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INTERPOLATION METHODS FOR ANISOTROPIC NET SPACES

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Abstract. In this paper, we study the interpolation properties of anisotropic net spaces $N_{\bar{p},\bar{q}}(M)$, where $\bar{p} = (p_1, ..., p_n)$, $\bar{q} = (q_1, ..., q_n)$. It is shown that, with respect to the multidimensional interpolation method, the following equality holds

$$(N_{\bar{p}_0,\bar{q}_0}(M), N_{\bar{p}_1,\bar{q}_1}(M))_{\bar{\theta},\bar{q}} = N_{\bar{p},\bar{q}}(M), \quad \frac{1}{\bar{p}} = \frac{1-\bar{\theta}}{\bar{p}_0} + \frac{\bar{\theta}}{\bar{p}_1}.$$

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1 Introduction

Let M be the set of all segments from $\mathbb{R}$. For a function $f(x)$, defined and integrable on each segment Q of M, we define the function

$$\bar{f}(t, M) = \sup_{\substack{Q\in M\\ |Q|>t}} \frac{1}{|Q|} \left| \int_Q f(x)dx \right|, \quad t > 0,$$

where the supremum is taken over all segments $Q \in M$, whose length is $|Q| > t$. The function $\bar{f}(t, M)$ is called the averaging of the function f over the net M.

We define the net spaces $N_{p,q}(M)$, $0 < p, q \leq \infty$ as the set of all functions f, such that for $q < \infty$

$$\|f\|_{N_{p,q}(M)} = \left(\int_0^\infty \left(t^{\frac{1}{p}} \bar{f}(t, M) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty,$$

and for $q = \infty$

$$\|f\|_{N_{p,\infty}(M)} = \sup_{t>0} t^{\frac{1}{p}} \bar{f}(t, M) < \infty.$$

These spaces were introduced in work [18]. Net spaces are an important research tool in the theory of Fourier series, in operator theory and in other areas [1]-[3], [19]-[23], [24], [28], [29].

It was shown in [17] that the scale of spaces $N_{p,q}(M)$ is closed under the real interpolation method, i.e. for $p_0 \neq p_1$ holds

$$(N_{p_0,q_0}(M), N_{p_1,q_1}(M))_{\theta,q} = N_{p,q}(M).$$

If in the definition of the space $N_{p,q}(M)$ instead of $\bar{f}(t, M)$ we consider the function

$$\sup_{\substack{Q\in M\\ |Q|>t}} \frac{1}{|Q|} \int_Q |f(x)|dx,$$

then the corresponding space, as can be seen from [9], coincides with the Morrey space $M^{\alpha}_{p,q}$, where $\alpha = \frac{1}{p} - \frac{1}{q}$, but for the scale of these spaces it is known that it is not closed under the real interpolation method (see [7], [25], [26]).

We consider the following generalization of the space $N_{p,q}(M)$ in the n-dimensional case.

Let $\tau \in \mathbb{Z}$, by G_τ we denote the set of all segments of the form $[0, 2^\tau\} + k^\tau$, $k \in \mathbb{Z}$. Let $G = \bigcup G_\tau$ be the set of all dyadic segments. Let M be a set of all parallelipeds of the form

$$Q = Q_1 \times \cdots \times Q_n$$

where $Q_i \in G$, $i = 1, \ldots, n$. We will call M *dyadic net.*

For the function $f(x) = f(x_1, ..., x_n)$ integrable on every set $Q \in M$ we define

$$\bar{f}(t; M) = \bar{f}(t_1, ..., t_n; M) = \sup_{|Q_i| \geq t_i} \frac{1}{|Q_n|} \left| \int_Q f(x_1, ..., x_n) dx_1 ... dx_n \right|, \quad t_i > 0,$$

where $|Q_i|$ is the length of the segment Q_i.

Let $0 < \bar{p} = (p_1, ..., p_n) < \infty$, $0 < \bar{q} = (q_1, ..., q_n) \leq \infty$. Denote by $N_{\bar{p},\bar{q}}(M)$ the set of all functions $f(x) = f(x_1, ..., x_n)$, for which

$$\|f\|_{N_{\bar{p},\bar{q}}(M)} = \left(\int_0^\infty \cdots \left(\int_0^\infty \left(t_1^{\frac{1}{p_1}} ... t_n^{\frac{1}{p_n}} \bar{f}(t_1, ..., t_n; M) \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \cdots \frac{dt_n}{t_n} \right)^{\frac{1}{q_n}} < \infty,$$

here and below, when $q = \infty$, the expression $\left(\int_0^\infty (\varphi(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}$ is understood as $\sup_{t>0} \varphi(t)$.

