ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2024, Volume 15, Number 2

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

EURASIAN MATHEMATICAL JOURNAL

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of the EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office through the provided web interface (www.enu.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/NewCode.pdf). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;

- compliance of the title of the paper to its content;

- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);

- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);

- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of the EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in the EMJ (free access).

Subscription

Subscription index of the EMJ 76090 via KAZPOST.

E-mail

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 473
3 Ordzonikidze St
117198 Moscow, Russia

Short communications

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 15, Number 2 (2024), 92 – 100

INEQUALITIES FOR TRIGONOMETRIC POLYNOMIALS IN PERIODIC MORREY SPACES

V.I. Burenkov, D.J. Joseph

Communicated by M.L. Goldman

Key words: Morrey spaces, periodic Morrey spaces, Bernstein's inequality, inequalities of different metrics and of different dimensions.

AMS Mathematics Subject Classification: 34A55, 34B05, 58C40.

Abstract. A detailed exposition of Bernstein's inequality, inequalities of different metrics and of different dimensions for trigonometic polynomials in Lebesgue spaces is given in the book of S.M. Nikol'skii [4]. In this paper, we state analogues of these inequalities in peridoic Morrey spaces.

DOI: https://doi.org/10.32523/2077-9879-2024-15-2-92-100

1 Introduction

Definition 1. Let $n \in \mathbb{N},\ \mu \in \mathbb{N}_0$. Let $\mathfrak{M}^*_\mu(\mathbb{R}^n)$ denote the set all real valued trigonometric polynomials of order less than or equal to μ:

$$T_\mu(x) = T_\mu(x_1, \dots, x_n) = \sum_{\substack{-\mu \le k_j \le \mu \\ j=1,\dots,n}} c_{k_j} e^{ik\cdot x} \tag{1.1}$$

$$= \sum_{-\mu \le k_1 \le \mu} \cdots \sum_{-\mu \le k_n \le \mu} c_{k_1,\dots,k_n} e^{i(k_1 x_1 + \dots k_n x_n)}.$$

where $x_1, \dots, x_n \in \mathbb{R},\ c_{k_1,\dots,k_n} \in \mathbb{C}$ are constant coefficients such that $c_{-k} = \bar{c}_k$ (hence $T_\mu(x) \in \mathbb{R}$ for any $x \in \mathbb{R}^n$).

Let hereafter $0 < p \le \infty$. A function $f \in L^*_p$ if it is 2π-periodic Lebesgue measurable and

$$\|f\|^*_{L_p} = \|f\|_{L_p(Q(0,\pi))} < \infty, \tag{1.2}$$

where $Q(x,r) = \{y \in \mathbb{R}^n : |x_j - y_j| < r, j = 1, \dots, n\}$.

In book [9] the following inequalities are proven for trigonometric polynomials $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$, where the space $\mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$ is $\mathfrak{M}^*_\mu(\mathbb{R}^n)$ equiped with the quasinorm $\|\cdot\|^*_{L_p}$.

1. (Bernstein's inequality) Let $1 \le p \le \infty$, then for any trigonometric polynomial $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$

$$\left\| \frac{\partial T_\mu}{\partial x_j} \right\|^*_{L_p} \le \mu \|T_\mu\|^*_{L_p}, \quad j = 1, \dots, n. \tag{1.3}$$

2. (Inequality of different metrics) Let $1 \le p < q \le \infty$, then for any trigonometric polynomials $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$

$$\|T_\mu\|^*_{L_q} \le 3^n \mu^{n(\frac{1}{p}-\frac{1}{q})} \|T_\mu\|^*_{L_p}. \tag{1.4}$$

3. (Inequality of different dimensions) Let $1 \le p \le \infty$, $1 \le m < n$, $x = (u,v)$, $u = (x_1, \dots, x_m) \in \mathbb{R}^m$, $v = (x_{m+1}, \dots, x_n) \in \mathbb{R}^{n-m}$, then for any trigonometric polynomial $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$

$$\Big\| \|T_\mu(u,v)\|_{L_{\infty,v}(\mathbb{R}^{n-m})} \Big\|^*_{L_{p,u}} \le 3^{n-m} \mu^{\frac{n-m}{p}} \|T_\mu\|^*_{L_p}, \tag{1.5}$$

in particular,

$$\|T_\mu(u,0)\|^*_{L_p} \le 3^{n-m} \mu^{\frac{n-m}{p}} \|T_\mu\|^*_{L_p}. \tag{1.6}$$

The purpose of this work is to present similar inequalities in which the space L^*_p is replaced by the periodic Morrey space $(M^\lambda_p)^*$.

