\frac{1}{p}}$ is finite. Then inequality (1.1) can be rewritten in the form: $\|Af\|_{q,u} \le C\|f\|_{p,v}$. The validity of this inequality is equivalent to the boundedness of matrix operator (1.2) from $l_{p,v}$ into $l_{q,u}$, while for the best constant $C > 0$ we have that $C = \|A\|_{p,v\to q,u}$, where $\|A\|_{p,v\to q,u}$ denotes the norm of operator (1.2) from $l_{p,v}$ to $l_{q,u}$.

Let $p' = \frac{p}{p-1}$. To prove the main result we need the following theorem proved in [4].

Theorem A. *Let $1 < p \le q < \infty$. Then for any non-negative $f \in l_{p,v}$ the inequality*

$$\left(\sum_{i=1}^{\infty} u_i^q \left(\sum_{j=1}^{i} f_j\right)^q\right)^{\frac{1}{q}} \le C \left(\sum_{i=1}^{\infty} v_i^p f_i^p\right)^{\frac{1}{p}}, \tag{1.4}$$

holds if and only if

$$A = \sup_{k\ge 1} \left(\sum_{n=k}^{\infty} u_n^q\right)^{\frac{1}{q}} \left(\sum_{j=1}^{k} v_j^{-p'}\right)^{\frac{1}{p'}} < \infty.$$

Moreover, $A \le C \le \widetilde{C}A$, where $\widetilde{C} = \left(1 + \frac{q}{p'}\right)^{\frac{1}{q}} \left(1 + \frac{p'}{q}\right)^{\frac{1}{p'}}$ and C is the best constant in (1.4).

Remark 1. In the case $p = q = 2$, we have that $\widetilde{C} = \left(1 + \frac{2}{2}\right)^{\frac{1}{2}} \left(1 + \frac{2}{2}\right)^{\frac{1}{2}} = 2$.

Note that the Hardy inequality has a long history (see [10]), and its various generalizations and applications have grown into a separate field called the “theory of Hardy-type inequalities”, with many papers published every year (see, e.g., most recent publications [7], [11] and [16]).

2 Main result

Theorem 2.1. *Let $1 < p \le q < \infty$ and a matrix $(a_{i,j})$ satisfy condition* (1.3). *Then for any non-negative $f \in l_{p,v}$ inequality* (1.1) *holds if and only if $B = \max\{B_1, B_2\} < \infty$, where*

$$B_1 = \sup_{k\ge 1} \left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q\right)^{\frac{1}{q}} \left(\sum_{j=1}^{k} v_j^{-p'}\right)^{\frac{1}{p'}},$$

$$B_2 = \sup_{k\ge 1} \left(\sum_{n=k}^{\infty} u_n^q\right)^{\frac{1}{q}} \left(\sum_{j=1}^{k} a_{k,j}^{p'} v_j^{-p'}\right)^{\frac{1}{p'}}.$$

Moreover, $B \le C \le \overline{C}B$, where $\overline{C} = \left(2(d+1)^q + (d+1)^{2q}(1 + d\,\widetilde{C}^q)\right)^{\frac{1}{q}}$ and C is the best constant in (1.1).

Proof. Necessity. Let inequality (1.1) hold. To estimate C from below, we follow the same steps as in paper [13]. Putting the test sequence $g = \{g_j\}_{j=1}^{\infty}$ such that $g_j = \begin{cases} v_j^{-p'}, & 1 \le j \le k, \\ 0, & j > k, \end{cases}$ for $k \ge 1$, into the right-hand side and then into the left-hand side of inequality (1.1), we get $B_1 \le C$. Putting one more test sequence $h = \{h_j\}_{j=1}^{\infty}$ such that $h_j = \begin{cases} a_{k,j}^{p'-1} v_j^{-p'}, & 1 \le j \le k, \\ 0, & j > k, \end{cases}$ for $k \ge 1$ into the both sides of inequality (1.1), we have $B_2 \le C$. Combining the obtained estimates, we find that

$$B \le C. \tag{2.1}$$

Sufficiency. Let $B < \infty$. For any $i \ge 1$ the set of positive numbers S_i is defined as follows: $S_i = \{k \in \mathbb{Z} : (d+1)^k \le (Af)_i\}$, where d is the number from (1.3). If $k(i) = \max S_i$, then

$$(d+1)^{k(i)} \le (Af)_i \le (d+1)^{k(i)+1}. \tag{2.2}$$

Let $m_1 = 1$ and $M_1 = \{i \in \mathbb{N} : k(i) = k(1) = k(m_1)\}$. We define m_2 as $m_2 = \sup M_1 + 1$. It is obvious that $m_2 > m_1$. Moreover, if the set M_1 is bounded from above, then $m_2 < \infty$ and $m_2 - 1 = \max M_1 = \sup M_1$. Suppose that for $s \ge 1$ the numbers $1 = m_1 < m_2 < ... < m_s < \infty$ are defined. We define the next number m_{s+1} as $m_{s+1} = \sup M_s + 1$, where $M_s = \{i \in \mathbb{N} : k(i) = k(m_s)\}$.