As can be seen from the definition of the space $N_{\bar{p},\bar{q}}(M)$, this is the space of functions that have different characteristics for each variable. These spaces are called *anisotropic net spaces.*

For spaces with a mixed metric, anisotropic spaces, the real interpolation method does not work. For the interpolation of mixed metric spaces, the interpolation method was introduced by D.L. Fernandez [11]-[13] and modified in [14], [17], [20], [21]. An interpolation theorem regarding this method for Lebesgue spaces $L_{\bar{p}}$ with a mixed metric was obtained in [22]: *let* $0 < \bar{p}_i < \infty$ *and* $p_0^i \neq p_1^i$, $i = 0, 1$, $0 < \bar{q} \leq \infty$, $0 < \bar{\theta} < 1$, *then*

$$(L_{\bar{p}_0}, L_{\bar{p}_1})_{\bar{\theta},\bar{q}} = L_{\bar{p},\bar{q}}, \quad \frac{1}{\bar{p}} = \frac{1 - \bar{\theta}}{\bar{p}_0} + \frac{\bar{\theta}}{\bar{p}_1},$$

where $L_{\bar{p},\bar{q}}$ *is the anisotropic Lorentz space.* (see [8])

Other applications of this method can be found in [6], [22].

The purpose of this paper is to obtain an interpolation theorem for anisotropic net spaces.

Given functions F and G, in this paper $F \asymp G$ means that $F \leq CG$ and $G \leq CF$, where C is a positive number, depending only on numerical parameters, that may be different on different occasions.

2 Lemmas

Let $\tau = (\tau_1, ..., \tau_n)$. The system of all sets $G_\tau = G_{\tau_1} \times \cdots \times G_{\tau_n} = \left\{ I_k = I^1_{k_1} \times \cdots \times I^n_{k_n} : \quad I^i_{k_i} \in G_{\tau_i} \right\}$ defines the partition of $\mathbb{R}^n$ into parallelepipeds, i.e. $\mathbb{R}^n = \bigcup\limits_{k \in \mathbb{Z}} I_k$.

Let $E = \{\epsilon = (\epsilon_1, ..., \epsilon_n) : \epsilon_i \in \{0, 1\}\}$ be the vertices of the unit cube in $\mathbb{R}^n$. For a locally integrable function $f(x_1, ..., x_n)$ and a set G_τ we define the functions $f_\epsilon(x)$, $\epsilon \in E$ as follows:

$$f_\epsilon(x) = \frac{1}{\prod\limits_{i=1}^{n} |I^i_{k_i}|} \int_{I^n_{k_n}} \cdots \int_{I^1_{k_1}} \Delta^\epsilon_x f(x_1', ..., x_n') dx_1' ... dx_n' \quad x \in I^1_{k_1} \times \cdots \times I^n_{k_n}, \tag{2.1}$$

where

$$\Delta_x^\epsilon f(x') = \Delta_{x_n}^{\epsilon_n} \dots \Delta_{x_1}^{\epsilon_1} f(x'),$$

$$\Delta_{x_i}^{\epsilon_i}\phi(x_i') = \begin{cases} \phi(x_i'), & \text{for} \quad \epsilon = 0, \\ \phi(x_i) - \phi(x_i'), & \text{for} \quad \epsilon = 1. \end{cases}$$

Note that $f(x) = \sum_{\epsilon\in E} f_\epsilon(x)$. These functions $\{f_\epsilon\}_{\epsilon\in E}$ will be called the expansion of the function $f(x)$, corresponding to the partition G_τ.