Note also that Bernstein's inequality, inequalities of different metrics and different dimensions for entire functions of exponential type for the spaces $L_p(\mathbb{R}^n)$ were proved by S.M. Nikolsky [9], and for the Morrey spaces in the works [2], [3]

2 Morrey spaces

The spaces $M^\lambda_p(\mathbb{R}^n)$, now called Morrey spaces, were first considered by Charles Morrey [8] in connection with the study of the regularity of solutions of partial differential equations.

Definition 2. Let $0 < p \le \infty$ and $0 \le \lambda \le \frac{n}{p}$, then $f \in M^\lambda_p(\mathbb{R}^n)$, if $f \in L^{loc}_p(\mathbb{R}^n)$ and

$$\|f\|_{M^\lambda_p(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \sup_{r>0} r^{-\lambda} \|f\|_{L_p(B(x,r))} < \infty, \tag{2.1}$$

where $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$.

Periodic analogues $(M^\lambda_p)^*(\mathbb{R}^n)$ of the Morrey space were considered in [10]

Definition 3. Let $0 < p \le \infty$ and $0 \le \lambda \le \frac{n}{p}$, then $f \in (M^\lambda_p)^*(\mathbb{R}^n)$, if it has period 2π, is Lebesgue measurable on $\mathbb{R}^n$ and

$$\|f\|^*_{M^\lambda_p} = \sup_{x\in Q(0,\pi)} \sup_{0<r\le\pi} r^{-\lambda} \|f\|_{L_p(Q(x,r))} < \infty. \tag{2.2}$$

We note some properties of these spaces.

1. It is immediately clear from the definition that for $\lambda = 0$

$$\|f\|^*_{M^0_p} = \|f\|^*_{L_p}.$$

2. For $\lambda = \frac{n}{p}$

$$\|f\|^*_{M^{\frac{n}{p}}_p} = \|f\|^*_{L_\infty},$$

3. If $\lambda < 0$ or $\lambda > \frac{n}{p}$, then the spaces $(M^\lambda_p)^*(\mathbb{R}^n)$ consist only of functions equivalent to 0 on $Q(0,\pi)$.

4. Note that the space $(M^\lambda_p)^*(\mathbb{R}^n)$ has the property of monotonicity with respect to the parameter λ:

$$(M^\lambda_p)^* \subset (M^\mu_p)^*, \ 0 \le \mu < \lambda \le \frac{n}{p}, \ 0 < p < \infty \tag{2.3}$$

and

$$\|f\|^*_{M_p^\mu} \le \pi^{\lambda-\mu}\|f\|^*_{M_p^\lambda}. \tag{2.4}$$

In particular, for $\mu = 0$

$$(M_p^\lambda)^*(\mathbb{R}^n) \subset (L_p)^*(\mathbb{R}^n)$$

and

$$\|f\|^*_{L_p} \le \pi^{\lambda}\|f\|^*_{M_p^\lambda}. \tag{2.5}$$

5. In [4] it is proven that for any $f \in (M_p^\lambda)^*$

$$\|f\|^*_{M_p^\lambda} = \|f\|^{**}_{M_p^\lambda} \equiv \sup_{x\in\mathbb{R}^n}\sup_{0<r\le\pi} r^{-\lambda}\|f\|^*_{L_p(Q(x,r))}. \tag{2.6}$$

6. (Shift invariance) For any $f \in (M_p^\lambda)^*$

$$\|f(y+h)\|^*_{M_p^\lambda} = \|f(y)\|^*_{M_p^\lambda} \quad \forall h \in \mathbb{R}^n. \tag{2.7}$$

3 Inequalities for trigonometric polynomials in periodic Morrey spaces

3.1 Bernstein's inequality

In the one-dimensional case, the interpolation formula for an arbitrary trigonometric polynomial T_μ of order $\mu > 0$ has the form (see [9]):

$$T'_\mu(x) = \frac{1}{4\mu}\sum_{k=1}^{2\mu}(-1)^{k+1}\frac{1}{\sin^2\frac{x_k}{2}}T_\mu(x+x_k), \tag{3.1}$$

where x_k are the zeros of the polynomial $\cos(nx)$.