Let $N = \{s \in \mathbb{N} : m_s < \infty\}$. For $s \in N$ the definition of m_s and (2.2) give that

$$(d+1)^{k(m_s)} \le (Af)_i \le (d+1)^{k(m_s)+1}, \quad m_s \le i \le m_{s+1} - 1, \tag{2.3}$$

and $\mathbb{N} = \bigcup_{s \in N} [m_s, m_{s+1})$. Hence,

$$\|Af\|_{q,u}^q = \sum_{s \in N} \sum_{j=m_s}^{m_{s+1}-1} u_j^q (Af)_j^q.$$

We assume that $\sum\limits_{j=m_s}^{m_{s+1}-1} u_j^q (Af)_j^q = 0$ if $m_s = \infty$. Then $\|Af\|_{q,u}^q$ can be presented as follows:

$$\|Af\|_{q,u}^q = \sum_{j=m_1}^{m_2-1} u_j^q (Af)_j^q + \sum_{j=m_2}^{m_3-1} u_j^q (Af)_j^q + \sum_{s \ge 3} \sum_{j=m_s}^{m_{s+1}-1} u_j^q (Af)_j^q. \tag{2.4}$$

Since $m_1 = 1 < \infty$, it belongs to N. Thus, from (2.3) we have

$$\sum_{j=m_1}^{m_2-1} u_j^q (Af)_j^q \le \sum_{j=1}^{m_2-1} u_j^q (d+1)^{(k(m_1)+1)q} \le (d+1)^q (d+1)^{k(m_1)q} \sum_{j=1}^{\infty} u_j^q$$

$$\le (d+1)^q (Af)_1^q \sum_{j=1}^{\infty} u_j^q \le (d+1)^q \left(\sum_{s=1}^{1} a_{1,s}^{p'} v_s^{-p'} \right)^{\frac{q}{p'}} \sum_{j=1}^{\infty} u_j^q \, \|f\|_{p,v}^q \le (d+1)^q B_2^q \, \|f\|_{p,v}^q. \tag{2.5}$$

If $m_2 = \infty$, then $m_s = \infty$ for all $s \ge 2$. Therefore, arguing as above, we get

$$\|Af\|_{q,u}^q \le (d+1)^q B_2^q \|f\|_{p,v}^q.$$

If $m_2 < \infty$, then $s = 2$ belongs to N. Thus, from (2.3) we have

$$\sum_{j=m_2}^{m_3-1} u_j^q (Af)_j^q \le (d+1)^q (d+1)^{k(m_2)q} \sum_{j=m_2}^{\infty} u_j^q \le (d+1)^q (Af)_{m_2}^q \sum_{j=m_2}^{\infty} u_j^q$$

$$= (d+1)^q \left(\sum_{i=1}^{m_2} a_{m_2,i} f_i\right)^q \sum_{j=m_2}^{\infty} u_j^q \le (d+1)^q \left(\sum_{i=1}^{m_2} a_{m_2,i}^{p'} v_i^{-p'}\right)^{\frac{q}{p'}} \sum_{j=m_2}^{\infty} u_j^q \left(\sum_{i=1}^{m_2} v_i^p f_i^p\right)^{\frac{q}{p}}$$

$$\le (d+1)^q \left(\left(\sum_{i=1}^{m_2} a_{m_2,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{j=m_2}^{\infty} u_j^q\right)^{\frac{1}{q}}\right)^q \|f\|_{p,v}^q \le (d+1)^q B_2^q \|f\|_{p,v}^q. \tag{2.6}$$

If $m_3 = \infty$, then from (2.4), (2.5) and (2.6) we get

$$\|Af\|_{q,u}^q \le 2(d+1)^q B_2^q \|f\|_{p,v}^q.$$

Let us consider $s \ge 3$ such that s belongs to N. Since $k(m_{s-2}) < k(m_{s-1}) < k(m_s)$, we have that $k(m_{s-2}) + 1 \le k(m_s) - 1$. Therefore, using (2.3) and (1.3), we obtain

$$(d+1)^{k(m_s)-1} = (d+1)^{k(m_s)} - d(d+1)^{k(m_s)-1} \le (d+1)^{k(m_s)} - d(d+1)^{k(m_{s-2})+1}$$

$$< (Af)_{m_s} - d(Af)_{m_{s-1}-1} = \sum_{i=1}^{m_s} a_{m_s,i} f_i - d \sum_{i=1}^{m_{s-1}-1} a_{m_{s-1}-1,i} f_i$$

$$= \sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i + \sum_{i=1}^{m_{s-1}-1} \left[a_{m_s,i} - d a_{m_{s-1}-1,i}\right] f_i$$

$$\le \sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i + \sum_{i=1}^{m_{s-1}-1} \left[d(a_{m_s,m_{s-1}-1} + a_{m_{s-1}-1,i}) - d a_{m_{s-1}-1,i}\right] f_i$$

$$= \sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i + d \sum_{i=1}^{m_{s-1}-1} a_{m_s,m_{s-1}-1} f_i.$$