Lemma 2.1. *Let $\tau = (\tau_1, ..., \tau_n) \in \mathbb{Z}^n$, G_τ be the partition of $\mathbb{R}^n$ into rectangles, $f(x_1, ..., x_n)$ be locally integrable on $\mathbb{R}^n$. $f = \sum_{\epsilon\in E} f_\epsilon(x)$ be the decomposition corresponding to the partition G_τ. Then for $\epsilon_i = 1$*

$$\frac{1}{|I_k^i|}\int_{I_k^i} f_\epsilon(x_1, ..., x_n)dx_i = \begin{cases} 0, & \epsilon_i = 1 \\ f_\epsilon(x_1, ..., x_n), & \epsilon_i = 0 \end{cases}, \qquad k \in \mathbb{Z}.$$

The proof follows from the definitions of the functions f_ϵ.

Lemma 2.2. *Let $\tau = (\tau_1, ..., \tau_n) \in \mathbb{Z}^n$, $\tau_i > 0$, G_τ be a partition of $\mathbb{R}^n$ into rectangles, $f(x_1, ..., x_n)$ be locally integrable on $\mathbb{R}^n$ and $\{f_\epsilon\}_{\epsilon\in E}$ be the decomposition of the function $f(x)$, corresponding to the partition G_τ. Then for an arbitrary $t \in \mathbb{Z}^n$*

$$\bar{f}_\epsilon(2^{t_1}, ..., 2^{t_n}; M)$$

$$\leq \begin{cases} 2^{|\epsilon|}\prod_{i=1}^n \min\{2^{\tau_i - t_i}, 1\}\bar{f}(2^{t_1\epsilon_1+\tau_1(1-\epsilon_1)}, ..., 2^{t_n\epsilon_n+\tau_n(1-\epsilon_n)}; M), & \textit{for} \quad t_i\epsilon_i < \tau_i, \quad i = \overline{0, n}, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.2}$$

where $|\epsilon| = \epsilon_1 + ... + \epsilon_n$.

Proof. Let $Q = Q_1 \times \cdots \times Q_n \in M$, $|Q_i| = 2^{s_i}$. Let us prove the following equality

$$\frac{1}{|Q|}\left|\int_Q f_\epsilon(x)dx\right| = \frac{1}{|Q_n|}\left|\int_{Q_n} \Delta_{x_n}^{\epsilon_n}\frac{1}{|Q_{n-1}|}\int_{Q_{n-1}} \Delta_{x_{n-1}}^{\epsilon_{n-1}} \cdots \frac{1}{|Q_1|}\int_{Q_1} \Delta_{x_1}^{\epsilon_1} f(x_1', ..., x_n')dx_1' \dots dx_n'\right|. \tag{2.3}$$

Since for $s_i \geq \tau_i$ the segment Q_i splits into segments from G_τ, then if for some index i, $s_i \geq \tau_i$ and $\epsilon_i = 1$ are satisfied, then

$$\frac{1}{|Q|}\left|\int_Q f_\epsilon(x)dx\right| = 0.$$

Therefore, we assume that if $\epsilon_i = 1$, then $s_i < \tau_i$. Further, in the case when $\epsilon_i = 0$ and $s_i < \tau_i$ we have

$$\frac{1}{|Q_i|}\int_{Q_i} \Delta_{x_i}^{\epsilon_i}\frac{1}{|Q_{i-1}|}\int_{Q_{i-1}} \Delta_{x_{i-1}}^{\epsilon_{i-1}} \cdots \frac{1}{|Q_1|}\int_{Q_1} \Delta_{x_1}^{\epsilon_1} f(x_1', ..., x_n')dx_1' \dots dx_i'$$

$$= \Delta_{x_i}^{\epsilon_i}\frac{1}{|Q_{i-1}|}\int_{Q_{i-1}} \Delta_{x_{i-1}}^{\epsilon_{i-1}} \cdots \frac{1}{|Q_1|}\int_{Q_1} \Delta_{x_1}^{\epsilon_1} f(x_1', ..., x_n')dx_1' \dots dx_{i-1}'.$$

And in the case when $\epsilon = 0$ and $s_i \geq \tau_i$

$$\frac{1}{|Q_i|}\int_{Q_i} \Delta_{x_i}^{\epsilon_i}\frac{1}{|Q_{i-1}|}\int_{Q_{i-1}} \Delta_{x_{i-1}}^{\epsilon_{i-1}} \cdots \frac{1}{|Q_1|}\int_{Q_1} \Delta_{x_1}^{\epsilon_1} f(x_1', ..., x_n')dx_1' \dots dx_i'$$

$$= \sum_{I_k^i \subset Q_i} \frac{1}{|Q_i|} \int_{I_k^i} \Delta_{x_i}^{\epsilon_i} \frac{1}{|Q_{i-1}|} \int_{Q_{i-1}} \Delta_{x_{i-1}}^{\epsilon_{i-1}} \dots \frac{1}{|Q_1|} \int_{Q_1} \Delta_{x_1}^{\epsilon_1} f(x_1', ..., x_n') dx_1' \dots dx_i'.$$