If $T_\mu(x) = \sin(\mu x)$ and $x = 0$, then we get

$$\mu = \frac{1}{4\mu}\sum_{k=1}^{2\mu}\frac{1}{\sin^2\frac{x_k}{2}}. \tag{3.2}$$

Theorem 3.1. *Let Z^* be a normed space of 2π-periodic functions in each variable, and let $\|\cdot\|^*_Z$ be a shift invariant norm, i.e. for any function $f \in Z^*$*

$$\|f(x+h)\|^*_Z = \|f\|^*_Z \quad \forall h \in \mathbb{R}^n. \tag{3.3}$$

Then for any trigonometric polynomials $T_\mu \in Z^*(\mathbb{R}^n)$

$$\left\|\frac{\partial T_\mu}{\partial x_j}\right\|^*_Z \le \mu\|T_\mu\|^*_Z,\ j = 1,\ldots,n. \tag{3.4}$$

The proof is based on representation (3.1).

Corollary 3.1. *Let $1 \le p \le \infty$, $0 \le \lambda \le \frac{n}{p}$, then for any trigonometric polynomial $T_\mu \in (M_p^\lambda)^*$*

$$\left\|\frac{\partial T_\mu}{\partial x_j}\right\|^*_{M_p^\lambda} \le \mu\|T_\mu\|^*_{M_p^\lambda}, \quad j = 1,\ldots,n. \tag{3.5}$$

3.2 Inequality of different metrics

Definition 4. Let $1 \le p \le \infty$, $0 \le \lambda \le \frac{n}{p}$, $r > 0$, $\mu, N \in \mathbb{N}$, $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$ and

$$((T_\mu))^*_{M^\lambda_{p,N}} = \sup_{x\in Q(0,\pi)} \sup_{0<r\le\pi} r^{-\lambda} \Bigg(\Big(\frac{r}{N}\Big)^n \sum_{k_1=-N}^{N-1} \cdots \sum_{k_n=-N}^{N-1}$$

$$\Big| T_\mu\Big(x_1 + \frac{r}{N}k_1, \ldots, x_n + \frac{r}{N}k_n\Big)\Big|^p\Bigg)^{1/p}.$$

Lemma 3.1. *Let $1 \le p \le \infty$, $n, \mu, N \in \mathbb{N}$, $0 \le \lambda \le \frac{n}{p}$, then for any trigonometric polynomial $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$*

$$\|T_\mu\|^*_{M^\lambda_p} \le ((T_\mu))^*_{M^\lambda_{p,N}} \le (1 + \frac{\pi}{N}\mu)^n \|T_\mu\|^*_{M^\lambda_p}. \tag{3.6}$$

Lemma 3.2. *Let $1 \le p \le q \le \infty$ $n, \mu, N \in \mathbb{N}$, $0 \le \lambda \le \frac{n}{q}$, then for any trigonometric polynomial $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$*

$$((T_\mu))^*_{M^{\lambda-n(\frac{1}{p}-\frac{1}{q})}_{q,N}} \le N^{n(\frac{1}{p}-\frac{1}{q})} ((T_\mu))^*_{M^\lambda_{p,N}}. \tag{3.7}$$

Theorem 3.2. *Let $1 \le p \le q \le \infty$, $n\big(\frac{1}{p}-\frac{1}{q}\big) \le \lambda \le \frac{n}{p}$, then for any trigonometric polynomial $T_\mu \in \mathfrak{M}^*_{\mu,p}(\mathbb{R}^n)$*

$$\|T_\mu\|^*_{M_q^{\lambda-n(\frac{1}{p}-\frac{1}{q})}} \le (1+\pi)^n \mu^{n(\frac{1}{p}-\frac{1}{q})} \|T_\mu\|^*_{M^\lambda_p}. \tag{3.8}$$

Consider the convolution of functions $\varphi, g \in L_1(Q(0,\pi))$ $2\pi-$ periodic in each variable

$$(\varphi * g)(x) = \int_{Q(0,\pi)} \varphi(x-y) g(y) dy, \; x \in \mathbb{R}^n. \tag{3.9}$$