The latter, together with (2.3), for $s \ge 3$ gives that

$$\sum_{s\ge3} \sum_{j=m_s}^{m_{s+1}-1} u_j^q (Af)_j^q < \sum_{s\ge3} \sum_{j=m_s}^{m_{s+1}-1} u_j^q (d+1)^{(k(m_s)+1)q} = (d+1)^{2q} \sum_{s\ge3} (d+1)^{(k(m_s)-1)q} \sum_{j=m_s}^{m_{s+1}-1} u_j^q$$

$$\le (d+1)^{2q} \sum_{s\ge3} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i + d \sum_{i=1}^{m_{s-1}-1} a_{m_s,m_{s-1}-1} f_i\right)^q \sum_{j=m_s}^{m_{s+1}-1} u_j^q$$

$$\le (d+1)^{2q} \left[\sum_{s\ge3} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i\right)^q \sum_{j=m_s}^{m_{s+1}-1} u_j^q\right.$$

$$\left. + d \sum_{s\ge3} \left(\sum_{i=1}^{m_{s-1}-1} a_{m_s,m_{s-1}-1} f_i\right)^q \sum_{j=m_s}^{m_{s+1}-1} u_j^q\right] = (d+1)^{2q}(I_1 + d\, I_2). \tag{2.7}$$

We estimate I_1 and I_2 separately. Using the Hölder and Jensen inequalities, we obtain

$$I_1 = \sum_{s\ge3} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i\right)^q \sum_{j=m_s}^{m_{s+1}-1} u_j^q \le \sum_{s\ge3} \left(\sum_{i=m_{s-1}}^{m_s} a_{m_s,i}^{p'} v_i^{-p'}\right)^{\frac{q}{p'}} \left(\sum_{i=m_{s-1}}^{m_s} v_i^p f_i^p\right)^{\frac{q}{p}}$$

$$\times \sum_{j=m_s}^{m_{s+1}-1} u_j^q \le \left(\sup_{k\ge 1} \left(\sum_{i=1}^{k} a_{k,i}^{p'} v_i^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{j=k}^{\infty} u_j^q \right)^{\frac{1}{q}} \right)^q \sum_{s\ge 3} \left(\sum_{i=m_{s-1}}^{m_s} v_i^p f_i^p \right)^{\frac{q}{p}}$$

$$\le B_2^q \left(\sum_{s\ge 3} \sum_{i=m_{s-1}}^{m_s} v_i^p f_i^p \right)^{\frac{q}{p}} \le B_2^q \|f\|_{p,v}^q. \tag{2.8}$$

Let us turn to the estimate of I_2. By Theorem A, we have

$$I_2 = \sum_{s\ge 3} a_{m_s,m_{s-1}-1}^q \sum_{j=m_s}^{m_{s+1}-1} u_j^q \left(\sum_{i=1}^{m_{s-1}-1} f_i \right)^q$$

$$\le \widetilde{C}^q \left(\sup_{k\ge 1} \left(\sum_{m_{s-1}-1\ge k} a_{m_s,m_{s-1}-1}^q \sum_{j=m_s}^{m_{s+1}-1} u_j^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^{k} v_j^{-p'} \right)^{\frac{1}{p'}} \right)^q \|f\|_{p,v}^q. \tag{2.9}$$

Since $a_{i,j}$ is non-decreasing in i and non-increasing in j, we deduce that

$$\sum_{m_{s-1}-1\ge k} a_{m_s,m_{s-1}-1}^q \sum_{j=m_s}^{m_{s+1}-1} u_j^q \le \sum_{m_{s-1}-1\ge k} \sum_{j=m_s}^{m_{s+1}-1} a_{j,k}^q u_j^q \le \sum_{j=k}^{\infty} a_{j,k}^q u_j^q.$$

Using the latter, from (2.9) we find

$$I_2 \le \widetilde{C}^q B_1^q \|f\|_{p,v}^q. \tag{2.10}$$

Combining (2.4), (2.5), (2.6), (2.7), (2.8), and (2.10), we get

$$\|Af\|_{q,u}^q \le (d+1)^q B_2^q \|f\|_{p,v}^q + (d+1)^q B_2^q \|f\|_{p,v}^q + (d+1)^{2q} (B_2^q \|f\|_{p,v}^q + d\,\widetilde{C}^q B_1^q \|f\|_{p,v}^q)$$

$$\le \left(2(d+1)^q + (d+1)^{2q} (1 + d\,\widetilde{C}^q) \right) B^q \|f\|_{p,v}^q. \tag{2.11}$$

Therefore, from (2.11) we obtain

$$C \le \left(2(d+1)^q + (d+1)^{2q} (1 + d\,\widetilde{C}^q) \right)^{\frac{1}{q}} B,$$

which, together with (2.1), gives that $B \le C \le \overline{C} B$. □

Remark 2. Taking into account Remark 1, in the case $p = q = 2$ and $d = 1$, we have that $\overline{C} = (2(1+1)^2 + (1+1)^4(1 + 1 \cdot 2^2))^{\frac{1}{2}} = 88^{\frac{1}{2}} \approx 9.38$.

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