By the above equalities, we have

$$\frac{1}{|Q|} \left| \int_Q f_\epsilon(x) dx \right| \le 2^{|\epsilon|} \prod_{i=1}^n \min\{2^{\tau_i - s_i}, 1\} \bar{f}(2^{s_1\epsilon_1 + \tau_1(1-\epsilon_1)}, ..., 2^{s_n\epsilon_n + \tau_n(1-\epsilon_n)}; M).$$

Taking into account that $s_i \ge t_i$ we get

$$\frac{1}{|Q|} \left| \int_Q f_\epsilon(x) dx \right| \le 2^{|\epsilon|} \prod_{i=1}^n \min\{2^{\tau_i - t_i}, 1\} \bar{f}(2^{t_1\epsilon_1 + \tau_1(1-\epsilon_1)}, ..., 2^{t_n\epsilon_n + \tau_n(1-\epsilon_n)}; M).$$

□

We will use the classical Hardy inequalities. Let us formulate them as a lemma.

Lemma 2.3 (Hardy's inequality)**.** *Let $1 \le q < \infty$, $\alpha > 0$, then the inequalities hold*

$$\left(\int_0^\infty \left(t^\alpha \int_t^\infty \varphi(s) ds \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \le \alpha^{-1} \left(\int_0^\infty \left(t^{1+\alpha} \varphi(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

$$\left(\int_0^\infty \left(t^{-\alpha} \int_0^t \varphi(s) ds \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \le \alpha^{-1} \left(\int_0^\infty \left(t^{1-\alpha} \varphi(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

3 Main result

Let us consider the interpolation method for anisotropic spaces proposed by Nursultanov E.D. [20]. This method is based on the ideas of G. Sparr [27] , D.L. Fernandez [11]-[13] and others [10], [15], [16]. Some results related to the interpolation of anisotropic net spaces were obtained in papers [4], [5].

Let $\mathbf{A_0} = (A_1^0, ..., A_n^0)$, $\mathbf{A_1} = (A_1^1, ..., A_n^1)$ be two anisotropic spaces, $E = \{\varepsilon = (\varepsilon_1, ..., \varepsilon_n) : \varepsilon_i = 0,$ or $\varepsilon_i = 1, \ \ i = 1, ..., n\}$. For arbitrary $\varepsilon \in E$ we define the space $\mathbf{A}_\varepsilon = (A_1^{\varepsilon_1}, ..., A_n^{\varepsilon_n})$ with the norm

$$\|f\|_{\mathbf{A}_\varepsilon} = \| \dots \|f\|_{A_1^{\varepsilon_1}} \dots \|_{A_n^{\varepsilon_n}}.$$

Let $0 < \bar{\theta} = (\theta_1, ..., \theta_n) < 1$, $0 < \bar{q} = (q_1, ..., q_n) \le \infty$. Via $\mathbf{A}_{\bar{\theta},\bar{q}} = (\mathbf{A_0}, \mathbf{A_1})_{\bar{\theta},\bar{q}}$ denote the linear subset $\sum_{\varepsilon \in E} \mathbf{A}_\varepsilon$, of all elements, for which

$$\|f\|_{\mathbf{A}_{\bar{\theta},\bar{q}}} = \left(\int_0^\infty \dots \left(\int_0^\infty \left(t_1^{-\theta_1} \dots t_n^{-\theta_n} K(t_1, ..., t_n; f) \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \dots \frac{dt_n}{t_n} \right)^{\frac{1}{q_n}} < \infty,$$

where

$$K(t, f; \mathbf{A_0}, \mathbf{A_1}) = \inf \left\{ \sum_{\varepsilon \in E} t^\varepsilon \|f_\varepsilon\|_{\mathbf{A}_\varepsilon} \ : \ f = \sum_{\varepsilon \in E} f_\varepsilon, \ f_\varepsilon \in \mathbf{A}_\varepsilon \right\},$$

where $t^\varepsilon = t_1^{\varepsilon_1} ... t_n^{\varepsilon_n}$.