Recall that $\forall k \in \mathbb{Z}^n$

$$c_k(\varphi * g) = (2\pi)^n c_k(\varphi) c_k(g). \tag{3.10}$$

If $c_k(\varphi) = (2\pi)^{-n}$ then

$$c_k(g) = c_k(\varphi * g). \tag{3.11}$$

Lemma 3.3. *Let $n \in \mathbb{N}$, $\mu \in \mathbb{N}, \varphi \in L_1(Q(0,\pi))$ be a 2π-periodic trigonometric polynomial in each variable. In order for any trigonometric polynomial T_μ of order μ to satisfy the equality*

$$T_\mu = \varphi * T_\mu, \tag{3.12}$$

it is necessary and sufficient condition that

$$c_k(\varphi) = (2\pi)^{-n} \;\forall k \in \mathbb{Z}^n : |k_j| \le \mu, \; j = 1, \ldots, n. \tag{3.13}$$

Definition 5. (Dirichlet kernel) Let

$$D_\mu(x) = \frac{1}{2} \sum_{k=-\mu}^{\mu} e^{ikx} = \frac{1}{2} + \sum_{k=1}^{\mu} \cos(kx) = \frac{\sin(\mu + \frac{1}{2})x}{2 \sin \frac{x}{2}} \tag{3.14}$$

and

$$\widetilde{D}_\mu(x) = \frac{1}{\pi} D_\mu(x). \tag{3.15}$$

Note that

$$\|\widetilde{D}_\mu\|^*_{L_2} = \sqrt{\frac{2\mu+1}{2\pi}} \tag{3.16}$$

and

$$\|\widetilde{D}_\mu\|^*_{L_\infty} = \frac{2\mu+1}{2\pi}. \tag{3.17}$$

From equalities (3.16) and (3.17) it follows that for any $2 < p < \infty$

$$\|\widetilde{D}_\mu\|^*_{L_p} \le \left(\frac{2\mu+1}{2\pi}\right)^{1-\frac{1}{p}}. \tag{3.18}$$

A special case of equality (3.12) is the well-known equality

$$T_\mu(x) = \widetilde{D}_\mu(x) * T_\mu(x).$$

Remark 1. If φ is a trigonometric polynomial of order μ in each variable, then equality (3.12) holds for any trigonometric polynomials T_μ of order μ in each variable if and only if

$$\varphi(x) = \frac{1}{(2\pi)^n}\sum_{\substack{|k_j|\le\mu\\ j=1,\dots,n}} e^{ik\cdot x} = \frac{1}{(2\pi)^n}\prod_{j=1}^n\sum_{|k_j|\le\mu} e^{ik_jx_j} = \frac{1}{\pi^n}\prod_{j=1}^n D_\mu(x_j) = \prod_{j=1}^n \widetilde{D}_\mu(x_j).$$

Remark 2. Let $\alpha, n \in \mathbb{N}$

$$\Delta_\alpha(j) = \{k \in \mathbb{Z}^n, |k_j| \le \alpha\}$$

and

$$\Delta_\alpha = \Delta_\alpha(1) \times \cdots \times \Delta_\alpha(n).$$

If φ is a trigonometric polynomial of order $\nu > \mu$ in each variable, then equality (3.12) holds for any trigonometric polynomials T_μ of order μ in each variable if and only if

$$\varphi(x) = \sum_{k\in\Delta_\nu} c_k e^{ik\cdot x} = \prod_{j=1}^n \widetilde{D}_\mu(x_j) + \sum_{k\in\Delta_\nu\setminus\Delta_\mu} c_k e^{ik\cdot x}. \tag{3.19}$$

$\Big($In particular, for $n = 1$ $\varphi(x) = \widetilde{D}_\mu(x) + \big(\sum_{k=-\nu}^{-\mu-1} + \sum_{k=\mu+1}^{\nu}\big) c_k e^{ik\cdot x}.\Big)$

Definition 6. Let, for $\mu \in \mathbb{N}$, J^*_μ denote the set of all 2π-periodic functions $\varphi \in L_1(Q(0,\pi))$, satisfying condition (3.13) (hence, having form (3.19) for some $\nu \in \mathbb{N}, \nu \ge \mu$).

According to Lemma 3.3 for such functions φ equality (3.12) holds.

Definition 7. (see [10]) Let $\mu, \nu \in \mathbb{N}$ and $\nu > \mu$. The Vallee Poussin kernels are defined as follows:

$$\mathfrak{V}_{\mu,\nu}(x) = (\nu-\mu)^{-1}\sum_{l=\mu}^{\nu-1} D_l(x), \ x \in \mathbb{R}, \tag{3.20}$$

in particular,

$$\mathfrak{V}_\mu(x) = \mathfrak{V}_{\mu,2\mu}(x), \quad \mu \ge 1, \quad \mathfrak{V}_0(x) = 1, \ x \in \mathbb{R}. \tag{3.21}$$

Remark 3. For $\nu > \mu$ we represent the Dirichlet kernel as

$$D_\nu(x) = \frac{1}{2} + \cos x + \cdots + \cos\mu x + (\cos(\mu+1)x + \cdots + \cos\nu x) \tag{3.22}$$

$$= D_\mu(x) + D_{\mu,\nu}(x), \tag{3.23}$$

where

$$D_{\mu,\nu}(x) = \sum_{l=\mu+1}^{\nu} \cos lx. \tag{3.24}$$

Then for $\nu > \mu + 1$

$$\mathfrak{V}_{\mu,\nu}(x) = D_\mu(x) + \frac{1}{\nu-\mu}\sum_{l=\mu+1}^{\nu-1} D_{\mu,l}(x). \tag{3.25}$$

Let us put

$$\widetilde{\mathfrak{V}}_{\mu,\nu}(x) = \frac{1}{\pi}\mathfrak{V}_{\mu,\nu}(x), \quad \widetilde{D}_{\mu,\nu}(x) = \frac{1}{\pi}D_{\mu,\nu}(x), \tag{3.26}$$

then

$$\widetilde{\mathfrak{V}}_{\mu,\nu}(x) = \widetilde{D}_\mu(x) + \frac{1}{\nu-\mu}\sum_{l=\mu+1}^{\nu-1} \widetilde{D}_{\mu,l}(x). \tag{3.27}$$

in particular,

$$\widetilde{\mathfrak{V}}_{\mu}(x) = \widetilde{D}_\mu(x) + \frac{1}{\mu}\sum_{l=\mu+1}^{2\mu-1} \widetilde{D}_{\mu,l}(x). \tag{3.28}$$

A special case of equality (3.12) is the equality

$$T_\mu(x) = \widetilde{\mathfrak{V}}_{\mu,\nu}(x) * T_\mu(x), \tag{3.29}$$

in particular,

$$T_\mu(x) = \widetilde{\mathfrak{V}}_{\mu}(x) * T_\mu(x).$$

Remark 4. Note that

$$\widetilde{D}_\mu(x),\ \widetilde{\mathfrak{V}}_{\mu,\nu},\ \nu > \mu,\ \widetilde{\mathfrak{V}}_\mu \in J_\mu^* \tag{3.30}$$

Theorem 3.3 (see, for example, [10]). *Let $\mu \in \mathbb{N}$, $1 \le p \le \infty$, then*

$$\|\widetilde{\mathfrak{V}}_\mu\|_{L_p}^* \le 3^n \mu^{n(1-1/p)}. \tag{3.31}$$

Theorem 3.4. *(Corollary of the Young-type inequality for periodic Morrey spaces, see* [4] *)*

Let

$$0 \le \lambda < \frac{n}{p}, 1 \le r, p < q \le \infty, \quad 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p},$$

$f_1 \in L_r(\mathbb{R}^n)$ *and* $f_2 \in (M_p^\lambda)^*$. *Then*

$$\|f_1 * f_2\|^*_{M_q^{\frac{p\lambda}{q}}} \le \|f_1\|^*_{L_r}(\|f_2\|^*_{M_p^\lambda})^{\frac{p}{q}}(\|f_2\|^*_{L_p})^{1-\frac{p}{q}}. \tag{3.32}$$

Theorem 3.5. *Let $1 \le r, p < q \le \infty$, $n, \mu \in \mathbb{N}$ $0 \le \lambda \le \frac{n}{p}, 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$. Then*

$$\|T_\mu\|^*_{M_q^{\frac{p\lambda}{q}}} \le c(\|T_\mu\|^*_{M_p^\lambda})^{\frac{p}{q}}(\|T_\mu\|^*_{L_p})^{1-\frac{p}{q}} \tag{3.33}$$

for any $T_\mu \in (M_p^\lambda)^*$, where

$$c = c(\mu, r) = \inf_{\varphi \in J_\mu^*} \|\varphi\|^*_{L_r}. \tag{3.34}$$

Corollary 3.2. *Let $1 \le p \le q \le \infty$, $n, \mu \in \mathbb{N}$ $0 \le \lambda \le \frac{n}{p}$, then for any $T_\mu \in (M_p^\lambda)^*$*

$$\|T_\mu\|^*_{M_q^{\frac{p\lambda}{q}}} \le 3^n \mu^{n(\frac{1}{p}-\frac{1}{q})}(\|T_\mu\|^*_{M_p^\lambda})^{\frac{p}{q}}(\|T_\mu\|^*_{L_p})^{1-\frac{p}{q}}. \tag{3.35}$$

Inequality (3.35) follows from inequalities (3.31) and (3.33) since $\widetilde{\mathfrak{V}}_\mu \in J_\mu^*$ and in (3.33) $c \le \|\widetilde{\mathfrak{V}}_\mu\|^*_{L_r}$.