Lemma 3.1. *Let $a_i > 1, i = 1, ..., n$ $0 < \bar{\theta} = (\theta_1, ..., \theta_n) < 1$, $0 < \bar{q} = (q_1, ..., q_n) \le \infty$. Then*

$$\|f\|_{\mathbf{A}_{\bar{\theta},\bar{q}}} \asymp \left(\sum_{k_n \in \mathbb{Z}} \dots \left(\sum_{k_1 \in \mathbb{Z}} \left(a_1^{-\theta_1 k_1} \dots a_n^{-\theta_n k_n} K(a_1^{k_1}, ..., a_n^{k_n}; f) \right)^{q_1} \right)^{\frac{q_2}{q_1}} \dots \right)^{\frac{1}{q_n}} = J_{\bar{\theta},\bar{q}}(f).$$

Proof. From the definition of the space $\mathbf{A}_{\bar\theta,\bar q}$ we have

$$\|f\|_{\mathbf{A}_{\bar\theta,\bar q}} = \left(\int_0^\infty \dots \left(\int_0^\infty \left(t_1^{-\theta_1}\dots t_n^{-\theta_n} K(t_1,...,t_n;f)\right)^{q_1} \frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}} \dots \frac{dt_n}{t_n}\right)^{\frac{1}{q_n}}$$

$$= \left(\sum_{k_n\in\mathbb{Z}} \int_{a_n^{k_n}}^{a_n^{k_n+1}} \dots \left(\sum_{k_n\in\mathbb{Z}} \int_{a_1^{k_1}}^{a_1^{k_1+1}} \left(t_1^{-\theta_1}\dots t_n^{-\theta_n} K(t_1,...,t_n;f)\right)^{q_1} \frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}} \dots \frac{dt_n}{t_n}\right)^{\frac{1}{q_n}}.$$

If the function $\Phi(t_i)$ is monotonically non-decreasing in the variable t_i then we get

$$\left(a_i^{-\theta_i(k_i+1)}\Phi(a_i^{-\theta_i k_i})\right)^{q_i} \ln a_i \le \int_{a_i^{k_1}}^{a_i^{k_1+1}} \left(t_i^{-\theta_i}\Phi(t_i)\right)^{q_i} \frac{dt_i}{t_i} \le \left(a_i^{-\theta_1 k_1}\Phi(a_i^{-\theta_1(k_1+1)})\right)^{q_i} \ln a_i.$$

Applying this relation and taking into account that $K(t_1,...,t_n;f)$ is non-decreasing in each variable, we obtain

$$C_1 J_{\bar\theta,\bar q}(f) \le \|f\|_{\mathbf{A}_{\bar\theta,\bar q}} \le C_2 J_{\bar\theta,\bar q}(f),$$

where

$$C_1 = \prod_{i=1}^n a_i^{-\theta_i}(\ln a_i)^{\frac{1}{q_i}},$$

and

$$C_2 = \prod_{i=1}^n a_i^{\theta_i}(\ln a_i)^{\frac{1}{q_i}}.$$

□

Theorem 3.1. *Let M be the dyadic net in $\mathbb{R}^n$, $0 < \bar p_1 = (p_1^1,...,p_n^1) < \bar p_0 = (p_1^0,...,p_n^0) < \infty$, $0 < \bar q_0, \bar q, \bar q_1 \le \infty$, $0 < \bar\theta = (\theta_1,...,\theta_n) < 1$, then*

$$(N_{\bar p_0,\bar q_0}(M), N_{\bar p_1,\bar q_1}(M))_{\bar\theta,\bar q} = N_{\bar p,\bar q}(M), \tag{3.1}$$

where $\frac{1}{\bar p} = \frac{1-\bar\theta}{\bar p_0} + \frac{\bar\theta}{\bar p_1}$.

Proof. Let us prove the continuous embedding

$$N_{\bar p,\bar q}(M) \hookrightarrow (N_{\bar p_0,\bar v}(M), N_{\bar p_1,\bar v}(M))_{\bar\theta,\bar q}, \tag{3.2}$$

where $\bar v = (v,...,v)$, $v = \min_{1\le i\le n} q_i$.

Let $\tau = (\tau_1,...,\tau_n) \in \mathbb{Z}^n$, G_τ be a partition of $\mathbb{R}^n$, $f \in N_{\bar p,\bar q}(M)$, $f = \sum_{\epsilon\in E} f_\epsilon(x)$ be the decomposition corresponding to the partition G_τ (f_ϵ is defined by the formula (2.1)).

Using Lemma 2.2, we get

$$\|f_\epsilon\|_{N_{\bar p_\epsilon,\bar v}} \asymp \left(\sum_{t_n\in Z}\dots\sum_{t_1\in Z}\left(2^{\frac{t_1}{p_1^{\epsilon_1}}}\dots 2^{\frac{t_n}{p_n^{\epsilon_n}}} \bar f_\epsilon(2^{t_1},...,2^{t_n};M)\right)^v\right)^{\frac{1}{v}}$$

$$\le 2^{|\epsilon|}\left(\sum_{\epsilon_i t_i<\tau_i}\left(\prod_{i=1}^n 2^{\frac{t_i}{p_i^{\epsilon_i}}} \min\{2^{\tau_i-t_i},1\} \bar f(2^{t_1\epsilon_1+\tau_1(1-\epsilon_1)},...,2^{t_n\epsilon_n+\tau_n(1-\epsilon_n)};M)\right)^v\right)^{\frac{1}{v}}.$$

Hence for $a_i > 1, i = \overline{1,n}$, we have

$$K(a_1^{\tau_1}, .., a_n^{\tau_n}, f; N_{\bar{p}_\epsilon,\bar{v}}, \epsilon \in E) = \sum_{\epsilon\in E} a_1^{\epsilon_1\tau_1} \dots a_n^{\epsilon_n\tau_n} \|f_\epsilon\|_{N_{\bar{p}_\epsilon,\bar{v}}}$$

$$\leq 2^n \sum_{\epsilon\in E} a_1^{\epsilon_1\tau_1} \dots a_n^{\epsilon_n\tau_n} \left(\sum_{\epsilon_i t_i<\tau_i} \left(\prod_{i=1}^n 2^{\frac{t_i}{p_i^{\epsilon_i}}} \min\{2^{\tau_i-t_i}, 1\} \bar{f}(2^{t_1\epsilon_1+\tau_1(1-\epsilon_1)}, ..., 2^{t_n\epsilon_n+\tau_n(1-\epsilon_n)}; M) \right)^v \right)^{\frac{1}{v}},$$

and

$$\|f\|_{(N_{\bar{p}_0,\bar{v}}(M), N_{\bar{p}_1,\bar{v}}(M))_{\bar{\theta},\bar{q}}} \asymp \left(\sum_{\tau_n\in Z} \dots \left(\sum_{\tau_1\in Z} \left(a_1^{-\theta_1\tau_1} \dots a_n^{-\theta_n\tau_n} K(a_1^{\tau_1}, ..., a_n^{\tau_n}, f) \right)^{q_1} \right)^{\frac{q_2}{q_1}} \dots \right)^{\frac{1}{q_n}}$$

$$\leq C \sum_{\epsilon\in E} \left(\sum_{\tau_n\in Z} \dots \left(\sum_{\tau_1\in Z} \left(a_1^{(\epsilon_1-\theta_1)\tau_1} \dots a_n^{(\epsilon_n-\theta_n)\tau_n} \times \right.\right.\right.$$

$$\left.\left.\left. \times \left(\sum_{\epsilon_i t_i<\tau_i} \left(\prod_{i=1}^n 2^{\frac{t_i}{p_i^{\epsilon_i}}} \min\{2^{\tau_i-t_i}, 1\} \bar{f}(2^{t_1\epsilon_1+\tau_1(1-\epsilon_1)}, ..., 2^{t_n\epsilon_n+\tau_n(1-\epsilon_n)}; M) \right)^v \right)^{\frac{1}{v}} \right)^{q_1} \right)^{\frac{q_2}{q_1}} \dots \right)^{\frac{1}{q_n}}, \quad (3.3)$$

where $C = 2^n 2^{\sum_{i=1}^n (1-\frac{1}{q_1})_+}$.