Corollary 3.3. *If $1 \le p \le 2$, $q \ge \frac{2p}{2-p}$, then for any $T_\mu \in L_p^*$*

$$\|T_\mu\|^*_{M_q^{\frac{p\lambda}{q}}} \le \left(\frac{2\mu+1}{2\pi}\right)^{n(\frac{1}{p}-\frac{1}{q})} (\|T_\mu\|^*_{M_p^\lambda})^{\frac{p}{q}} (\|T_\mu\|^*_{L_p})^{1-\frac{p}{q}}, \tag{3.36}$$

in particular, for $0 \le \lambda \le \frac{n}{2}$

$$\|T_\mu\|^*_{LM_2^{\frac{\lambda}{2}}} \le \left(\frac{2\mu+1}{2\pi}\right)^{\frac{n}{2}} (\|T_\mu\|^*_{LM_1^\lambda} \|T_\mu\|^*_{L_1})^{\frac{1}{2}}, \tag{3.37}$$

and

$$\|T_\mu\|^*_{L_\infty} \le \left(\frac{2\mu+1}{2\pi}\right)^{\frac{n}{2}} \|T_\mu\|^*_{L_2}. \tag{3.38}$$

Inequality (3.38) *follows from inequalities* (3.31), (3.33) *and* (3.16) *since $\widetilde{D}_\mu \in J_\mu^*$ and in* (3.33) *$c \le \|\widetilde{D}_\mu\|^*_{L_2}$. In the last inequality the constant is sharp, the equality is attained for $T_\mu(x) = \prod_{l=1}^n \widetilde{D}_\mu(x_l)$. Regarding generalizations, see* [7].

Corollary 3.4. *By inequality* (2.5) *inequalities* (3.33)-(3.36) *imply that*

$$\|T_\mu\|^*_{M_q^{\frac{p\lambda}{q}}} \le c\pi^{\lambda(1-\frac{p}{q})} \|T_\mu\|^*_{M_p^\lambda}, \tag{3.39}$$

$$\|T_\mu\|^*_{M_q^{\frac{p\lambda}{q}}} \le 3^n \pi^{\lambda(1-\frac{p}{q})} \mu^{n(\frac{1}{p}-\frac{1}{q})} \|T_\mu\|^*_{M_p^\lambda}, \tag{3.40}$$

$$\|T_\mu\|^*_{M_q^{\frac{p\lambda}{q}}} \le \left(\frac{2\mu+1}{2\pi}\right)^{n(\frac{1}{p}-\frac{1}{q})} \pi^{\lambda(1-\frac{p}{q})} \|T_\mu\|^*_{M_p^\lambda}. \tag{3.41}$$

Remark 5. Inequality (3.35) is a periodic analogue of the inequality of different metrics for entire functions of exponential type (see [2],[3]).

Remark 6. Note that inequailies (2.4) and (3.40) imply that

$$\|T_\mu\|^*_{M_q^{\lambda-n(\frac{1}{p}-\frac{1}{q})}} \le \pi^{(\lambda p-n)(\frac{1}{p}-\frac{1}{q})} \|T_\mu\|^*_{M_q^{\frac{\lambda p}{q}}} \le 3^n (\pi\mu)^{n(\frac{1}{p}-\frac{1}{q})} \|T_\mu\|^*_{M_p^\lambda}. \tag{3.42}$$

So, inequality (3.40) has a better exponent $\frac{p\lambda}{q}$ compared with the exponent $\lambda - n(\frac{1}{p}-\frac{1}{q})$ in (3.8). However, for some values of λ, p, q the constant $(1+\pi)^n$ in (3.8) is better than the constant $3^n \pi^{\lambda(1-\frac{p}{q})}$ in (3.40).