Let $\epsilon \in E$, using the definition of v and the generalized Minkowski inequality, we obtain

$$\left(\sum_{\tau_n\in Z} \dots \left(\sum_{\tau_1\in Z} \left(a_1^{(\epsilon_1-\theta_1)\tau_1} \dots a_n^{(\epsilon_n-\theta_n)\tau_n} \times \right.\right.\right.$$

$$\left.\left.\left. \times \left(\sum_{\epsilon_i t_i<\tau_i} \left(\prod_{i=1}^n 2^{\frac{t_i}{p_i^{\epsilon_i}}} \min\{2^{\tau_i-t_i}, 1\} \bar{f}(2^{t_1\epsilon_1+\tau_1(1-\epsilon_1)}, ..., 2^{t_n\epsilon_n+\tau_n(1-\epsilon_n)}; M) \right)^v \right)^{\frac{1}{v}} \right)^{q_1} \right)^{\frac{q_2}{q_1}} \dots \right)^{\frac{1}{q_n}},$$

$$\leq \left(\sum_{\tau_n\in Z} \left(a_n^{(\epsilon_n-\theta_n)\tau_n} \left(\sum_{\epsilon_n t_n<\tau_n} \left(2^{\frac{t_n}{p_n^{\epsilon_n}}} \min\{2^{\tau_n-t_n}, 1\} F_{n-1}(2^{t_n\epsilon_n+\tau_n(1-\epsilon_n)}) \right)^v \right)^{\frac{1}{v}} \right)^{q_n} \right)^{\frac{1}{q_n}},$$

where

$$F_{n-1}(y) = \left(\sum_{\tau_{n-1}\in Z} \dots \left(\sum_{\tau_1\in Z} \left(a_1^{(\epsilon_1-\theta_1)\tau_1} \dots a_n^{(\epsilon_n-\theta_n)\tau_n} \times \right.\right.\right.$$

$$\left.\left.\left. \times \left(\sum_{\epsilon_i t_i<\tau_i} \left(\prod_{i=1}^{n-1} 2^{\frac{t_i}{p_i^{\epsilon_i}}} \min\{2^{\tau_i-t_i}, 1\} \bar{f}(2^{t_1\epsilon_1+\tau_1(1-\epsilon_1)}, ..., y; M) \right)^v \right)^{\frac{1}{v}} \right)^{q_1} \right)^{\frac{q_2}{q_1}} \dots \right)^{\frac{1}{q_n}}.$$

Let $a_n = 2^{\frac{1}{p_n^0} - \frac{1}{p_n^1}}$. If $\epsilon_n = 0$, then we have

$$\left(\sum_{\tau_n\in Z} \left(a_n^{(\epsilon_n-\theta_n)\tau_n} \left(\sum_{\epsilon_n t_n<\tau_n} \left(2^{\frac{t_n}{p_n^{\epsilon_n}}} \min\{2^{\tau_n-t_n}, 1\} F_{n-1}(2^{t_n\epsilon_n+\tau_n(1-\epsilon_n)}) \right)^v \right)^{\frac{1}{v}} \right)^{q_n} \right)^{\frac{1}{q_n}}$$

$$= \left(\sum_{\tau_n \in Z} \left(2^{-\theta_n \tau_n (\frac{1}{p_n^0} - \frac{1}{p_n^1})} \left(\sum_{t_n \in Z} \left(2^{\frac{t_n}{p_n^0}} \min\{2^{\tau_n - t_n}, 1\} F_{n-1}(2^{\tau_n}) \right)^v \right)^{\frac{1}{v}} \right)^{q_n} \right)^{\frac{1}{q_n}}$$

$$= \left(\sum_{\tau_n \in Z} \left(2^{-\theta_n \tau_n (\frac{1}{p_n^0} - \frac{1}{p_n^1})} F_{n-1}(2^{\tau_n}) \left(\sum_{t_n = -\infty}^{\tau_n} \left(2^{\frac{t_n}{p_n^0}} \right)^v + \sum_{t_n = \tau_n + 1}^{\infty} \left(2^{\frac{t_n}{p_n^0}} 2^{\tau_n - t_n} \right)^v \right)^{\frac{1}{v}} \right)^{q_n} \right)^{\frac{1}{q_n}}$$

$$\asymp \left(\sum_{\tau_n \in Z} \left(2^{-\theta_n \tau_n (\frac{1}{p_n^0} - \frac{1}{p_n^1})} F_{n-1}(2^{\tau_n}) 2^{\frac{\tau_n}{p_n^0}} \right)^{q_n} \right)^{\frac{1}{q_n}} = \left(\sum_{\tau_n \in Z} \left(2^{\frac{\tau_n}{p_n}} F_{n-1}(2^{\tau_n}) \right)^{q_n} \right)^{\frac{1}{q_n}}.$$

In the last relation, we used the equality $\frac{1}{p_n} = \frac{1-\theta_n}{p_n^0} + \frac{\theta}{p_n^1}$.