3.3 Inequality of different dimensions

Definition 8. Let

$$0 < p_1, p_2 \le \infty, \quad m_1, m_2 \in \mathbb{N}$$
$$0 \le \lambda_1 \le \frac{m_1}{p_1}, \quad 0 \le \lambda_2 \le \frac{m_2}{p_2}.$$

Let us define the space

$$(M_{p_1}^{\lambda_1})^*(\mathbb{R})^{m_1} \times (M_{p_2}^{\lambda_2})^*(\mathbb{R}^{m_2}) \tag{3.43}$$

with a mixed quasinorm as the set of all measurable functions f on $\mathbb{R}^{m_1+m_2}$ for which

$$\|T_\mu\|^*_{M_{p_1}^{\lambda_1}(\mathbb{R}^{m_1}) \times M_{p_2}^{\lambda_2}(\mathbb{R}^{m_2})} = \big\| \|T_\mu(u_1,u_2)\|^*_{M_{p_1,u_1}^{\lambda_1}(\mathbb{R}^{m_1})} \big\|^*_{M_{p_2,u_2}^{\lambda_2}(\mathbb{R}^{m_2})}$$
$$= \sup_{y \in Q_{m_2}(0,\pi)} \sup_{0<\rho\le\pi} \rho^{-\lambda_2} \| \sup_{x \in Q_{m_1}(0,\pi)} \sup_{0<r\le\pi} r^{-\lambda_1} \|T_\mu(u_1,u_2)\|_{L_{p_1,u_1}(Q(x,r))} \|_{L_{p_2,u_2}(Q(x,r))}, \tag{3.44}$$

where $Q_{m_1}(0,\pi) = \{u_1 \in \mathbb{R}^{m_1} : |u_{1j}| < \pi, j = 1, \dots, m_1\}$ and $Q_{m_2}(0,\pi)$ is defined similarly.

Let us note some properties of these spaces.

Lemma 3.4. *Let* $0 < p \le \infty,\ m_1, m_2 \in \mathbb{N}, 0 < \lambda_1 \le \frac{m_1}{p}, 0 < \lambda_2 \le \frac{m_2}{p},\ f_1 \in (M_p^{\lambda_1})^*(\mathbb{R}^{m_1})\ f_2 \in (M_p^{\lambda_2})^*(\mathbb{R}^{m_2})$ $f_1 \sim 0$ *on* $\mathbb{R}^{m_2}$ $f_2 \sim 0$ *on* $\mathbb{R}^{m_1}$, *then*

$$\|f_1 f_2\|^*_{M_p^{\lambda_1}(\mathbb{R}^{m_1})\times M_p^{\lambda_2}(\mathbb{R}^{m_2})} = \|f_1\|^*_{M_p^{\lambda_1}(\mathbb{R}^{m_1})} \|f_2\|^*_{M_p^{\lambda_2}(\mathbb{R}^{m_2})} \tag{3.45}$$

Lemma 3.5. *Let* $0 < p \le \infty,\ m_1, m_2 \in \mathbb{N},\ 0 \le \lambda_1 \le \frac{m_1}{p},\ 0 \le \lambda_2 \le \frac{m_2}{p}$. *Then*

$$(M_p^{\lambda_1})^*(\mathbb{R}^{m_1}) \times (M_p^{\lambda_2})^*(\mathbb{R}^{m_2}) \subset (M_p^{\lambda_1+\lambda_2})^*(\mathbb{R}^{m_1+m_2}), \tag{3.46}$$

and

$$\|f\|^*_{M_p^{\lambda_1+\lambda_2}(\mathbb{R}^{m_1+m_2})} \le \|f\|^*_{M_p^{\lambda_1}(\mathbb{R}^{m_1})\times M_p^{\lambda_2}(\mathbb{R}^{m_2})} \tag{3.47}$$

for any $f \in (M_p^{\lambda_1})^*(\mathbb{R}^{m_1}) \times (M_p^{\lambda_2})^*(\mathbb{R}^{m_2})$.