If $\epsilon_n = 1$, then we get

$$\left(\sum_{\tau_n \in Z} \left(a_n^{(\epsilon_n - \theta_n)\tau_n} \left(\sum_{\epsilon_n t_n < \tau_n} \left(2^{\frac{t_n}{p_n^{\epsilon_n}}} \min\{2^{\tau_n - t_n}, 1\} F_{n-1}(2^{t_n \epsilon_n + \tau_n (1-\epsilon_n)}) \right)^v \right)^{\frac{1}{v}} \right)^{q_n} \right)^{\frac{1}{q_n}}$$

$$= \left(\sum_{\tau_n \in Z} \left(2^{(1-\theta)_n \tau_n (\frac{1}{p_n^0} - \frac{1}{p_n^1})} \left(\sum_{t_n = -\infty}^{\tau_n - 1} \left(2^{\frac{t_n}{p_n^1}} F_{n-1}(2^{t_n}) \right)^v \right)^{\frac{1}{v}} \right)^{q_n} \right)^{\frac{1}{q_n}}$$

$$\leq C \left(\sum_{\tau_n \in Z} \left(2^{\frac{\tau_n}{p_n}} F_{n-1}(2^{\tau_n}) \right)^{q_n} \right)^{\frac{1}{q_n}}.$$

where C>0 is independent of f. Here we have used Hardy's inequality and the equality $\frac{1}{p_n} = \frac{1-\theta_n}{p_n^0} + \frac{\theta}{p_n^1}$.

Further, applying to $F_{n-1}(2^{\tau_n})$ the same procedure as above, after $n-1$ steps we obtain the estimate of the form

$$\left(\sum_{\tau_n \in Z} \left(a_n^{(\epsilon_n - \theta_n)\tau_n} \left(\sum_{\epsilon_n t_n < \tau_n} \left(\min\{2^{\tau_n - t_n (1 - \frac{1}{p_n^{\epsilon_n}})}, 2^{\frac{t_n}{p_n^{\epsilon_n}}}\} F_{n-1}(2^{t_n \epsilon_n + \tau_n (1-\epsilon_n)}) \right)^v \right)^{\frac{1}{v}} \right)^{q_n} \right)^{\frac{1}{q_n}}$$

$$\leq C \left(\sum_{\tau_n \in Z} \dots \left(\sum_{\tau_1 \in Z} \left(2^{\frac{\tau_n}{p_n}} \dots 2^{\frac{\tau_1}{p_1}} \bar{f}_{n-1}(2^{\tau_1}, ..., 2^{\tau_n}; M) \right)^{q_1} \right)^{\frac{q_2}{q_1}} \dots \right)^{\frac{1}{q_n}} \asymp \|f\|_{N_{\bar{p},\bar{q}}(M)}.$$

where C>0 is independent of f.

Substituting the resulting relation into (3.3) we get (3.2). Thus, taking into account that $v = \min_{1 \leq i \leq n} q_i$, we get the continuous embedding

$$N_{\bar{p},\bar{q}}(M) \hookrightarrow (N_{\bar{p}_0,\bar{v}}(M), N_{\bar{p}_1,\bar{v}}(M))_{\bar{\theta},\bar{q}} \hookrightarrow (N_{\bar{p}_0,\bar{q}_0}(M), N_{\bar{p}_1,\bar{q}_1}(M))_{\bar{\theta},\bar{q}}.$$

The reverse continuous embedding $(N_{\bar{p}_0,\bar{q}_0}(M), N_{\bar{p}_1,\bar{q}_1}(M))_{\bar{\theta},\bar{q}} \hookrightarrow N_{\bar{p},\bar{q}}(M)$ was proved in [20] (see Theorem 1). □

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