If $0 < \lambda_1 + \lambda_2 < \frac{m_1+m_2}{p}$, *then inclusion* (3.46) *is strict.*

Theorem 3.6. *Let* $1 \le p < \infty,\ m, n \in \mathbb{N},\ m < n,\ 0 \le \lambda \le \frac{n}{p}$, *then*

$$\|T_\mu\|^*_{L_\infty(\mathbb{R}^{n-m})\times M_p^{\lambda}(\mathbb{R}^m)} \le 3^{n-m}\mu^{\frac{n-m}{p}} \|T_\mu\|^*_{L_{p,v}(\mathbb{R}^{n-m})\times M_p^{\lambda}(\mathbb{R}^m)}, \tag{3.48}$$

in particular, if $x = (u, v),\ u = (x_1 \dots x_m),\ v = (x_{m+1}, \dots, x_n)$, *then*

$$\|T_\mu(u, 0)\|^*_{M_p^{\lambda}(\mathbb{R}^m)} \le 3^{n-m}\mu^{\frac{n-m}{p}} \|T_\mu\|^*_{L_p(\mathbb{R}^{n-m})\times M_p^{\lambda}(\mathbb{R}^m)}. \tag{3.49}$$

Remark 7. If $\lambda = 0$, then it is obvious that

$$L_p^*(\mathbb{R}^{n-m}) \times (M_p^0)^*(\mathbb{R}^m) = L_p^*(\mathbb{R}^{n-m}) \times L_p^*(\mathbb{R}^m) = L_p^*(\mathbb{R}^n) \tag{3.50}$$

however, for $0 < \lambda \le \frac{m}{p}$ according to Lemma 3.5

$$L_p^*(\mathbb{R}^{n-m}) \times (M_p^{\lambda})^*(\mathbb{R}^m) \subset (M_p^{\lambda})^*(\mathbb{R}^n), \tag{3.51}$$

but

$$L_p^*(\mathbb{R}^{n-m}) \times (M_p^{\lambda})^*(\mathbb{R}^m) \ne (M_p^{\lambda})^*(\mathbb{R}^n). \tag{3.52}$$

Acknowledgments

The research of Sections 3.1 and 3.3 of the first author V.I. Burenkov was financially supported by the Russian Science Foundation (project no. 24-11-00170, https://rscf.ru/project/24-11-00170/) and the research of Section 3.2 of the first author V.I. Burenkov was financially supported by the Russian Science Foundation (project no. 22-11-00042, https://rscf.ru/project/22-11-00042).

References

[1] V.I. Burenkov, *Sobolev spaces on domains*, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], 137, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1997 , 312 pp.

[2] V.I. Burenkov, D.J. Joseph, *Inequalities for entire functions of exponential type in Morrey spaces*, Eurasian Math. J., 13 (2022), no. 3, 92-99.

[3] V.I. Burenkov, D.J. Joseph, πïSIntegral Inequalities for Entire Functions of Exponential Type in Morrey SpacesπïS, Proc. Steklov Inst. Math., 323 (2023), 81πïS100 crossref; (in Russian).

[4] V.I. Burenkov, T.V. Tararykova *An analogue of Young's inequality for convolutions of functions for general Morrey-type spaces*, Tr. MIAN, 293, MAIK, M., 2016, 113πïS132 (in Russian).

[5] M.L. Gol'dman, *Description of traces for some function spaces*, Tr. MIAN USSR, 1979, volume 150, 99πïS127 (in Russian).

[6] I.I. Ibragimov, *Extremal properties of entire functions of finite order*, Izdat. Akad. Nauk Azerbaijan. SSR, Baku, 1962. 316 pp. (in Russian).

[7] I.I. Ibragimov, *Extremum problems in the class of trigonometric polynomials*, Dokl. Akad. Nauk. SSSR, 121:3 1958 415-417. (in Russian).

[8] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. 43 (1938), 126-166.

[9] S.M. Nikol'skii, *Approximation of functions of several variables and embedding theorems.*, Izd. 2, revised. and additional .. - Moscow: Nauka, 1977. -456 pp. (in Russian).

[10] V.N. Temlyakov, *Multivariate approximation*, Cambridge Monogr. Appl. Comput. Math., 32, Cambridge University Press, Cambridge, 2018 , 550 pp

Victor Ivanovich Burenkov
V.A. Steklov Mathematical Institute
Russian Academy of Sciences
42 Gubkin St
117966 Moscow, Russian Federation
and
V.A. Trapeznikov Institute of Control Sciences
Russian Academy of Sciences
65 Profsoyuznaya St
117997 Moscow, Russian Federation
E-mail: burenkov@cf.ac.uk.

Daryl James Joseph
S.M. Nikol'skii Mathematical Institute
RUDN University
6 Miklukho Maklay St
117198 Moscow, Russian Federation
E-mail: dj_144life@hotmail.com

Received: 06.06.2024